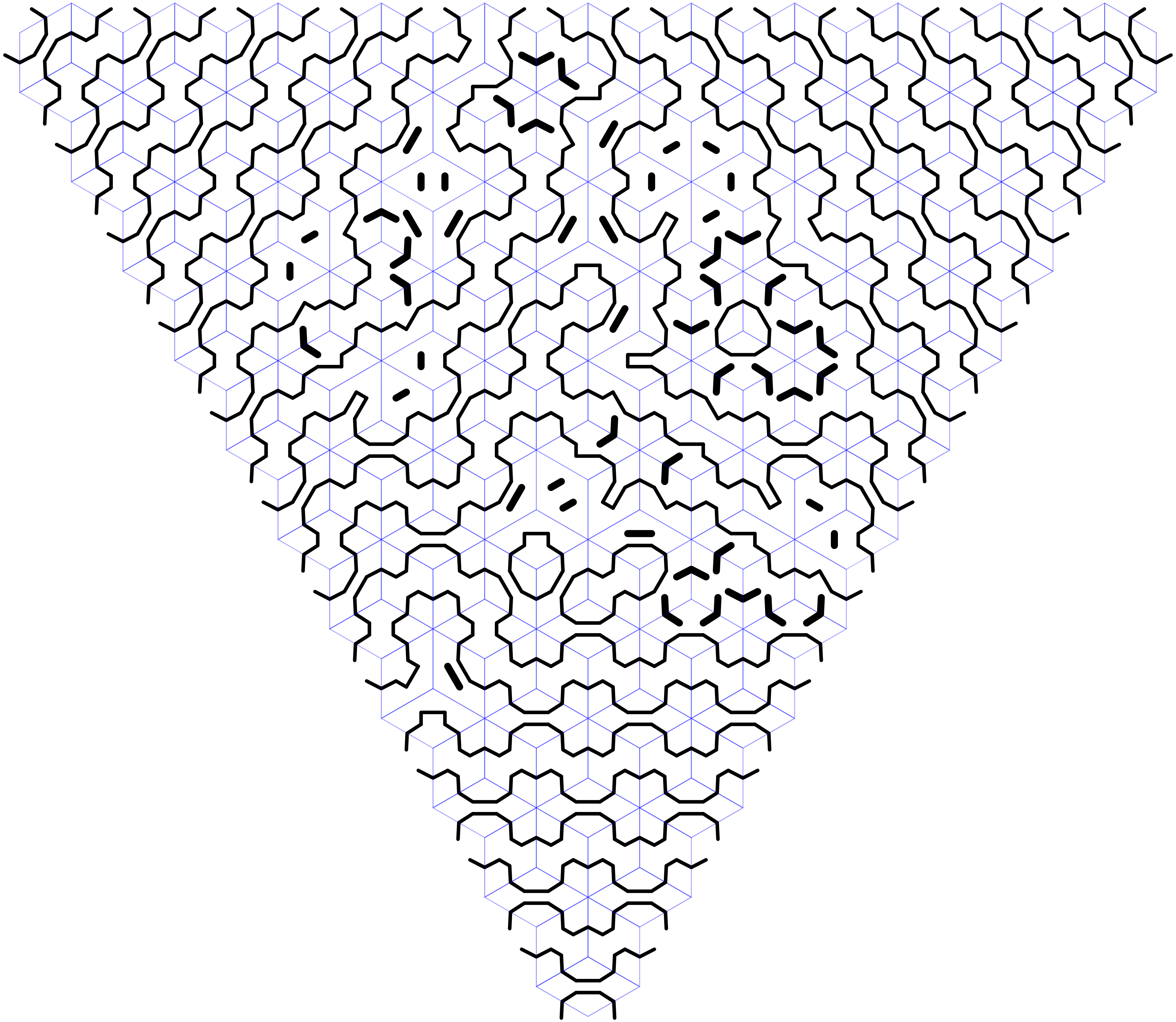


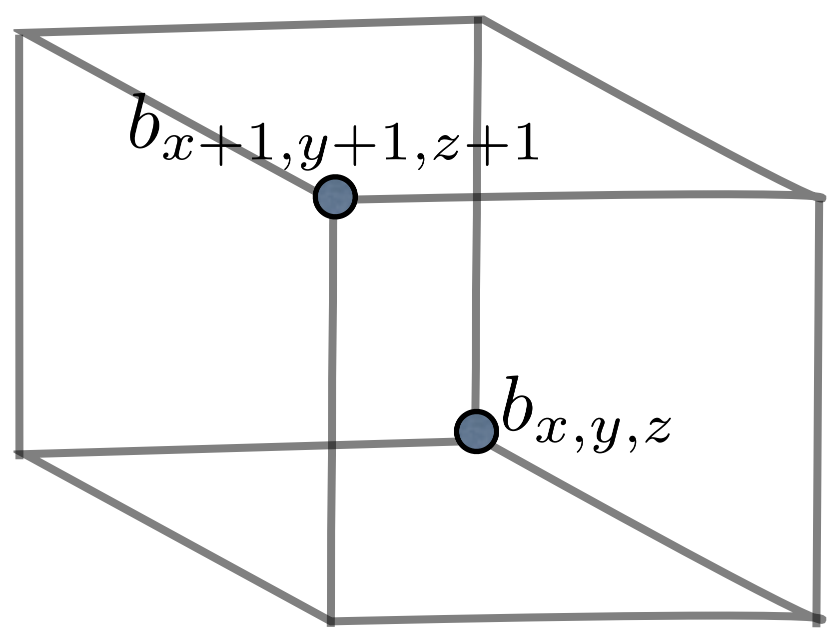
The Hexahedron Recurrence and the Ising Model

R. Kenyon (Brown)

R. Pemantle (UPenn)

Cube recurrence = Miwa equation

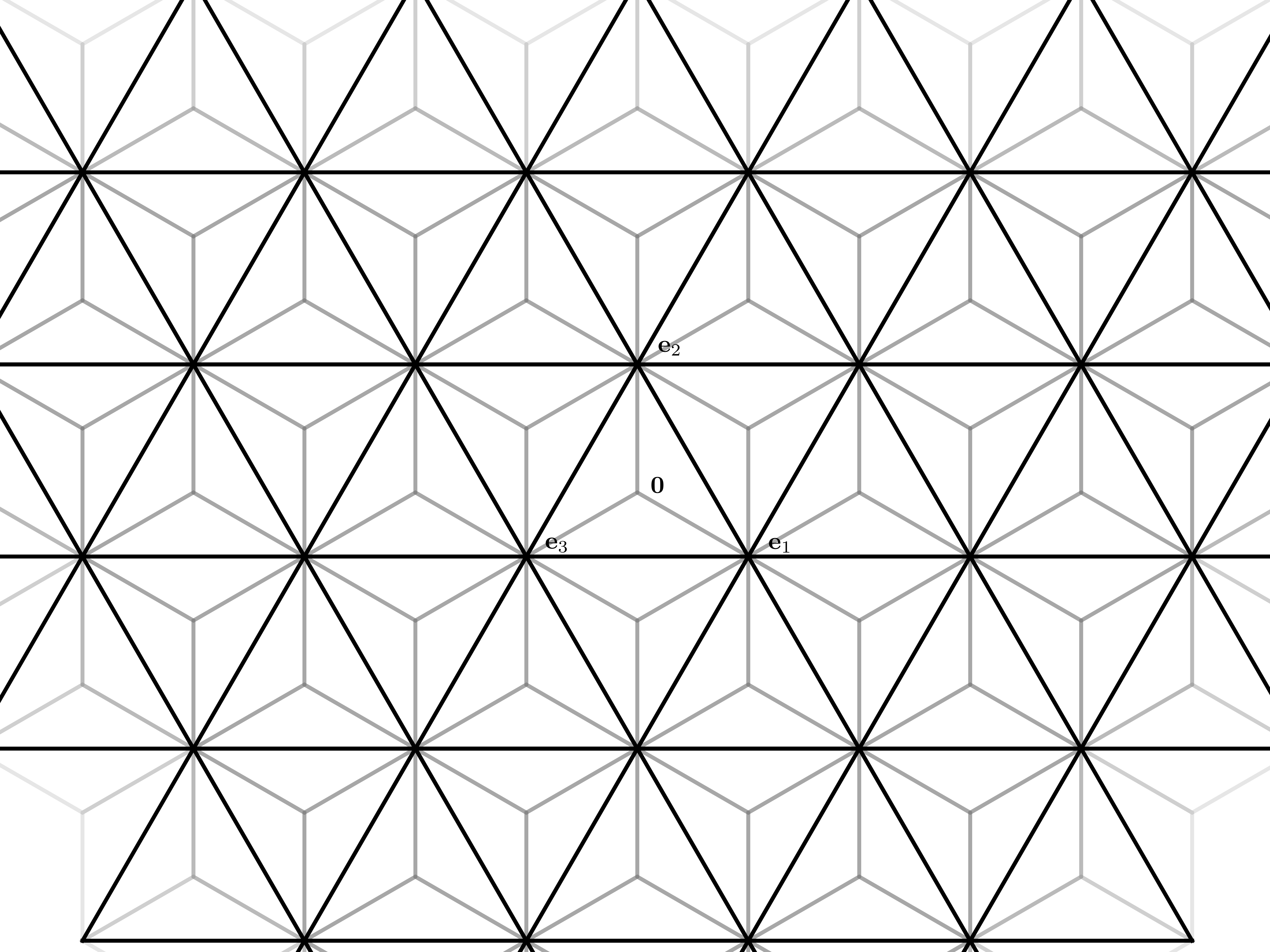
$$b_{x+1,y+1,z+1} = \frac{b_{x+1,y,z}b_{x,y+1,z+1} + b_{x,y+1,z}b_{x+1,y,z+1} + b_{x,y,z+1}b_{x+1,y+1,z}}{b_{x,y,z}}$$



From the values on $0 \leq x+y+z \leq 2$ and the recurrence, get all $b_{x,y,z}$.

The Laurent property holds:

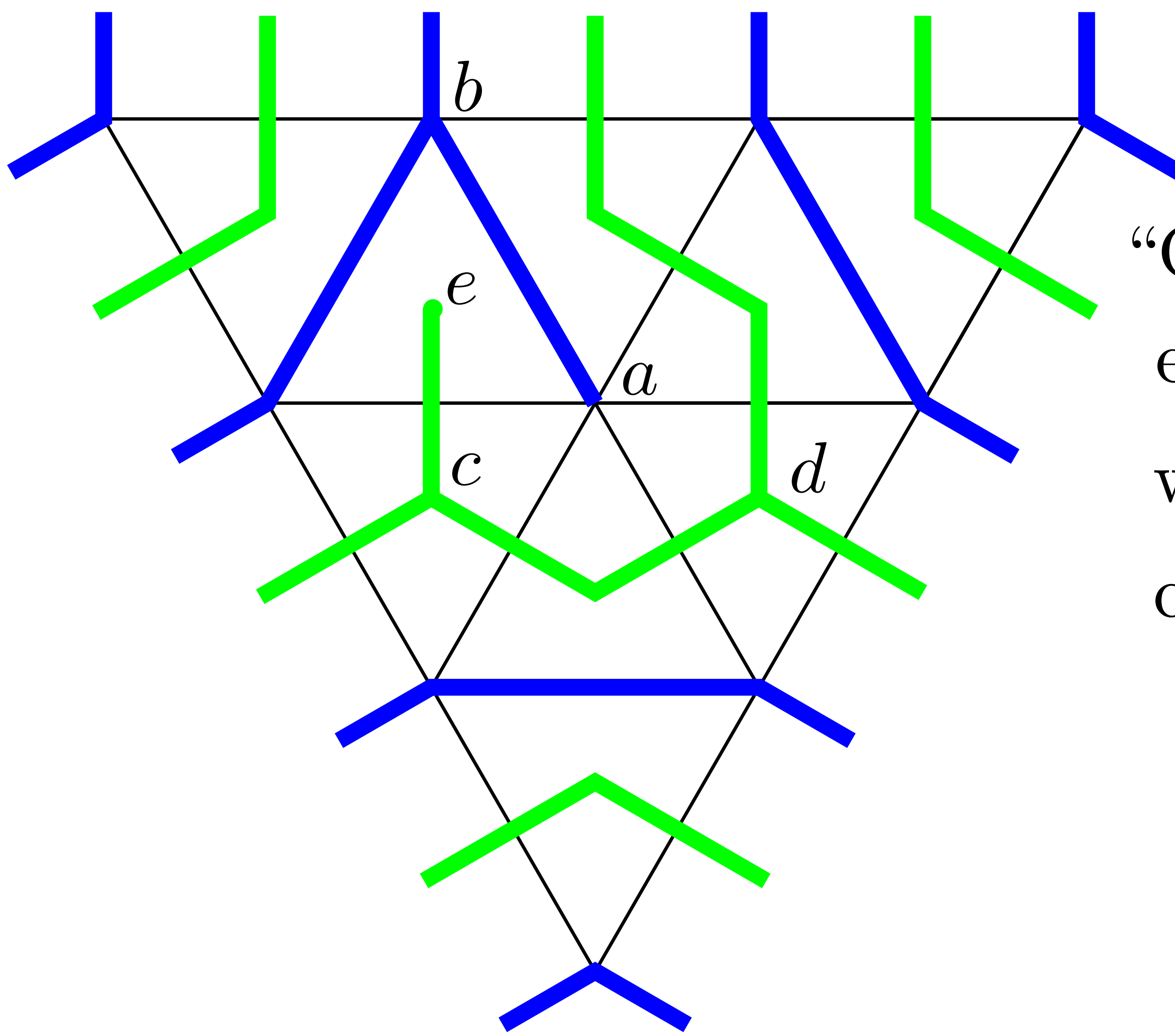
$b_{i,j,k}$ is a Laurent polynomial in the initial variables.


e₂
0
e₃
e₁

Theorem (Carroll-Speyer):

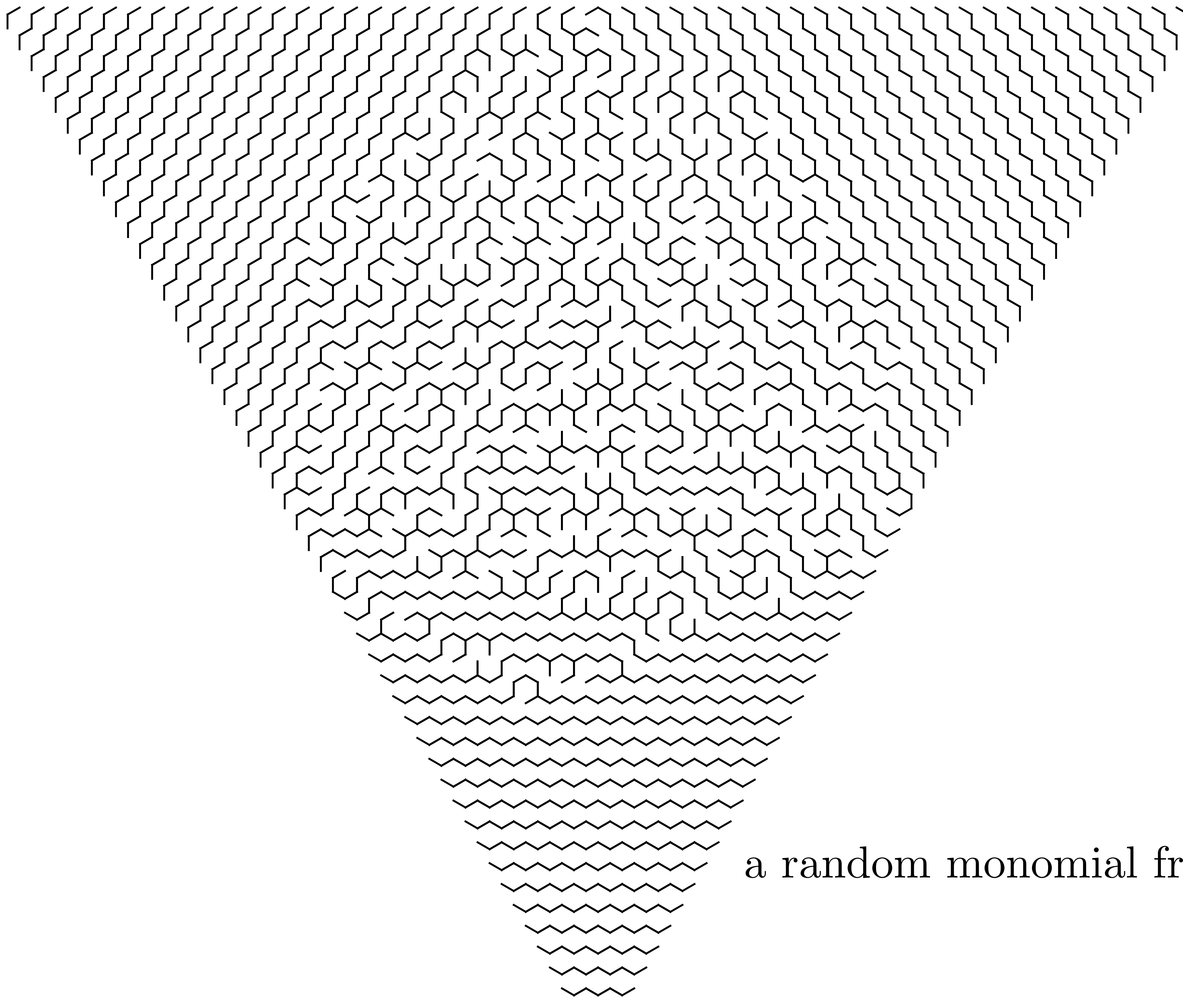
Terms in the Laurent expansion are in bijection with "cube groves".

$$b_{2,1,1} = \cdots + \frac{bcd}{ae} + \ldots$$



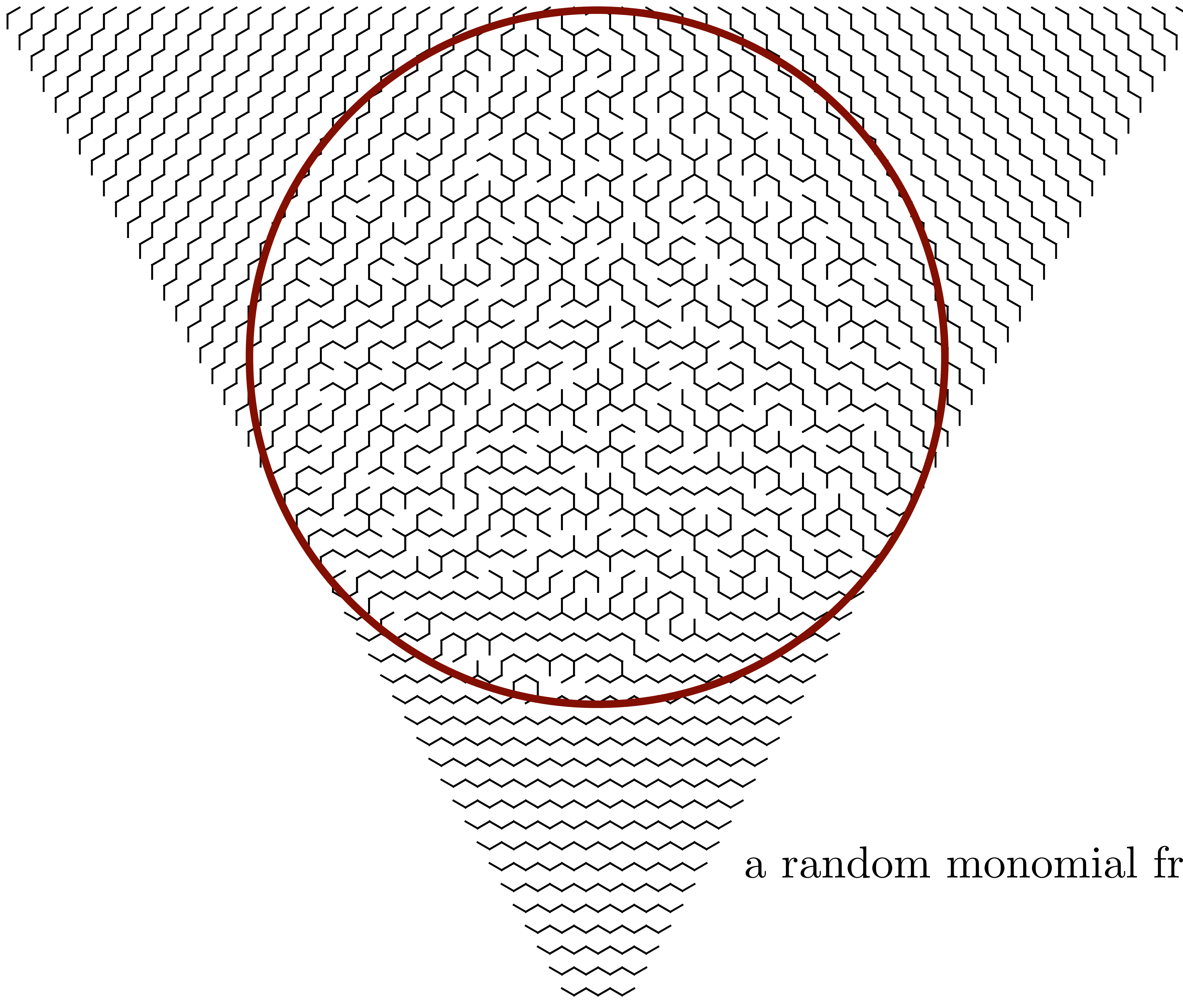
"Cube groves" are essential spanning forests with certain boundary connections.

For large groves, arctic circle theorem [Peterson-Speyer]



a random monomial from $b_{n,n,n}$

For large groves, arctic circle theorem [Peterson-Speyer]



a random monomial from $b_{n,n,n}$

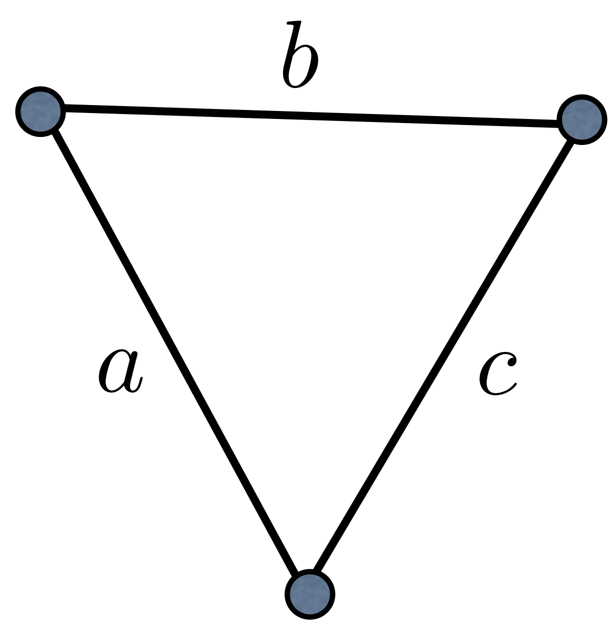
Ising model

G a graph, $c : E \to \mathbb{R}_{>0}$ edge weights.

Configuration space $\Omega = \{1, -1\}^G$

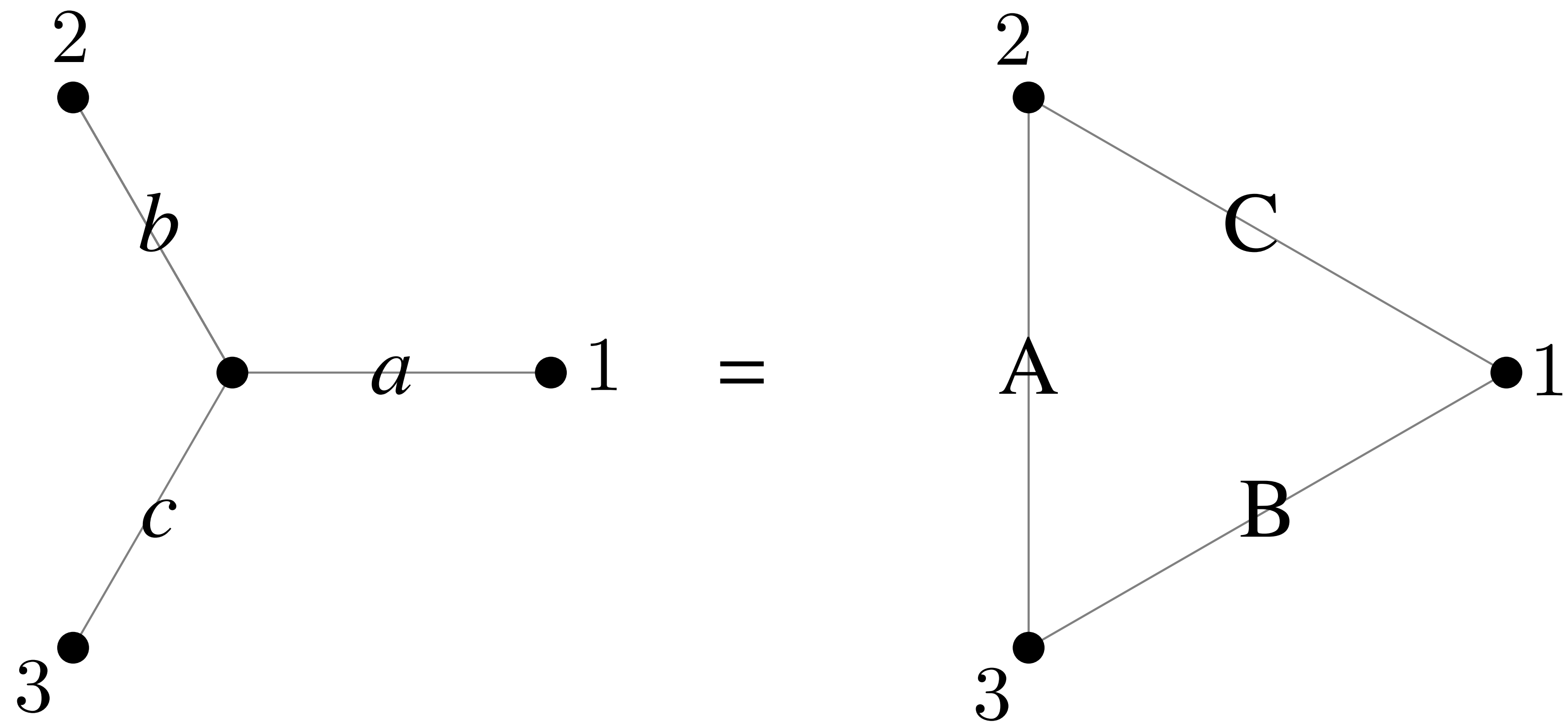
Partition function

$$Z = \sum_{\sigma \in \Omega} \prod_{i \sim j:\ \sigma_i = \sigma_j} c_{ij}$$



$$Z = 2abc + 2a + 2b + 2c$$

Ising model Y-Delta transformation



spins		
1 2 3		
+ + +	$abc + 1$	ABC
− + +	$a + bc$	A
+ − +	$b + ac$	B
+ + −	$c + ab$	C

these should be proportional

"Before" and "after" are proportional (Ising measure preserved) iff

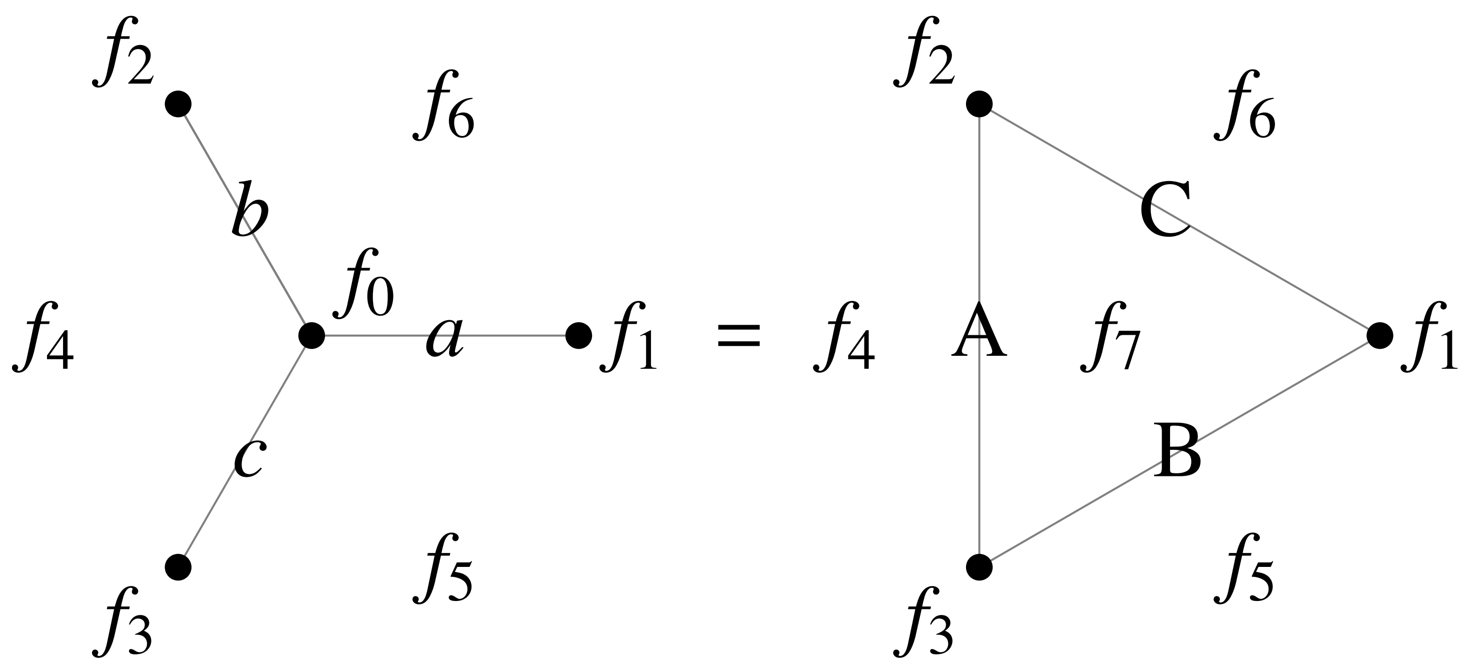
$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}}$$

$$B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}}$$

$$C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}$$

Remarkable fact about the Ising Y-Delta move (Kashaev):

Define new variables f on vertices and faces: "activities"



The activities are related to edge weights as:

$$\left(\frac{a - 1/a}{2}\right)^2 = \frac{f_0 f_1}{f_5 f_6}, \qquad etc.$$

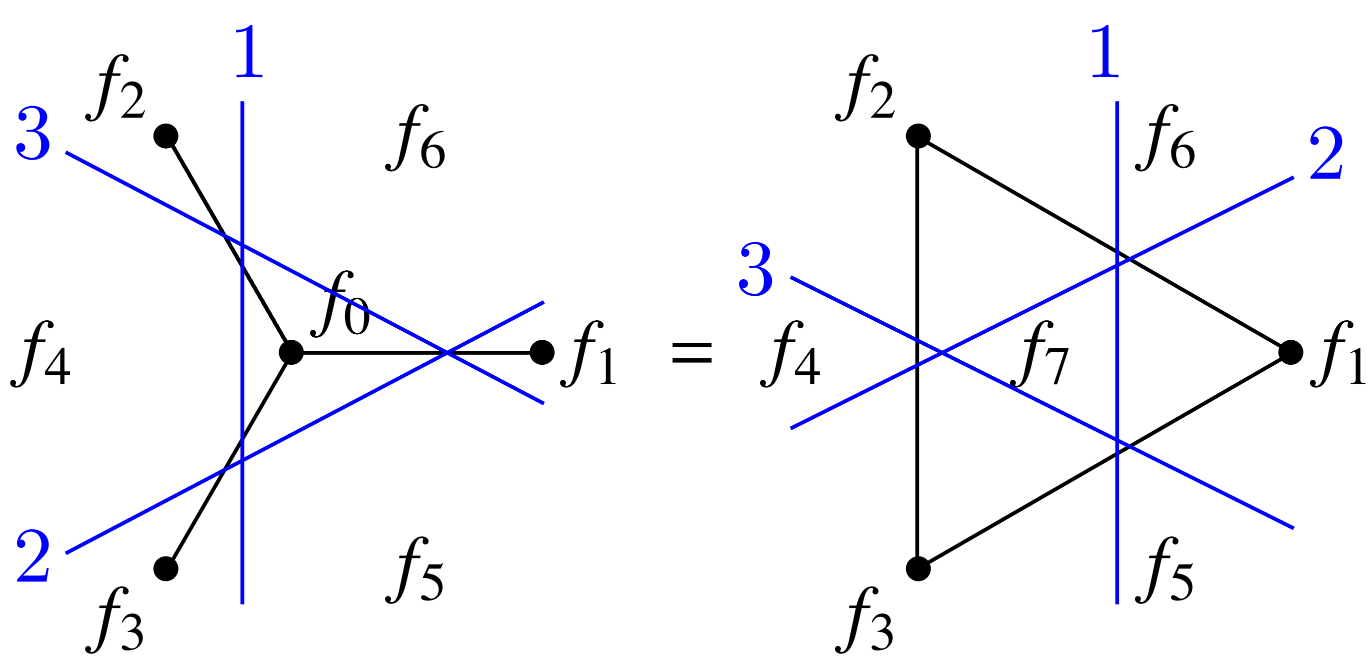
Then: only need to change a *single* weight $f_0 \to f_7$.

Theorem [Kashaev] The fs satisfy

$$f_0^2 f_7^2 + f_1^2 f_4^2 + f_2^2 f_5^2 + f_3^2 f_6^2 - 2(f_1 f_2 f_4 f_5 + f_1 f_4 f_3 f_6 + f_2 f_3 f_5 f_6) \\ -2 f_0 f_7 (f_1 f_4 + f_2 f_5 + f_3 f_6) - 4(f_0 f_4 f_5 f_6 + f_7 f_1 f_2 f_3) = 0.$$

This is the algebraic identity satisfied by the principal minors of a symmetric matrix:

$$S_\phi^2 S_{123}^2 + S_1^2 S_{23}^2 + S_2^2 S_{13}^2 + S_3^2 S_{12}^2 - 2(S_1 S_2 S_{23} S_{13} + S_1 S_3 S_{12} S_{23} + S_2 S_3 S_{12} S_{13}) \\ -2 S_\phi S_{123}(S_1 S_{23} + S_2 S_{13} + S_3 S_{12}) - 4(S_\phi S_{12} S_{13} S_{23} + S_{123} S_1 S_2 S_3) = 0.$$



We say $f : \mathbb{Z}^3 \to \mathbb{C}$ **satisfies the Kashaev recurrence** if $P(f_{i,j,k}, f_{i+1,j,k}, \ldots, f_{i+1,j+1,k+1}) = 0$ for all $(i,j,k) \in \mathbb{Z}^3$.

By defining $f_{i,j,k}$ on $0 \leq i+j+k \leq 2$ we can use P to define it everywhere.

...take positive square root(?)

Example. Let $f_{i,j,k} = f_{i+j+k}$ (translation-invariant.)

The Kashaev recurrence becomes:

$$f_n = \frac{2f_{n-2}^3 + 3f_{n-3}f_{n-1}f_{n-2} + 2\left(f_{n-2}^2 + f_{n-3}f_{n-1}\right)^{3/2}}{f_{n-3}^2}$$

$f_0, f_1, f_2, \ldots$

$1,\ 1,\ 1,\ 5+4\sqrt{2},$

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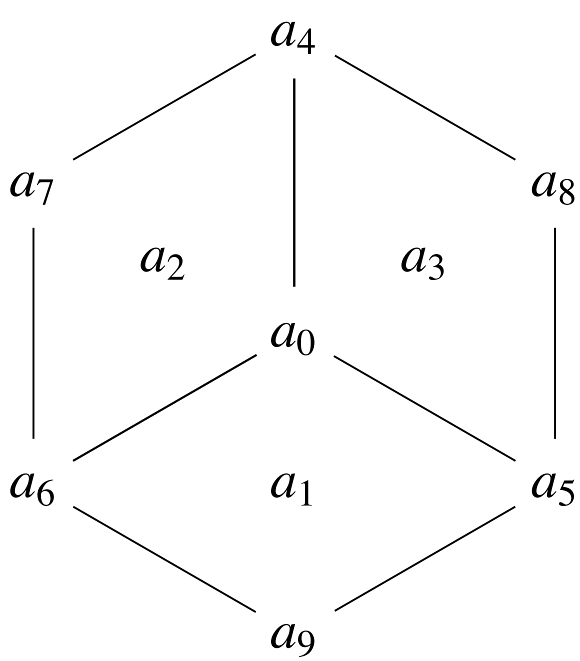
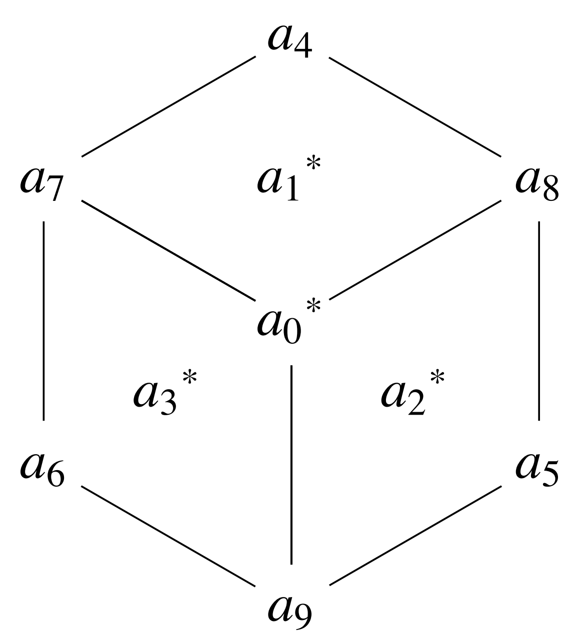
Obs. For any translation-invariant initial conditions, only get single levels of square roots.

Obs.2. For any *non-translation-invariant* initial conditions, only get single levels of square roots.

We define a new recurrence generalizing the Kashaev recurrence:

The hexahedron recurrence





$$
\begin{aligned}
a_1^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_4a_7}{a_0a_1}\\
a_2^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_5a_8}{a_0a_2}\\
a_3^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_6a_9}{a_0a_3}\\
a_0^* &= \frac{a_1^2a_2^2a_3^2 + a_1a_2a_3(2a_4a_5a_6 + a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9) + (a_5a_6 + a_0a_7)(a_4a_5 + a_0a_9)(a_4a_6 + a_0a_8)}{a_0^2a_1a_2a_3}.
\end{aligned}
$$

Theorem: The Kashaev rec. is a special case of the hexahedron rec. where

$$a_{\text{face}} = \sqrt{a_{\text{vtx}}a_{\text{vtx}} + a_{\text{vtx}}a_{\text{vtx}}}.$$

Main results:

Theorem [KP]: The hexahedron recurrence is a composition of six cluster mutations (urban renewals).

Corollary [FZ]: Laurent phenomenon.

Theorem [KP]: Terms in the Laurent polynomials count “taut” double dimer configurations on certain graphs.

Theorem [KP]:

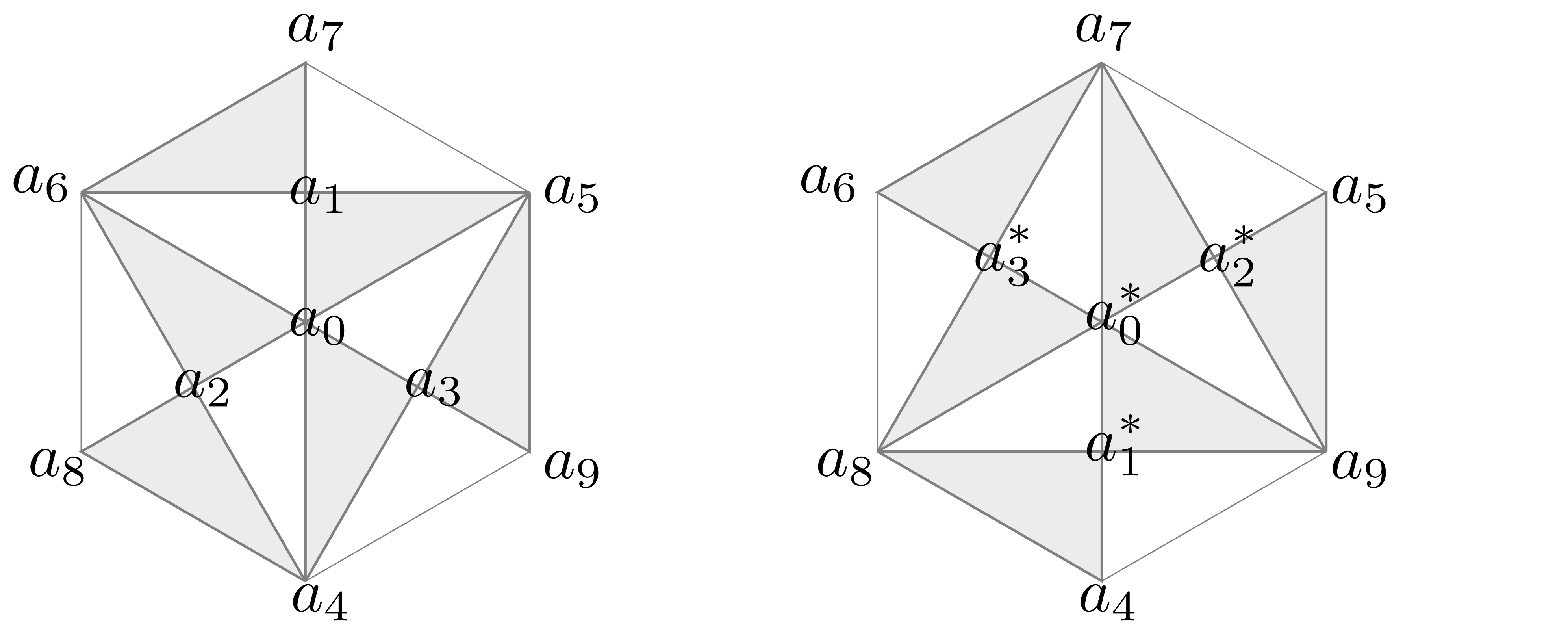
The Kashaev recurrence is a special case of the hexahedron recurrence

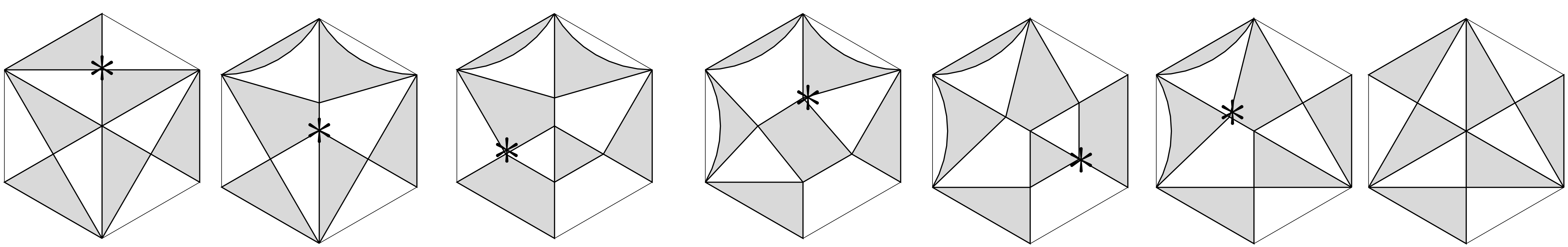
Corollary [KP]: Laurent phenomenon, positive coefficients.

Open question:

Identify monomials of Kashaev recurrence with some combinatorial objects.

The hexahedron recurrence



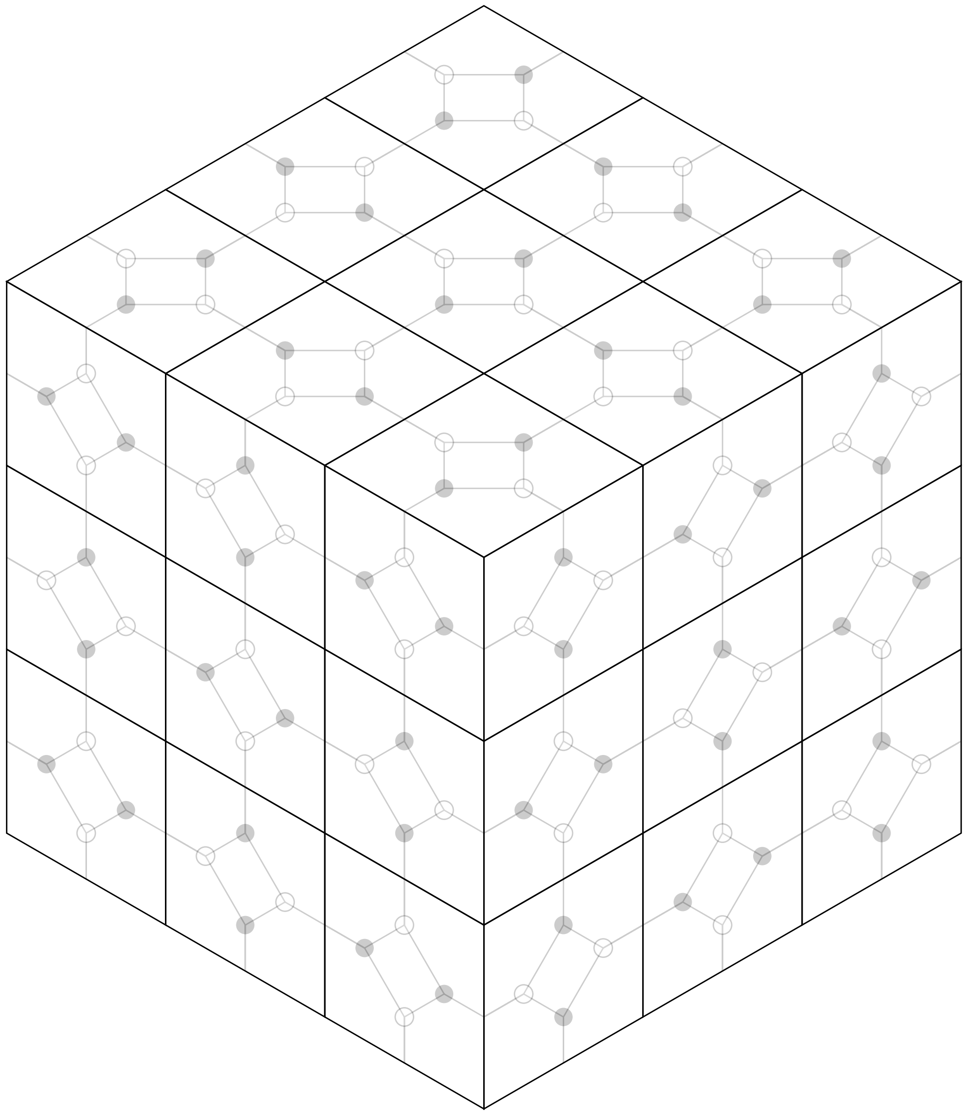


...is a composition of six cluster mutations

What do the terms in the hexahedron recurrence count?

(Fact: planar “alternating” quivers $\leftrightarrow$ dimer covers.)

Certain “double-dimer” configurations on the cubic corner graph:



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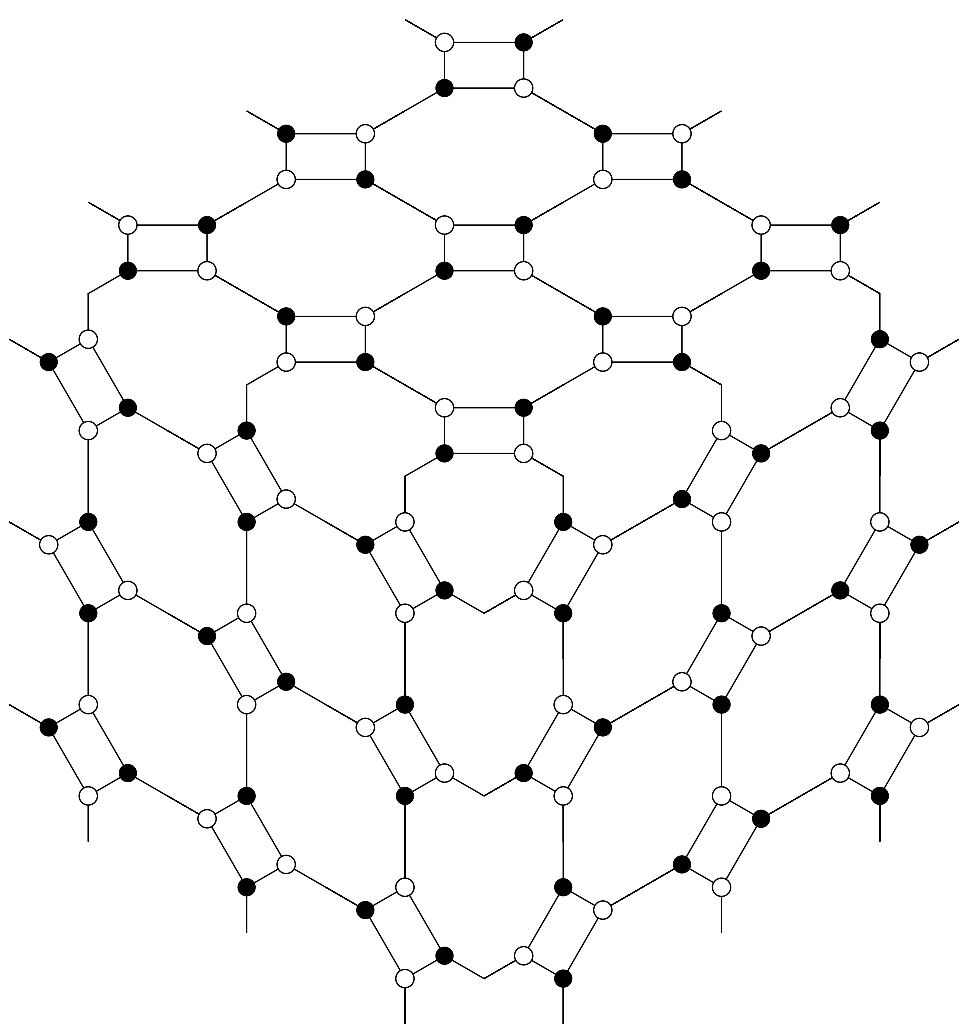
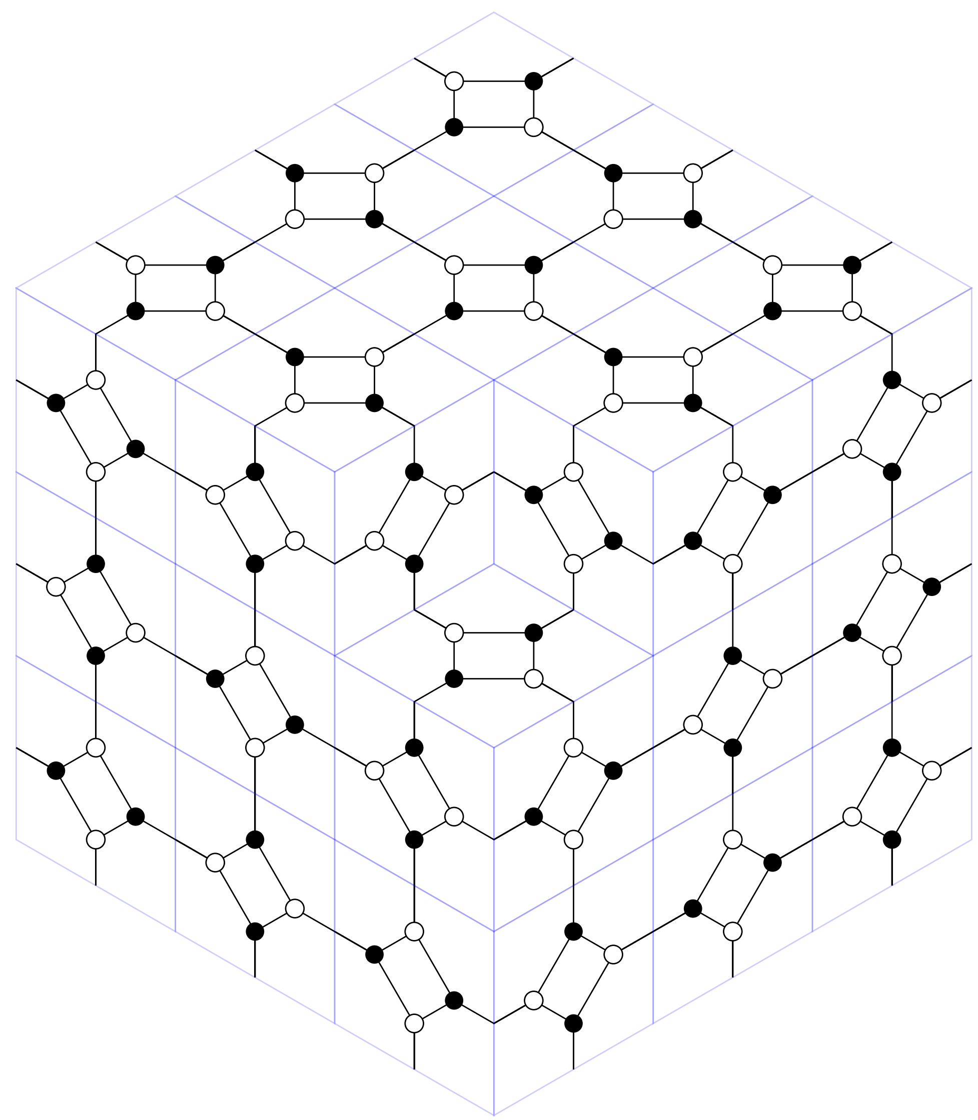
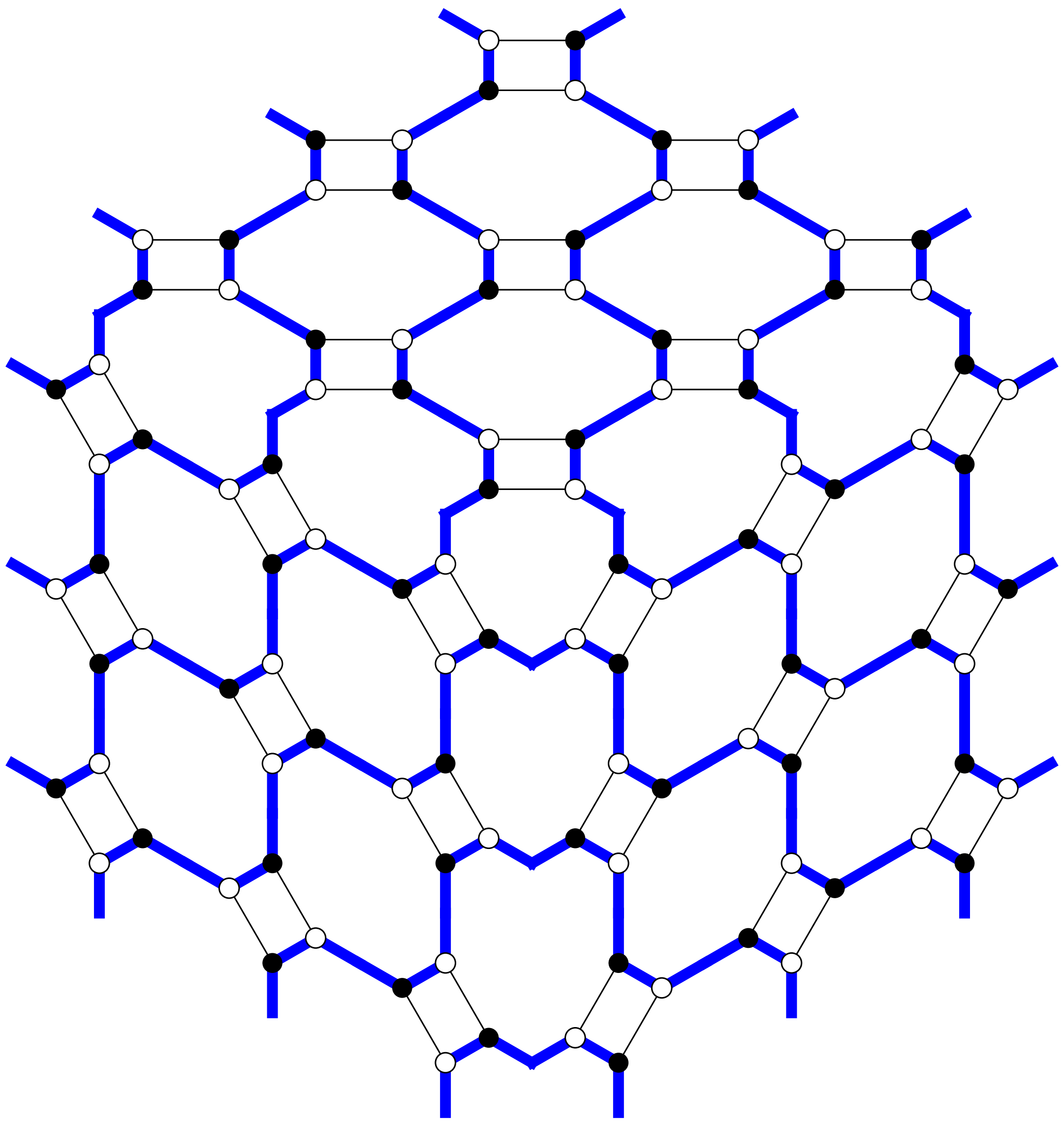


Γ_n is obtained by "removing" cubes $x + y + z < n$.

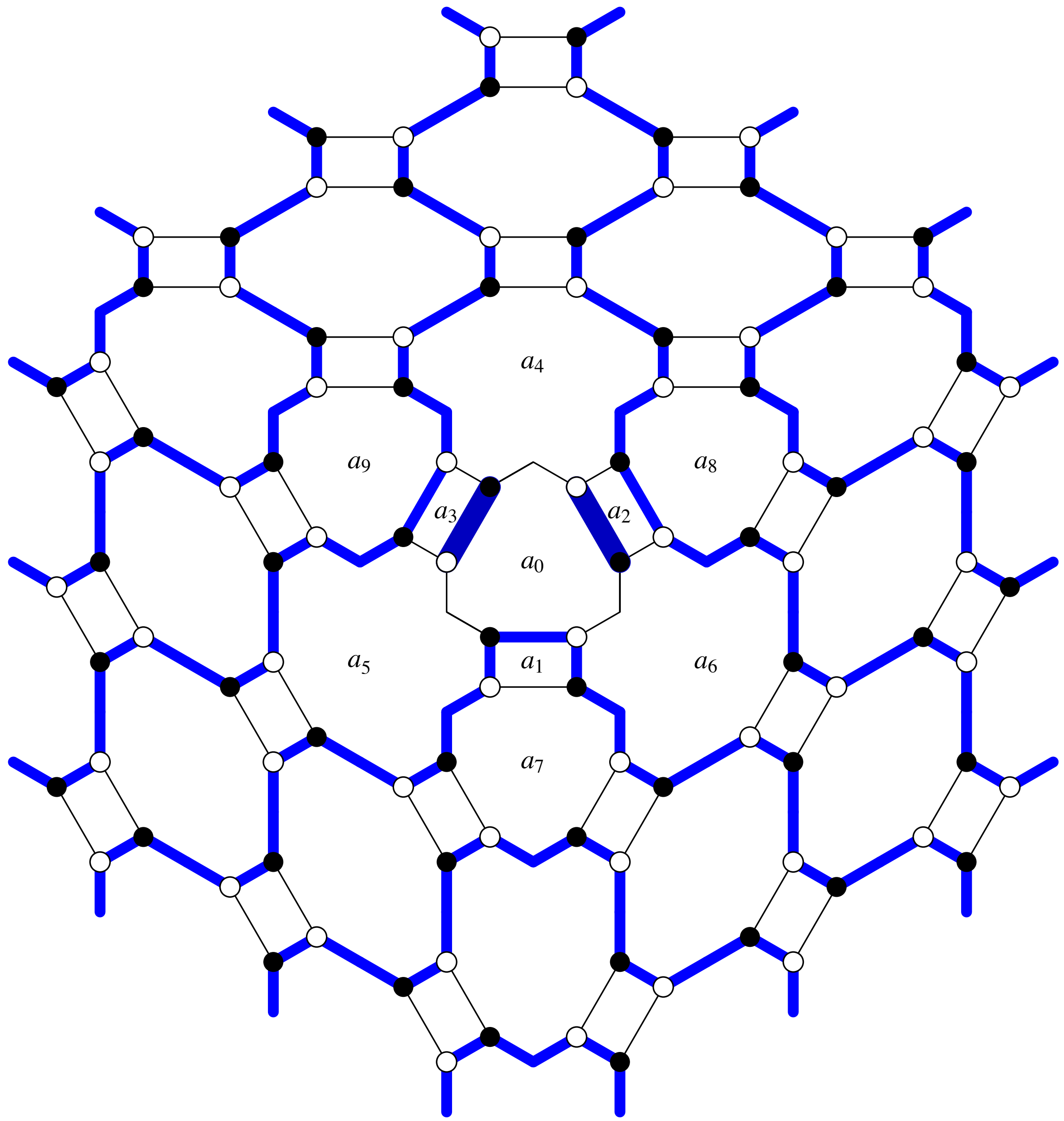
Γ_1 :





The initial configuration m_0 on Γ_0.

When you "remove" a cube the configuration changes *locally* (and randomly) and preserving topology.

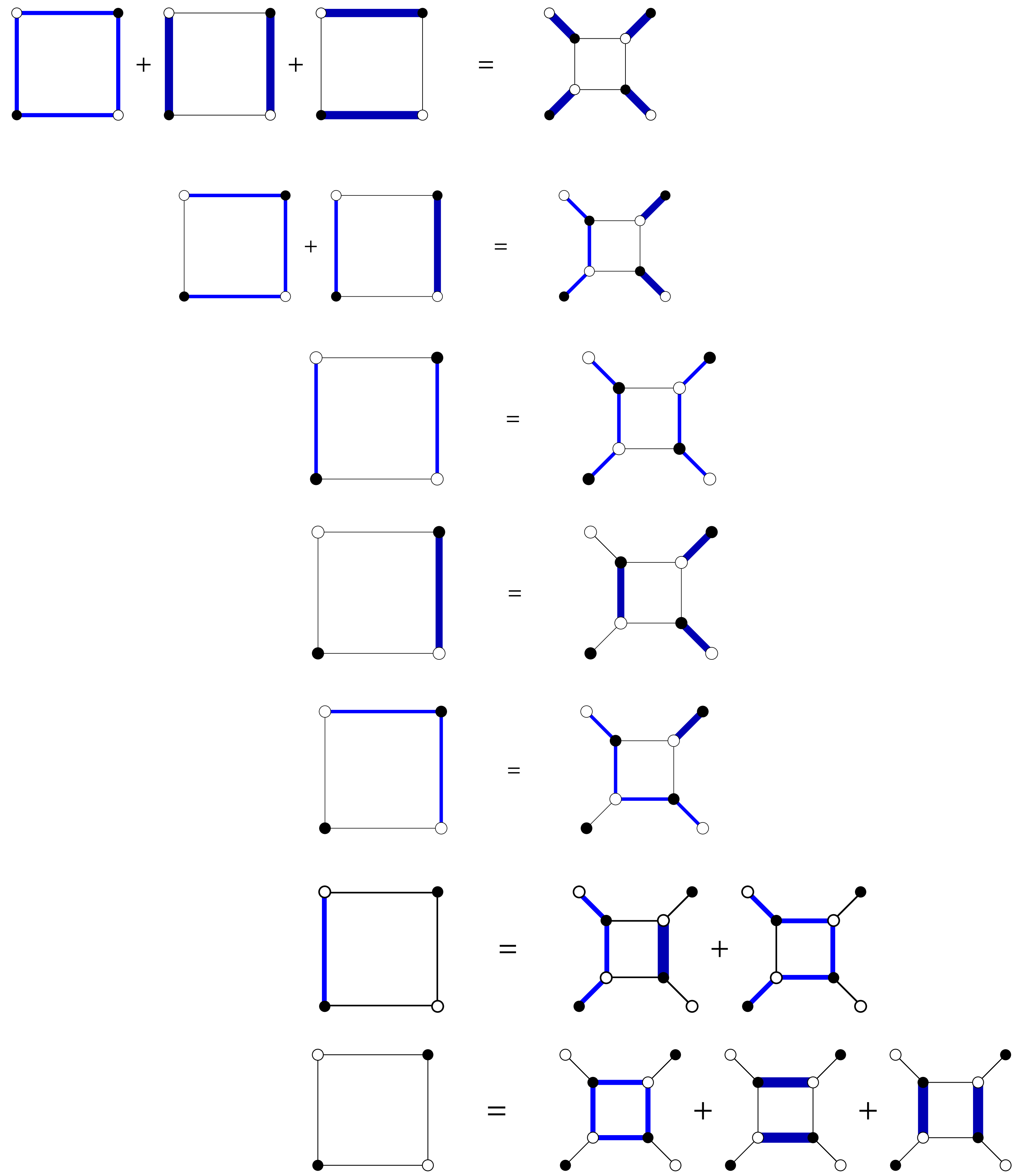


$$\text{weight} = \frac{a_4^2 a_5 a_6 a_7}{a_0 a_1 a_2 a_3}$$

A taut configuration on Γ_1 (same connections as m_0).

Thm: $a_{i,i,i}$ is a Laurent polynomial whose monomials are in bijection with "taut" double-dimer covers of Γ_i.

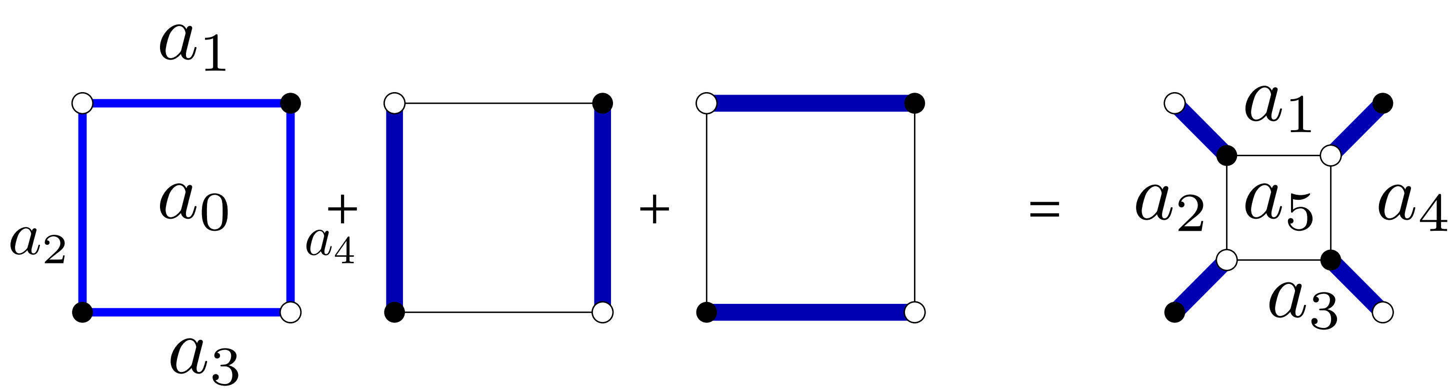
Proof: Check that mutation preserves topology...



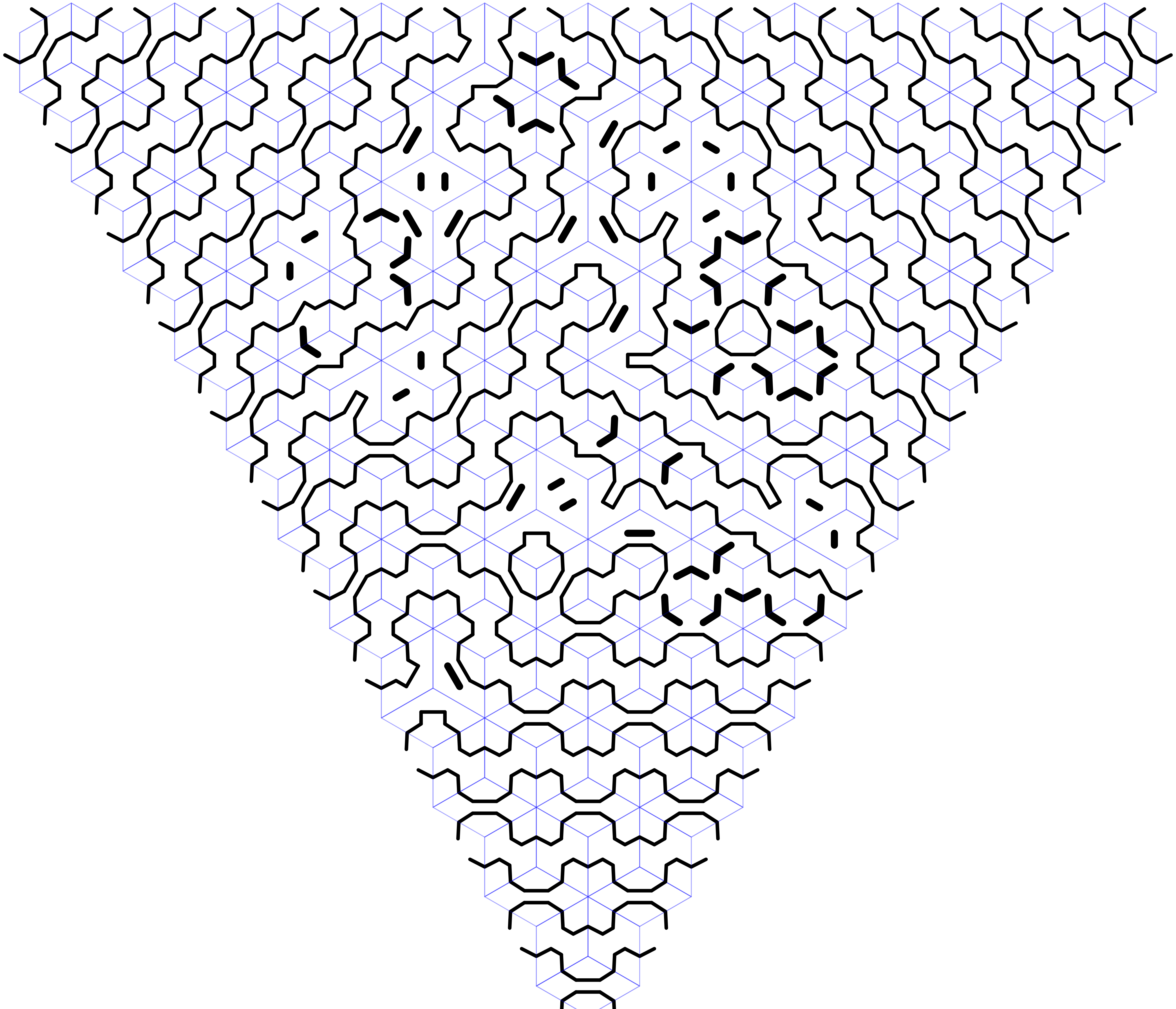
magic formula:

the monomial for a configuration is $$2^c \prod_{\text{faces } f} a_f^{L-2-\#edges}.$$

for example:



$$\frac{2}{a_0^2 a_1 a_2 a_3 a_4} + \frac{1}{a_0^2 a_2^2 a_4^2} + \frac{1}{a_0^2 a_1^2 a_3^2} = \frac{a_5^2}{a_1^2 a_2^2 a_3^2 a_4^2}$$



Arctic circle theorem: use $a_{i,j,k} = 3^{(i+j+k)^2/2}$ (vertex variables)

$a_{i,j,k} = 2 \cdot 3^{(i+j+k+1)^2/2}$ (face variables)

then $a'_{i,j,k}$ satisfies a linear recurrence.

Let $G(x,y,z) = \sum a'_{i,j,k} x^i y^j z^k$.

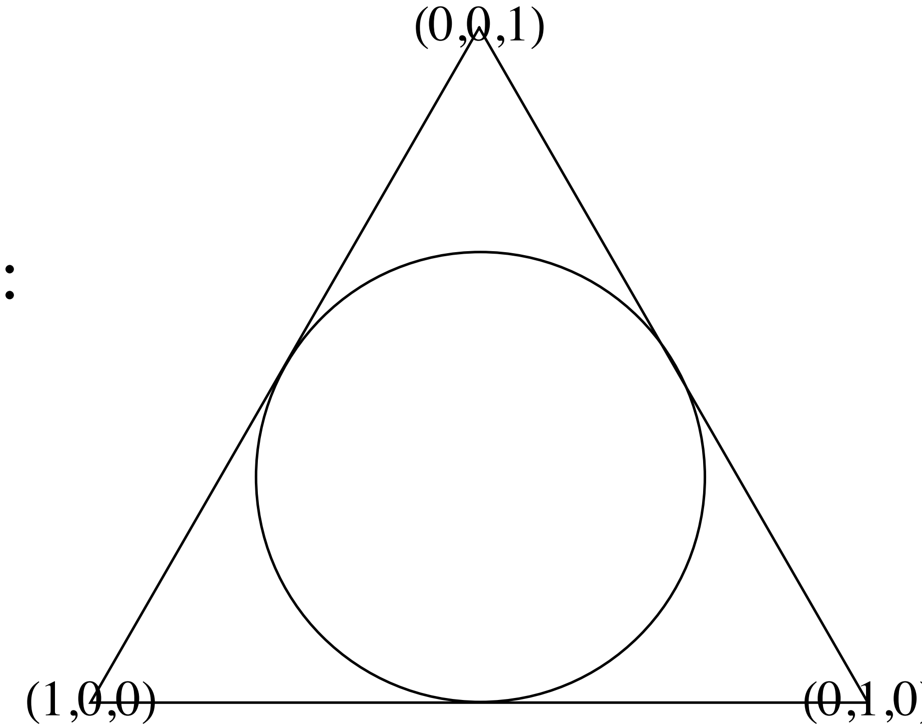
Then $G(x,y,z)$ satisfies a linear recurrence with characteristic polynomial:

$$P(x,y,z) = xyz + 1 - \tfrac{1}{3}(xy + xz + yz + x + y + z).$$

Analyze growth of coefficients of $1/P$:

- polynomial inside inscribed circle
- exponential decay outside inscribed circle QED.

Specific “3^n” initial conditions:



More generic initial conditions