Moments of Askey-Wilson polynomials

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- Let $T_n(\cos\theta) = \cos n\theta$. (**Tchebyshev polynomial of 1st kind**)
- $T_n(x)$ are orthogonal w.r.t. $w(x) = (1-x^2)^{-1/2}$.

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}dx = \begin{cases} 0 & \text{if } n \neq m \\ \text{nonzero} & \text{if } n = m \end{cases}$$

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- $\mu_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}$

Three-term recurrence and Viennot's theorem

Theorem (Favard, 1935)

Monic orthogonal polynomials $P_n(x)$ *satisfy a* ***three-term recurrence***

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \geq 0$$

for $b_0, b_1, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$ *.*

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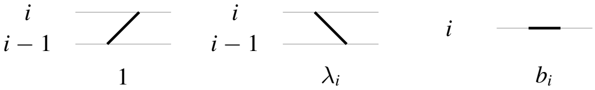

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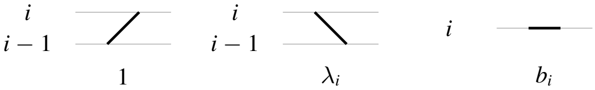

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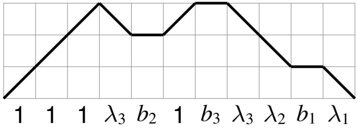

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- The **Hermite polynomials** $H_n(x)$ satisfy
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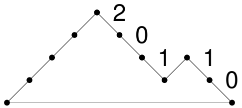

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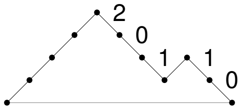

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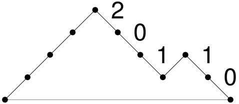

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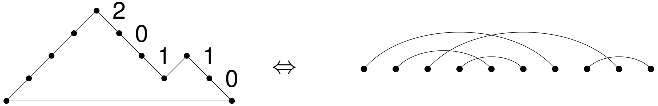

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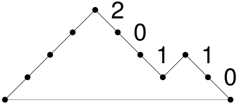

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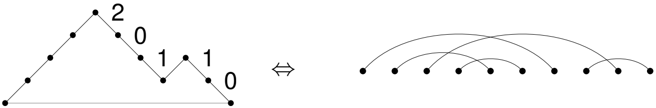

- $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n-1)!! = 1 \cdot 3 \cdots (2n-1)$.

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- Yes!

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Theorem (Azor, Gillis, and Victor, 1982)

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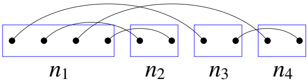

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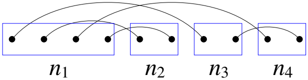

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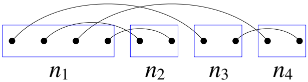

- $\mathcal{L}(H_n(x)H_m(x)) = 0$ if $n \neq m$
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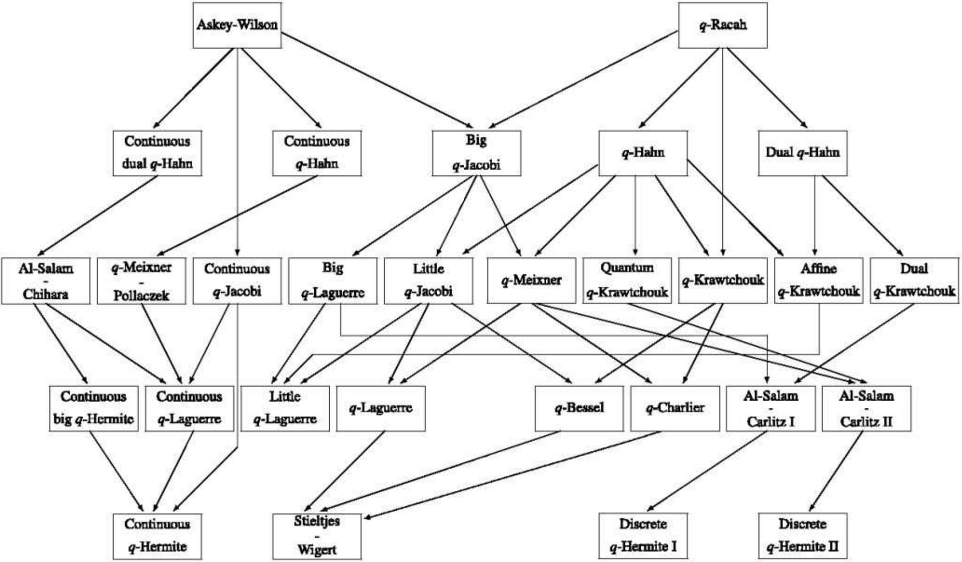
Askey scheme



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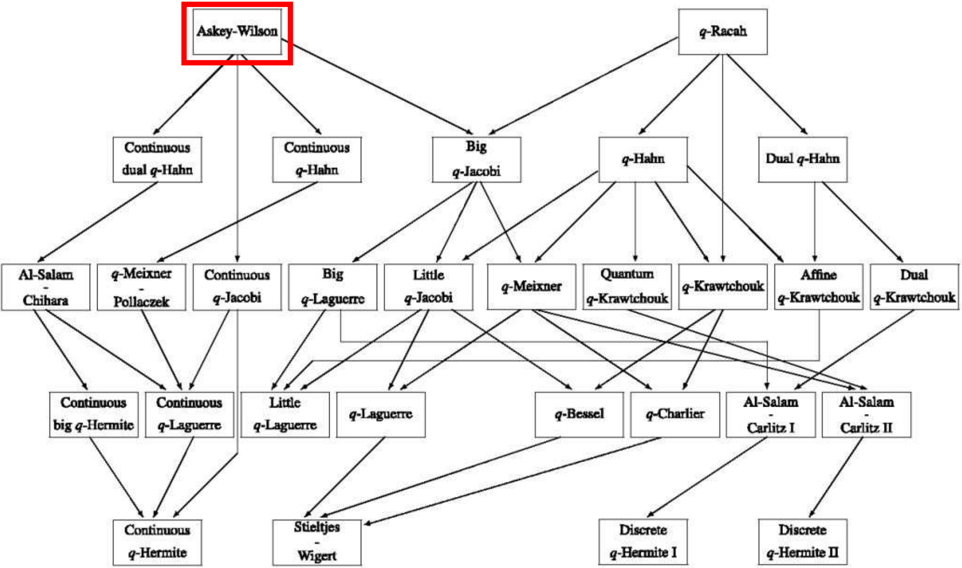

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Askey-Wilson polynomials

- **Askey-Wilson polynomials** $P_n(x) = P_n(x; a, b, c, d; q)$ with $x = \cos\theta$

$$P_n(x; a, b, c, d|q) = \frac{(ab, ac, ad)_n}{a^n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right].$$

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- $\mu_n(a, b, c, d; q)$ is symmetrical in a, b, c, d.

Known formulas for Askey-Wilson moments

Theorem (Corteel, Stanley, Stanton, Williams, 2010)

$$\mu_n(a,b,c,d;q) = \frac{1}{2^n} \sum_{m=0}^{n} \frac{(ab,ac,ad)_m}{(abcd)_m} q^m \sum_{j=0}^{m} \frac{q^{-j^2} a^{-2j} (aq^j + q^{-j}/a)^n}{(q, q^{1-2j}/a^2)_j (q, q^{2j+1}a^2)_{m-j}}.$$

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Theorem (Ismail and Rahman, 2011)

$$\mu_n(a,b,c,d;q) = \frac{(ab,qac,qad)_n}{(2a)^n(q,qa^2,abcd)_n}\sum_{k=0}^{n}\frac{1-a^2q^{2k}}{1-a^2}\cdot\frac{(a^2,q^{-n})_k}{(q,a^2q^{n+1})_k}(1+a^2q^{2k})^n$$
$$\times\, q^{k(n+1)}\frac{(1-ac)(1-ad)}{(1-acq^k)(1-adq^k)}\,{}_4\phi_3\left[\begin{matrix} q^{k-n},q,cd,aq^{k+1}/b \\ acq^{k+1},adq^{k+1},q^{1-n}/ab\end{matrix}\,\middle|\, q,q\right]$$

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Proposition (K., Stanton, 2012)

$2^n(abcd)_n\mu_n(a,b,c,d;q)$ *is a polynomial in* a,b,c,d,q *with integer coefficients.*

Main purpose

Give 3 combinatorial methods for computing μ_n.

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- Motzkin paths

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- Motzkin paths
- staircase tableaux
- q-Hermite polynomials and matchings

Motzkin paths

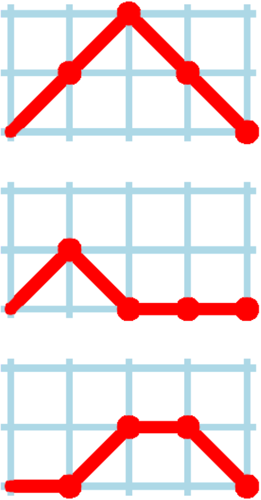

Image stolen from Wikipedia

Motzkin paths

- The Askey-Wilson polynomials $P_n = P_n(x; a, b, c, d; q)$ satisfy

$$P_{n+1} = (x - b_n)P_n - \lambda_n P_{n-1},$$

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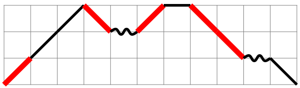
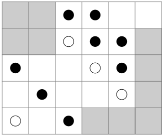
- **Doubly striped skew shapes**: generalization of Dongsu Kim's striped skew shapes.



$\Leftrightarrow$



The $c = d = 0$ case: Al-Salam-Chihara polynomials

Theorem (K., Stanton, 2012)

$$2^n \mu_n(a,b,0,0;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q$$

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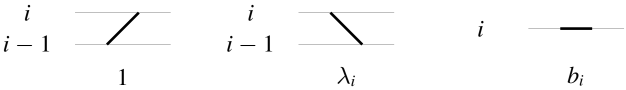
- Our proof is the first combinatorial proof of CJRP.

Open problem

- If $d = 0$,

$$b_n = (a+b+c)q^n - abcq^{2n} - abcq^{2n-1}$$
$$\lambda_n = (1-q^n)(1-abq^{n-1})(1-bcq^{n-1})(1-caq^{n-1}).$$



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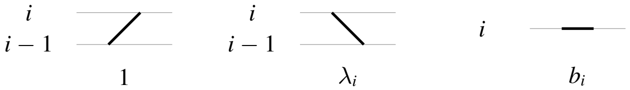
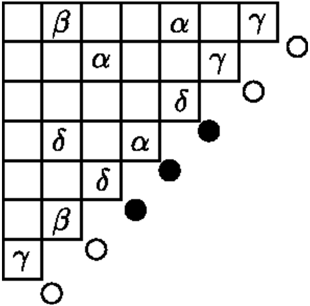


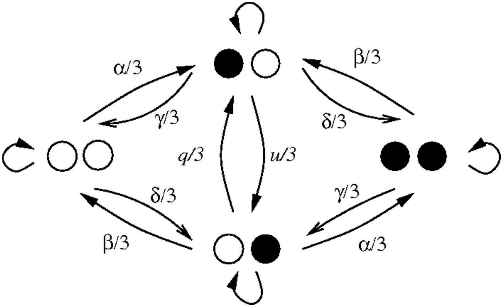
Problem

Find a combinatorial proof using Motzkin paths of the following identity:

$$\sum_{P \in \mathrm{Mot}_n} \mathrm{wt}(P) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right)$$
$$\times \sum_{u+v+w+2t=k} a^u b^v c^w (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ v \end{bmatrix}_q \begin{bmatrix} v+w+t \\ w \end{bmatrix}_q \begin{bmatrix} w+u+t \\ u \end{bmatrix}_q$$

Staircase tableaux





Staircase tableaux

- A **staircase tableau of size n** is a filling of the Young diagram of the staircase partition $(n, n-1, \ldots, 1)$ with $\alpha, \beta, \gamma, \delta$ satisfying certain conditions.

		β				γ
	γ			α	α	
				δ		
	δ		γ			
		β				
	δ					
β						

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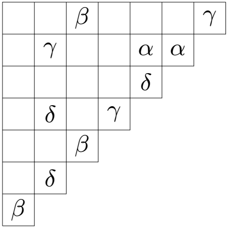

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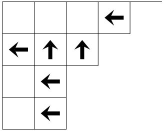
- Have connection with asymmetric exclusion process (ASEP) and moments of Askey-Wilson.
- If there are no γ and δ, we get **permutation tableaux**.



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Theorem (Corteel, Stanley, Stanton, Williams, 2010)

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$$\times (ac)^{C(T)} (bd)^{D(T)} \big((1+ai)(1+ci)\big)^{n-A(T)-C(T)} \big((1-bi)(1-di)\big)^{n-B(T)-D(T)}.$$

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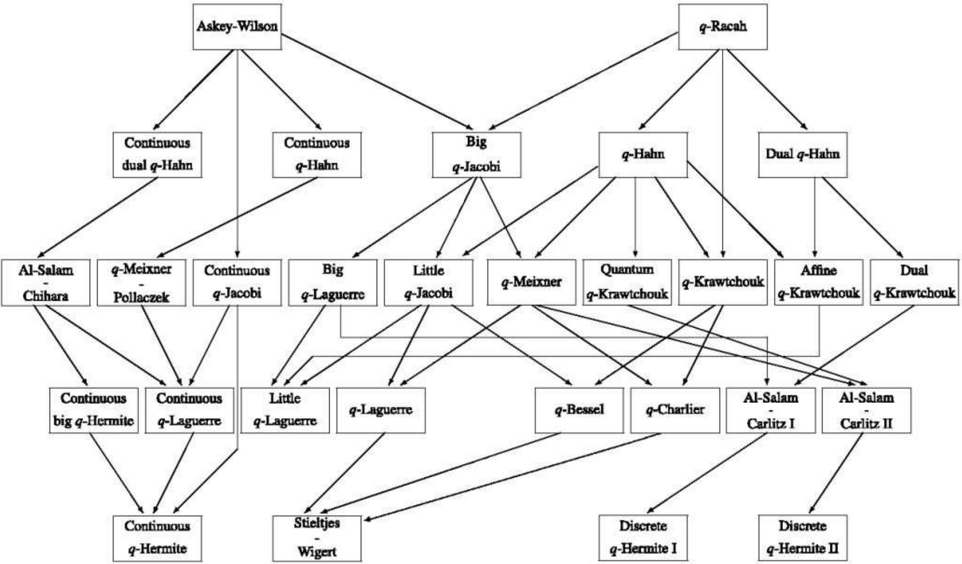
Theorem (K., Stanton, 2012)

We have

$$\left[a^nb^nc^nd^nq^{\binom{n}{2}}\right]2^n(abcd)_n\mu_n(a,b,c,d;q) = \mathrm{Cat}\left(\frac{n}{2}\right),$$
$$\left[a^{n-1}b^nc^nd^nq^{\binom{n}{2}}\right]2^n(abcd)_n\mu_n(a,b,c,d;q) = -\,\mathrm{Cat}\left(\frac{n+1}{2}\right),$$
$$\left[a^{n-1}b^{n-1}c^nd^nq^{\binom{n}{2}}\right]2^n(abcd)_n\mu_n(a,b,c,d;q) = \mathrm{Cat}\left(\frac{n+2}{2}\right) - \mathrm{Cat}\left(\frac{n}{2}\right),$$

where $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ *if* n *is a nonnegative integer, and* $\mathrm{Cat}(n) = 0$ *otherwise.*

q-Hermite polynomials



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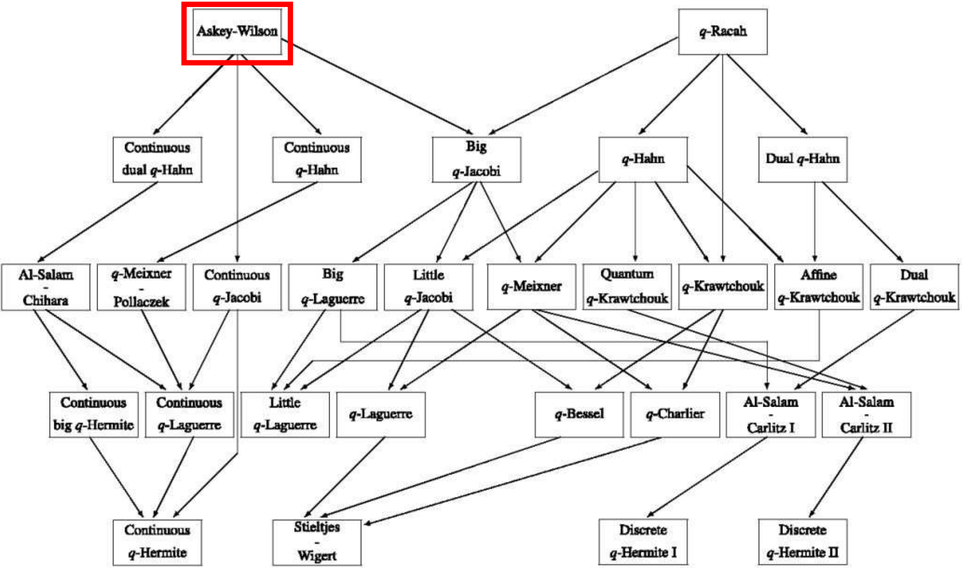

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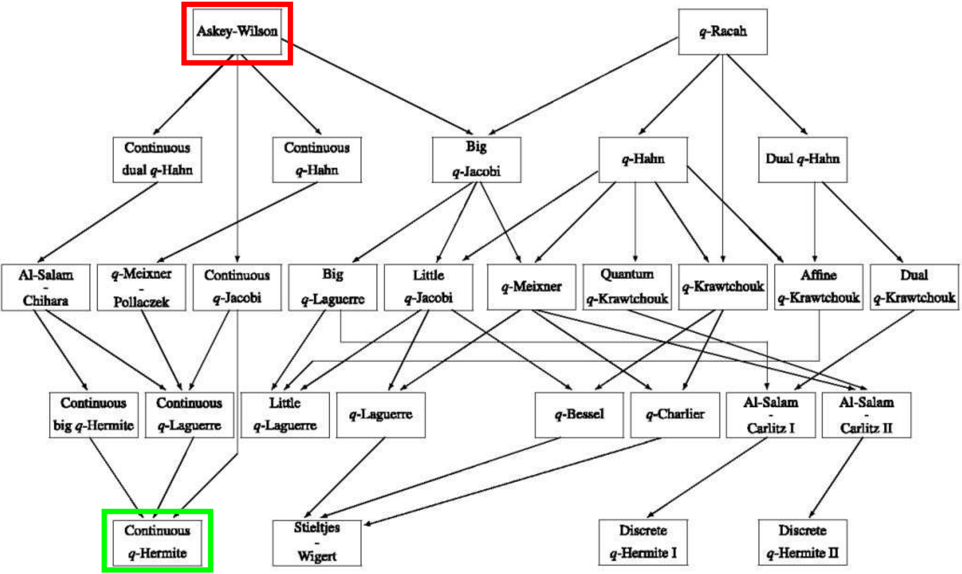

Back to the definition of $\mu_n(a, b, c, d; q)$

- Recall

$$\mu_n(a, b, c, d; q) = C \int_{-1}^{1} x^n w(x) \frac{dx}{\sqrt{1 - x^2}} = C \int_0^{\pi} (\cos\theta)^n w(\cos\theta) d\theta,$$

where

$$w(\cos\theta, a, b, c, d; q) = \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta})_\infty}.$$

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$$= \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}} H_{n_1}H_{n_2}H_{n_3}H_{n_4}(e^{2i\theta}, e^{-2i\theta})_\infty$$

- We can write

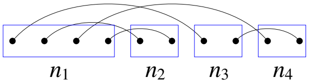
$$I_0 = \frac{(q)_\infty}{2\pi}\int_0^\pi w(\cos\theta)d\theta = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}} \mathcal{L}(H_{n_1}H_{n_2}H_{n_3}H_{n_4})$$

Combinatorial description for I_0

Theorem (Ismail, Stanton, and Viennot (1985))

$$I_0 = \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{\widetilde{a}^{n_1}\widetilde{b}^{n_2}\widetilde{c}^{n_3}\widetilde{d}^{n_4}}{[n_1]_q![n_2]_q![n_3]_q![n_4]_q!} \sum_{\sigma \in \mathcal{PM}(n_1,n_2,n_3,n_4)} q^{\mathrm{cr}(\sigma)}$$

where $\widetilde{a} = a/\sqrt{1-q}$, $\widetilde{b} = b/\sqrt{1-q}$, $\widetilde{c} = c/\sqrt{1-q}$, $\widetilde{d} = d/\sqrt{1-q}$ *and* $\mathcal{PM}(n_1,n_2,n_3,n_4)$ *is the set of perfect matchings on* $[n_1] \uplus [n_2] \uplus [n_3] \uplus [n_4]$ *without homogeneous edges.*



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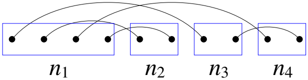

Question

How about I_n?

Combinatorial description for I_n

- I_0 is the generating function for perfect matchings with 4 sections:

$$I_0 = \frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta)d\theta = \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{a^{n_1} b^{n_2} c^{n_3} d^{n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}} \mathcal{L}(H_{n_1} H_{n_2} H_{n_3} H_{n_4})$$

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- I_n is the generating function for perfect matchings with 5 sections:

$$I_n = \frac{(q)_\infty}{2\pi}\int_0^\pi (\cos\theta)^n w(\cos\theta)d\theta = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}\mathcal{L}(x^n H_{n_1}H_{n_2}H_{n_3}H_{n_4})$$

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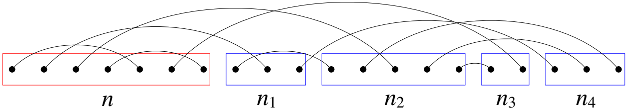
Theorem (K., Stanton, 2012)

$$I_n = \left(\frac{\sqrt{1-q}}{2}\right)^n \sum_{n_1,n_2,n_3,n_4\geq 0}\frac{\widetilde{a}^{n_1}\widetilde{b}^{n_2}\widetilde{c}^{n_3}\widetilde{d}^{n_4}}{[n_1]_q![n_2]_q![n_3]_q![n_4]_q!}\sum_{\sigma\in\mathcal{PM}_n(n_1,n_2,n_3,n_4)} q^{\mathrm{cr}(\sigma)}$$

where $\mathcal{PM}_n(n_1,n_2,n_3,n_4)$ *is the set of perfect matchings on* $[n]\uplus[n_1]\uplus[n_2]\uplus[n_3]\uplus[n_4]$ *with homogeneous edges only in the first section.*

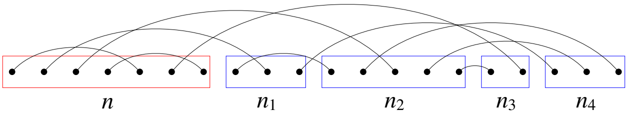


Combinatorial intepretation for $\mu_n(a,b,c,d;q)$

Theorem (K., Stanton, 2012)

$$2^n \mu_n(a,b,c,d;q) = (1-q)^{n/2} I_n / I_0$$

where I_n is the generating function for



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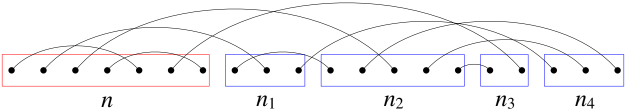

Theorem (K., Stanton, 2012)

$$2^n \mu_n(a,b,c,d;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{\alpha+\beta+\gamma+\delta+2t=k} a^\alpha b^\beta c^\gamma d^\delta \frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}}$$

$$\times (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} \alpha+\beta+\gamma+t \\ \alpha \end{bmatrix}_q \begin{bmatrix} \beta+\gamma+\delta+t \\ \beta,\gamma,\delta+t \end{bmatrix}_q \begin{bmatrix} \delta+\alpha+t \\ \delta \end{bmatrix}_q$$

Corollaries

Corollary (K., Stanton, 2012)

$$2^n \mu_n(a,b,c,0;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right)$$

$$\times \sum_{u+v+w+2t=k} a^u b^v c^w (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ v \end{bmatrix}_q \begin{bmatrix} v+w+t \\ w \end{bmatrix}_q \begin{bmatrix} w+u+t \\ u \end{bmatrix}_q .$$

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$[n+1]_q! 2^n \mu_n(a,b,q/a,q/b;q)$ *is a Laurent polynomial in* a *and* b *whose coefficients are positive polynomials in* q.

Open problems

Problem

Find a combinatorial proof of

$$2^n \mu_n(a,b,q/a,q/b;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \frac{1}{[k+1]_q} \sum_{\substack{|A|+|B|\le k \\ A+B\equiv k \mod 2}} a^A b^B q^{\frac{k-A-B}{2}}$$

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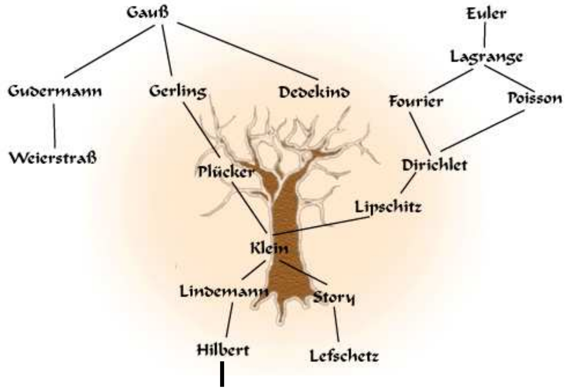
$[n+1]_q! 2^n \mu_n(a,b,q/a,q/b;q)$ *is a Laurent polynomial in* a *and* b *whose coefficients are positive polynomials in* q*.*

- If $ac = q^i$ and $bd = q^j$,

Conjecture

For positive integers i and j,
$2^n [n+i+j-1]_q! \mu_n(a,b,q^i/a,q^j/b;q)$ is a Laurent polynomial in a, b whose coefficients are positive polynomials in q.

Math Genealogy



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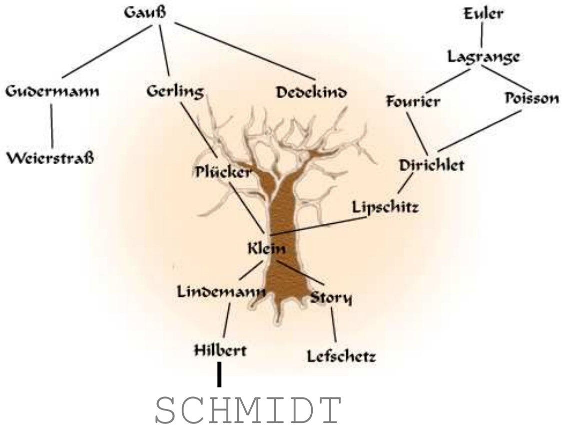

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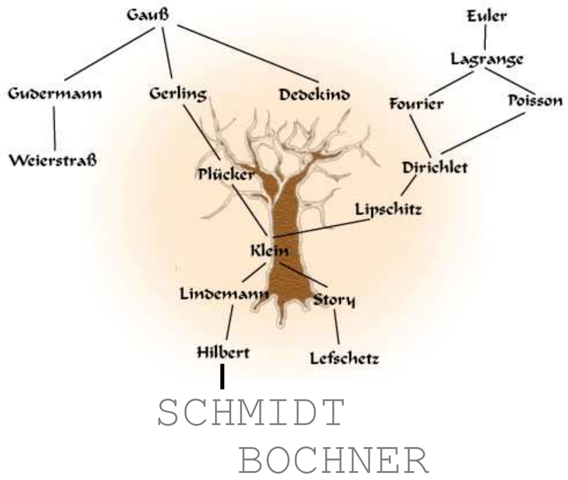

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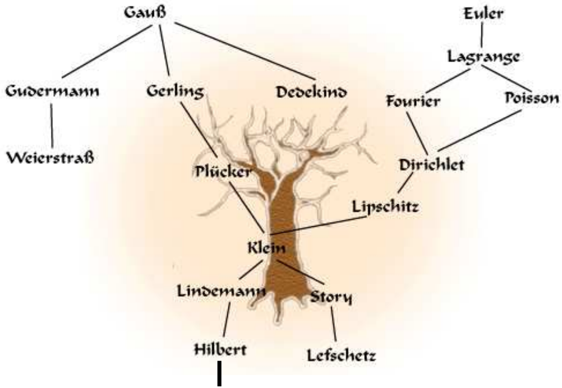

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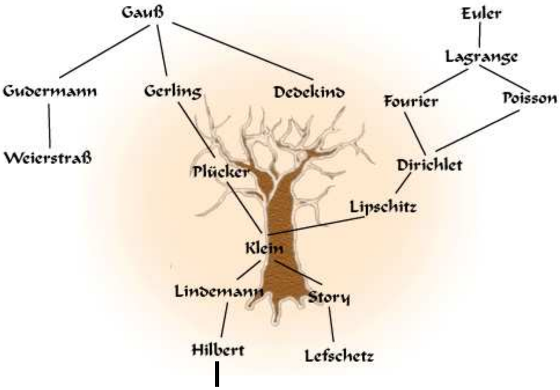

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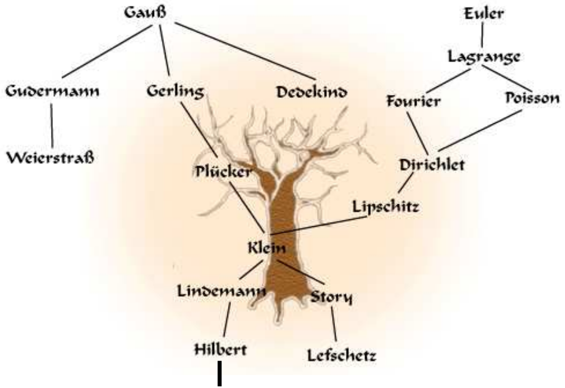

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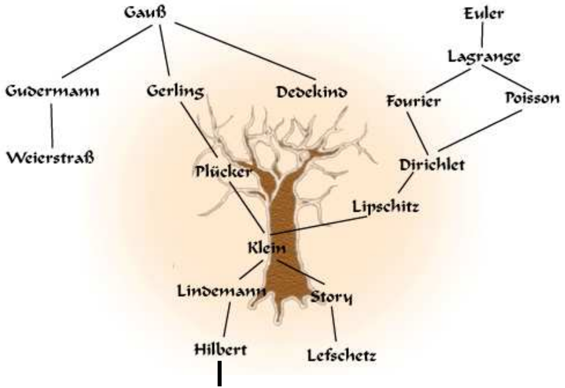

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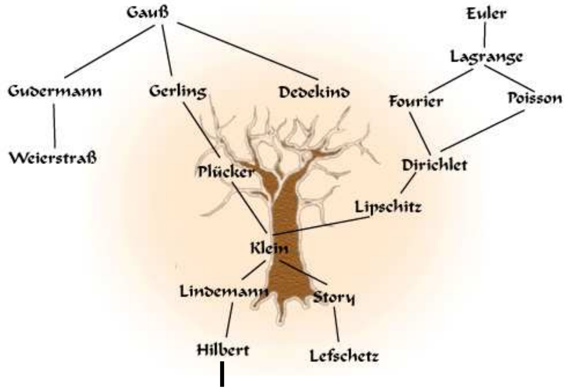

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