A uniform model for Kirillov-Reshetikhin crystals

Cristian Lenart[1] Satoshi Naito[2] Daisuke Sagaki[3]
Anne Schilling[4] Mark Shimozono[5]

[1]State University of New York, Albany

[2]Tokyo Institute of Technology

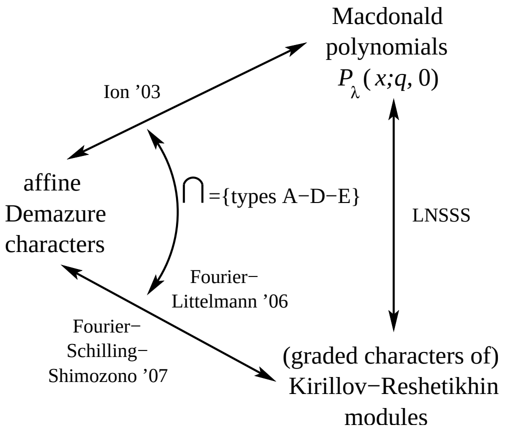
[3]University of Tsukuba

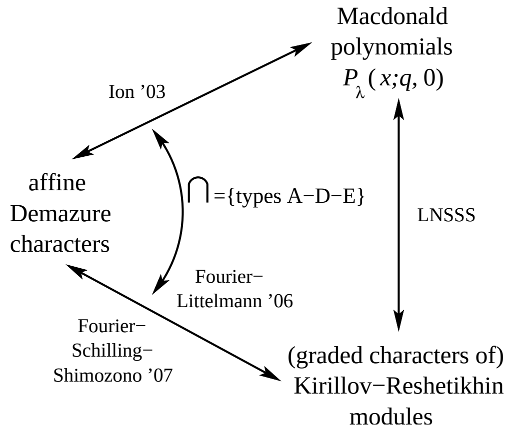
[4]University of California, Davis

[5]Virginia Tech

FPSAC, June 26, 2013

Based on arXiv:1211.2042 and a forthcoming sequel.


Macdonald
polynomials
$P_\lambda(x;q,0)$
Ion '03
affine
Demazure
characters
$\bigcap$={types A−D−E}
LNSSS
Fourier−
Littelmann '06
Fourier−
Schilling−
Shimozono '07
(graded characters of)
Kirillov−Reshetikhin
modules



Theorem (LNSSS 2012)

For all untwisted affine root systems,

$$P_\lambda(x; q, 0) = X_\lambda(x; q)\,,$$

where $X_\lambda(x; q)$ is the (graded) character of a tensor product of one-column Kirillov-Reshetikhin (KR) modules.

Summary

Ram−Yip formula for $P_\lambda(x;q,t)$

uniform models for KR crystals

(the quantum alcove model)

$X_\lambda(x;q) = P_\lambda(x;q,0)$

Summary

Ram−Yip formula for $P_\lambda(x;q,t)$

uniform models for KR crystals
(the quantum alcove model)

$X_\lambda(x;q) = P_\lambda(x;q,0)$

computational applications

energy function

combinatorial R−matrix

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$.

Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\widetilde{e}_i$, $\widetilde{f}_i$.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$.

Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\widetilde{e}_i$, $\widetilde{f}_i$.

Fact. V has a crystal basis B (vertices) $\implies$ in the limit $q \to 0$ we have

$$\widetilde{f}_i, \widetilde{e}_i \; : \; B \to B \sqcup \{0\}\,,$$
$$\widetilde{f}_i\, b = b' \iff \widetilde{e}_i\, b' = b \iff b \to b'\,.$$

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) of affine Lie algebras $\widehat{\mathfrak{g}}$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) of affine Lie algebras $\widehat{\mathfrak{g}}$.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) of affine Lie algebras $\widehat{\mathfrak{g}}$.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, so they are denoted $B^{p,q}$.

Kirillov–Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) of affine Lie algebras $\widehat{\mathfrak{g}}$.

The corresponding crystals have arrows $\widetilde{f}_0, \widetilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, so they are denoted $B^{p,q}$. We only consider column shapes $B^{p,1}$.

Tensor products of KR crystals

Definition. Given a composition $\mathbf{p} = (p_1, p_2, \ldots)$, let

$$B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes B^{p_2,1} \otimes \ldots .$$

Tensor products of KR crystals

Definition. Given a composition $\mathbf{p} = (p_1, p_2, \ldots)$, let

$$B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes B^{p_2,1} \otimes \ldots .$$

The crystal operators are defined on $B^{\otimes \mathbf{p}}$ by a tensor product rule.

Tensor products of KR crystals

Definition. Given a composition $\mathbf{p} = (p_1, p_2, \ldots)$, let

$$B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes B^{p_2,1} \otimes \ldots .$$

The crystal operators are defined on $B^{\otimes \mathbf{p}}$ by a tensor product rule.

Fact. $B^{\otimes \mathbf{p}}$ is connected (with the 0-arrows).

Models for KR crystals: type $A_{n-1}^{(1)}$ $(\widehat{\mathfrak{sl}}_n)$

Fact. We have as classical crystals (without the 0-arrows):

$$B^{p,1} \simeq B(\omega_p)\,, \qquad \text{where } \omega_p = (1,\ldots,1,0,\ldots,0) = (1^p)\,.$$

Models for KR crystals: type $A_{n-1}^{(1)}$ $(\widehat{\mathfrak{sl}}_n)$

Fact. We have as classical crystals (without the 0-arrows):

$$B^{p,1} \simeq B(\omega_p)\,, \qquad \text{where } \omega_p = (1,\ldots,1,0,\ldots,0) = (1^p)\,.$$

The vertices of this crystal are labeled by strictly increasing fillings of the Young diagram/column (1^p) with $1,\ldots,n$.

Models for KR crystals: type $A_{n-1}^{(1)}$ ($\widehat{\mathfrak{sl}}_n$)

Fact. We have as classical crystals (without the 0-arrows):

$$B^{p,1} \simeq B(\omega_p)\,, \qquad \text{where } \omega_p = (1,\ldots,1,0,\ldots,0) = (1^p)\,.$$

The vertices of this crystal are labeled by strictly increasing fillings of the Young diagram/column (1^p) with $1,\ldots,n$.

The action of the crystal operators:

$$1 \xrightarrow{\widetilde{f}_1} 2 \xrightarrow{\widetilde{f}_2} \ldots n-1 \xrightarrow{\widetilde{f}_{n-1}} n \xrightarrow{\widetilde{f}_0} 1\,.$$

Models for KR crystals: types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$

Fact. There are more involved type-specific models (based on Kashiwara–Nakashima columns).

Models for KR crystals: types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$

Fact. There are more involved type-specific models (based on Kashiwara–Nakashima columns).

Goal. Uniform model for all types $A_{n-1}^{(1)} - G_2^{(1)}$, based on the corresponding finite root systems $A_{n-1} - G_2$.

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);
- it decreases by 1 along certain 0-arrows.

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B \; : \; B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);
- it decreases by 1 along certain 0-arrows.

Goal. A more efficient uniform calculation, based only on the combinatorial data associated with a crystal vertex

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);
- it decreases by 1 along certain 0-arrows.

Goal. A more efficient uniform calculation, based only on the combinatorial data associated with a crystal vertex (type A: Lascoux–Schützenberger charge statistic).

Setup: finite root systems

Root system $\Phi \subset V = \mathbb{R}^r$.

Setup: finite root systems

Root system $\Phi \subset V = \mathbb{R}^r$.

Reflections s_α, $\alpha \in \Phi$.

Setup: finite root systems

Root system $\Phi \subset V = \mathbb{R}^r$.

Reflections s_α, $\alpha \in \Phi$.

Example. Type A_{n-1}.

$V = (\varepsilon_1 + \ldots + \varepsilon_n)^\perp$ in $\mathbb{R}^n = \langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ $(r = n - 1)$.

$\Phi = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j) \ : \ 1 \leq i \neq j \leq n\}$.

The Weyl group

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle .$$

The Weyl group

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle \, .$$

Length function: $\ell(w)$.

The Weyl group

$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle$.

Length function: $\ell(w)$.

Example. Type A_{n-1}.

$W = S_n$, $s_{\varepsilon_i - \varepsilon_j} = (i, j)$ is the transposition t_{ij} .

The Weyl group

$$W = \langle s_\alpha \; : \; \alpha \in \Phi \rangle \,.$$

Length function: $\ell(w)$.

Example. Type A_{n-1}.

$W = S_n$, $s_{\varepsilon_i - \varepsilon_j} = (i, j)$ is the transposition t_{ij}.

The quantum Bruhat graph on W is the directed graph with labeled edges

$$w \xrightarrow{\alpha} w s_\alpha \,,$$

The Weyl group

$W = \langle s_\alpha \;:\; \alpha \in \Phi \rangle$.

Length function: $\ell(w)$.

Example. Type A_{n-1}.

$W = S_n$, $s_{\varepsilon_i - \varepsilon_j} = (i, j)$ is the transposition t_{ij} .

The quantum Bruhat graph on W is the directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha ,$$

where

$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(Bruhat graph)}, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1 \qquad (\mathrm{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle) .$$

The Weyl group

$W = \langle s_\alpha \; : \; \alpha \in \Phi \rangle \, .$

Length function: $\ell(w)\,.$

Example. Type A_{n-1}.

$W = S_n \, , \quad s_{\varepsilon_i - \varepsilon_j} = (i,j)$ is the transposition t_{ij} .

The quantum Bruhat graph on W is the directed graph with labeled edges

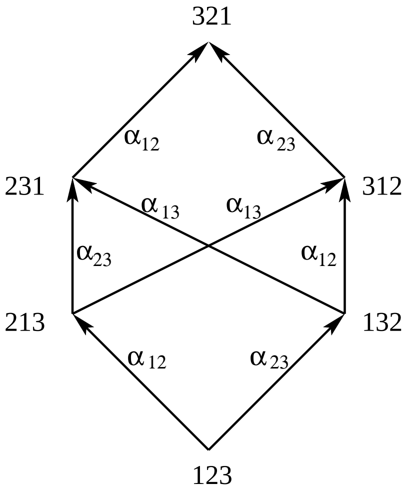
$$w \stackrel{\alpha}{\longrightarrow} ws_\alpha \, ,$$

where

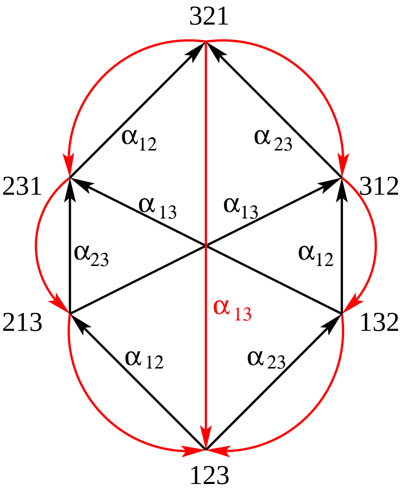
$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(Bruhat graph)}\, , \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1 \qquad (\mathrm{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle)\, .$$

Comes from the multiplication of Schubert classes in the quantum cohomology of flag varieties $QH^*(G/B)$ (Fulton and Woodward).

Bruhat graph for S_3:



Quantum Bruhat graph for S_3:



The quantum alcove model

Definition. Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m)\,.$$

The quantum alcove model

Definition. Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m)\,.$$

Fact. A λ-chain corresponds to a sequence of alcoves.

The quantum alcove model

Definition. Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m) \,.$$

Fact. A λ-chain corresponds to a sequence of alcoves.

Let $r_i := s_{\beta_i}$.

The quantum alcove model

Definition. Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \dots, \beta_m)\,.$$

Fact. A λ-chain corresponds to a sequence of alcoves.

Let $r_i := s_{\beta_i}$.

We consider subsets of positions in Γ

$$J = (j_1 < \dots < j_s) \subseteq \{1, \dots, m\}\,.$$

The quantum alcove model

Definition. Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \dots, \beta_m)\,.$$

Fact. A λ-chain corresponds to a sequence of alcoves.

Let $r_i := s_{\beta_i}$.

We consider subsets of positions in Γ

$$J = (j_1 < \dots < j_s) \subseteq \{1, \dots, m\}\,.$$

We identify J with the chain in W

$$w_0 = Id, \;\; \dots, \;\; w_i := r_{j_1} \dots r_{j_i}, \;\; \dots, \;\; w_s = w_{\mathrm{end}}\,.$$

The quantum alcove model (cont.)

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

The quantum alcove model (cont.)

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \, .$$

A position j_i is called a positive (resp. negative) folding if

$$w_{i-1} \xrightarrow{\beta_{j_i}} w_i$$

is an up (resp. down) step.

The quantum alcove model (cont.)

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

A position j_i is called a positive (resp. negative) folding if

$$w_{i-1} \xrightarrow{\beta_{j_i}} w_i$$

is an up (resp. down) step.

Let

$$J^- := \{j_i \;:\; w_{i-1} > w_i\}\,.$$

The quantum alcove model (cont.)

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \stackrel{\beta_{j_1}}{\longrightarrow} w_1 \stackrel{\beta_{j_2}}{\longrightarrow} \ldots \stackrel{\beta_{j_s}}{\longrightarrow} w_s \,.$$

A position j_i is called a positive (resp. negative) folding if

$$w_{i-1} \stackrel{\beta_{j_i}}{\longrightarrow} w_i$$

is an up (resp. down) step.

Let

$$J^- := \{j_i \;:\; w_{i-1} > w_i\}\,.$$

Let $\mathcal{A}(\Gamma) = \mathcal{A}(\lambda)$ be the collection of all admissible subsets.

The quantum alcove model (cont.)

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \, .$$

A position j_i is called a positive (resp. negative) folding if

$$w_{i-1} \xrightarrow{\beta_{j_i}} w_i$$

is an up (resp. down) step.

Let

$$J^- := \{j_i \; : \; w_{i-1} > w_i\} \, .$$

Let $\mathcal{A}(\Gamma) = \mathcal{A}(\lambda)$ be the collection of all admissible subsets.

Construction. (L. and Lubovsky, generalization of L.-Postnikov, Gaussent-Littelmann) *Crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$ *and* $\widetilde{f}_0$ *on* $\mathcal{A}(\lambda)$.

The main result

Theorem. (LNSSS) *Given* $\mathbf{p} = (p_1, p_2, \dots)$ *and an arbitrary Lie type, let*

$$\lambda = \omega_{p_1} + \omega_{p_2} + \dots .$$

The main result

Theorem. (LNSSS) *Given* $\mathbf{p} = (p_1, p_2, \ldots)$ *and an arbitrary Lie type, let*

$$\lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

The (combinatorial) crystal $\mathcal{A}(\lambda)$ *is isomorphic to the tensor product of KR crystals* $B^{\otimes \mathbf{p}}$.

Proof sketch

Projected level 0 LS paths $\xrightarrow[=]{(1)}$ Quantum LS paths $\xrightarrow[\longleftrightarrow]{(2)}$ Quantum alcove model

Proof sketch

$$\text{Projected level 0 LS paths} \xrightarrow[=]{(1)} \text{Quantum LS paths} \xrightarrow[\longleftrightarrow]{(2)} \text{Quantum alcove model}$$

Level 0 LS (Lakshmibai–Seshadri) paths: certain piecewise-linear paths with "directions" in $W_{\mathrm{af}} \cdot \lambda$.

Proof sketch

$$\begin{array}{ccccc} \text{Projected} & \xrightarrow[=]{(1)} & \text{Quantum} & \xrightarrow[\longleftrightarrow]{(2)} & \text{Quantum} \\ \text{level 0 LS paths} & & \text{LS paths} & & \text{alcove model} \end{array}$$

Level 0 LS (Lakshmibai–Seshadri) paths: certain piecewise-linear paths with "directions" in $W_{\mathrm{af}} \cdot \lambda$.

Projected level 0 LS paths: project to the finite weight lattice.

Proof sketch

Projected level 0 LS paths	$\xrightarrow[=]{(1)}$	Quantum LS paths	$\xrightarrow[\longleftrightarrow]{(2)}$	Quantum alcove model

Level 0 LS (Lakshmibai–Seshadri) paths: certain piecewise-linear paths with “directions” in $W_{\mathrm{af}} \cdot \lambda$.

Projected level 0 LS paths: project to the finite weight lattice.

Fact. $B^{\otimes \lambda}$ realized in terms of projected level 0 LS paths (Naito-Sagaki ’03-’08).

Proof sketch

Projected level 0 LS paths $\xrightarrow[=]{(1)}$ Quantum LS paths $\xrightarrow[\longleftrightarrow]{(2)}$ Quantum alcove model

Level 0 LS (Lakshmibai–Seshadri) paths: certain piecewise-linear paths with "directions" in $W_{\mathrm{af}} \cdot \lambda$.

Projected level 0 LS paths: project to the finite weight lattice.

Fact. $B^{\otimes\lambda}$ realized in terms of projected level 0 LS paths (Naito-Sagaki '03-'08).

Quantum LS paths: the "directions" in $W \cdot \lambda \simeq W/W_\lambda$ are related by paths in the parabolic quantum Bruhat graph $\mathrm{QB}(W/W_\lambda)$.

Ingredients in the proof

Projected level 0 LS paths $\xrightarrow[=]{(1)}$ Quantum LS paths $\xrightarrow[\longleftrightarrow]{(2)}$ Quantum alcove model

Ingredients in the proof

Projected level 0 LS paths $\xrightarrow[=]{(1)}$ Quantum LS paths $\xrightarrow[\longleftrightarrow]{(2)}$ Quantum alcove model

Ingredients for (1): we lift $\mathrm{QB}(W/W_\lambda)$ to

Ingredients in the proof

Projected level 0 LS paths	$\xrightarrow[=]{(1)}$	Quantum LS paths	$\xrightarrow[\longleftrightarrow]{(2)}$	Quantum alcove model

Ingredients for (1): we lift $\mathrm{QB}(W/W_\lambda)$ to

- Littelmann's poset of level 0 weights $W_{\mathrm{af}} \cdot \lambda$

Ingredients in the proof

$$\begin{array}{c}\text{Projected}\\ \text{level 0 LS paths}\end{array} \xrightarrow[=]{(1)} \begin{array}{c}\text{Quantum}\\ \text{LS paths}\end{array} \xrightarrow[\longleftrightarrow]{(2)} \begin{array}{c}\text{Quantum}\\ \text{alcove model}\end{array}$$

Ingredients for (1): we lift $\mathrm{QB}(W/W_\lambda)$ to

- Littelmann's poset of level 0 weights $W_{\mathrm{af}} \cdot \lambda \implies$ description of covers;

Ingredients in the proof

$$\begin{matrix}\text{Projected} \\ \text{level 0 LS paths}\end{matrix} \xrightarrow[=]{(1)} \begin{matrix}\text{Quantum} \\ \text{LS paths}\end{matrix} \xrightarrow[\longleftrightarrow]{(2)} \begin{matrix}\text{Quantum} \\ \text{alcove model}\end{matrix}$$

Ingredients for (1): we lift $\mathrm{QB}(W/W_\lambda)$ to

- Littelmann's poset of level 0 weights $W_{\mathrm{af}} \cdot \lambda \implies$ description of covers;
- the Bruhat order on W_{af} (parabolic version of "quantum to affine", cf. Peterson '97, Lam–Shimozono '10).

Ingredients in the proof

$$\text{Projected level 0 LS paths} \xrightarrow[=]{(1)} \text{Quantum LS paths} \xrightarrow[\longleftrightarrow]{(2)} \text{Quantum alcove model}$$

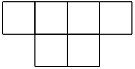
Ingredients for (1): we lift $\mathrm{QB}(W/W_\lambda)$ to

- Littelmann's poset of level 0 weights $W_{\mathrm{af}} \cdot \lambda \implies$ description of covers;
- the Bruhat order on W_{af} (parabolic version of "quantum to affine", cf. Peterson '97, Lam–Shimozono '10).

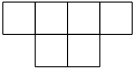
Ingredients for (2):

- we study various other properties of the parabolic quantum Bruhat graph.

Example in type A_2.

$$\mathbf{p} = (1, 2, 2, 1) =$$  $$; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0).$$

Example in type A_2.

$\mathbf{p} = (1,2,2,1) =$  ; $\quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$

A λ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ \ (1,2),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (1,2),\ \ (1,3)\ \ ).$$

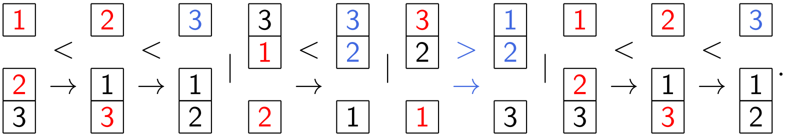
Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left(\ \underline{(1,2)}, \ \underline{(1,3)} \mid \underline{(2,3)}, \ (1,3) \mid (2,3), \ \underline{(1,3)} \mid \underline{(1,2)}, \ \underline{(1,3)} \ \right).$$

Example. Let $J = \{1,2,3,6,7,8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \mid\ \underline{(2,3)},\ (1,3)\ \mid\ (2,3),\ \underline{(1,3)}\ \mid\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

Claim: J is admissible. Indeed, the corresponding path in the quantum Bruhat graph is



Example. Let $J = \{1,2,3,6,7,8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \middle|\ \underline{(2,3)},\ (1,3)\ \middle|\ (2,3),\ \underline{(1,3)}\ \middle|\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

Claim: J is admissible. Indeed, the corresponding path in the quantum Bruhat graph is

$$\begin{array}{c} \boxed{1} < \boxed{2} < \boxed{3} \\ \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \end{array} \Bigg| \begin{array}{c} \begin{smallmatrix}3\\1\end{smallmatrix} < \begin{smallmatrix}3\\2\end{smallmatrix} \\ \boxed{2} \to \boxed{1} \end{array} \Bigg| \begin{array}{c} \begin{smallmatrix}3\\2\end{smallmatrix} > \begin{smallmatrix}1\\2\end{smallmatrix} \\ \boxed{1} \to \boxed{3} \end{array} \Bigg| \begin{array}{c} \boxed{1} < \boxed{2} < \boxed{3} \\ \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \end{array}.$$

The corresponding element in $B^{\otimes \mathbf{p}} = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$ represented via column-strict fillings:

$$\boxed{3} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{3}.$$

The energy function in arbitrary Lie type

Definition. Given the λ-chain

$$\Gamma = (\beta_1, \ldots, \beta_m)\,,$$

define the height sequence $(h_1, \ldots, h_m)$ by

$$h_i := \#\{j \geq i \; : \; \beta_j = \beta_i\}\,.$$

The energy function in arbitrary Lie type

Definition. Given the λ-chain

$$\Gamma = (\beta_1, \ldots, \beta_m)\,,$$

define the height sequence $(h_1, \ldots, h_m)$ by

$$h_i := \#\{j \geq i \;:\; \beta_j = \beta_i\}\,.$$

Then, for $J \in \mathcal{A}(\lambda)$, define the statistic

$$\mathrm{height}(J) := \sum_{j \in J^-} h_j\,.$$

The energy function in arbitrary Lie type

Definition. Given the λ-chain

$$\Gamma = (\beta_1, \ldots, \beta_m)\,,$$

define the height sequence $(h_1, \ldots, h_m)$ by

$$h_i := \#\{j \geq i \;:\; \beta_j = \beta_i\}\,.$$

Then, for $J \in \mathcal{A}(\lambda)$, define the statistic

$$\mathrm{height}(J) := \sum_{j \in J^-} h_j\,.$$

Theorem. (LNSSS) *Given $J \in \mathcal{A}(\lambda)$, which is identified with $B^{\otimes \mathbf{p}}$, we have*

$$D_B(J) = -\mathrm{height}(J)\,.$$

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

We considered the λ-chain Γ and $J = \{1, 2, 3, 6, 7, 8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rl} \Gamma = & (\,\underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid (2,3), \underline{(1,3)} \mid \underline{(1,2)}, \underline{(1,3)}\,)\,, \\ (h_i) = & (\quad 2, \quad 4 \quad \mid \quad 2, \quad 3 \quad \mid \quad 1, \quad 2 \quad \mid \quad 1, \quad 1 \quad)\,. \end{array}$$

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.
We considered the λ-chain Γ and $J = \{1,2,3,6,7,8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rl} \Gamma = & (\, \underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid (2,3), \underline{(1,3)} \mid \underline{(1,2)}, \underline{(1,3)} \,)\,, \\ (h_i) = & (\quad 2, \quad 4 \quad \mid \quad 2, \quad 3 \quad \mid \quad 1, \quad 2 \quad \mid \quad 1, \quad 1 \quad)\,. \end{array}$$

We have

$$\mathrm{height}(J) = 2\,.$$

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

We considered the λ-chain Γ and $J = \{1,2,3,6,7,8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rcccccccccccc} \Gamma = & (& \underline{(1,2)}, & \underline{(1,3)} & | & \underline{(2,3)}, & (1,3) & | & (2,3), & \underline{(1,3)} & | & \underline{(1,2)}, & \underline{(1,3)} \ ), \\ (h_i) = & (& 2, & 4 & | & 2, & 3 & | & 1, & 2 & | & 1, & 1 \ ). \end{array}$$

We have

$$\mathrm{height}(J) = 2\,.$$

Remarks. (1) In type A, the height statistic translates into the Lascoux–Schützenberger charge statistic on Young tableaux (L.).

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

We considered the λ-chain Γ and $J = \{1, 2, 3, 6, 7, 8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rcccccccccccc} \Gamma = & (& \underline{(1,2)}, & \underline{(1,3)} & \mid & \underline{(2,3)}, & (1,3) & \mid & (2,3), & \underline{(1,3)} & \mid & \underline{(1,2)}, & \underline{(1,3)} &), \\ (h_i) = & (& 2, & 4 & \mid & 2, & 3 & \mid & 1, & 2 & \mid & 1, & 1 &). \end{array}$$

We have

$$\mathrm{height}(J) = 2 \,.$$

Remarks. (1) In type A, the height statistic translates into the Lascoux–Schützenberger charge statistic on Young tableaux (L.).

(2) A similar charge statistic was defined in type C (L. and Schilling), and one is being developed in type B (Briggs and L.).