Particles jumping on a cycle, a process on permutations and words

(multi-TASEP on a ring)

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FPSAC, June 28, 2013

Introduction

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2 7
6 5
4 8
3

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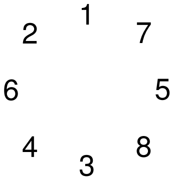

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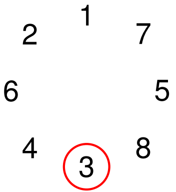

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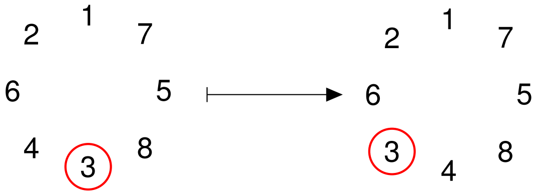

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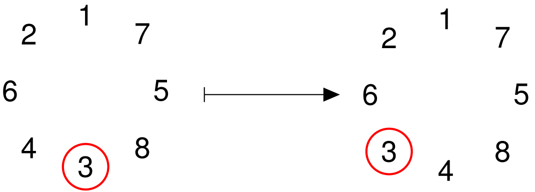

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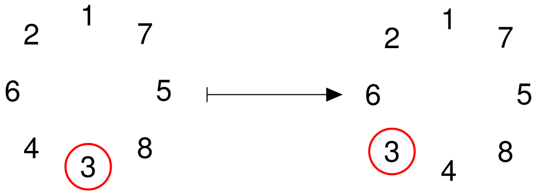

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This is a TASEP (Totally Asymetric Simple Exclussion Process).

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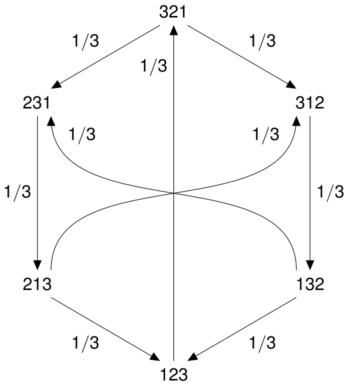

Figure : The cyclic-TASEP Markov chain for $n = 3$.

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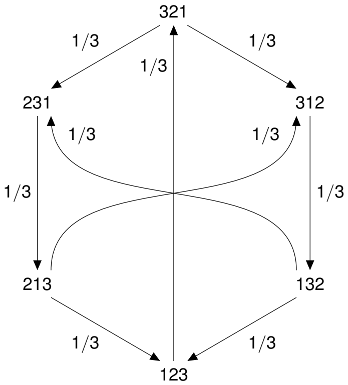

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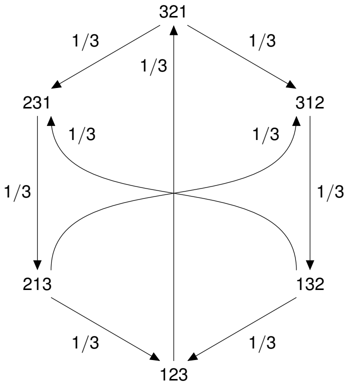

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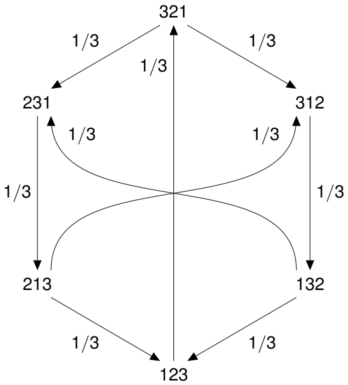

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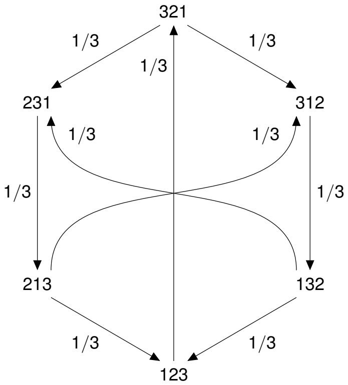

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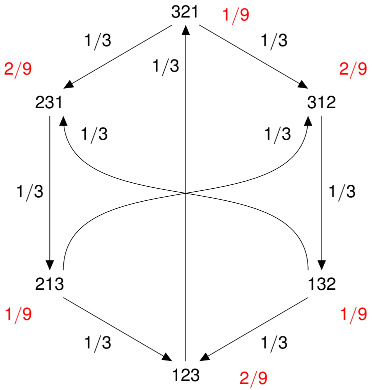

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Let $\sigma_0 = n\ n-1 \ldots 2\ 1$ be the reverse permutation.
Then $p_{\sigma_0} = \frac{1}{2}, \frac{1}{9}, \frac{1}{96}, \frac{1}{2500}, \ldots$

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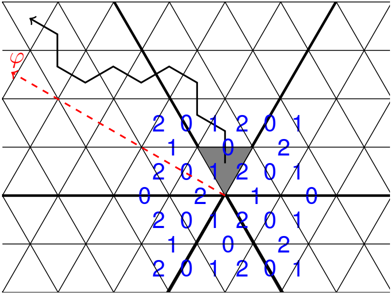

A reduced random walk in the alcoves of the $\tilde{A}_2$ arrangement.
The shown walk has reduced word $\cdots s_1 s_0 s_2 s_0 s_1 s_2 s_0 s_2 s_1 s_0$.
The thick lines divide V into Weyl chambers.

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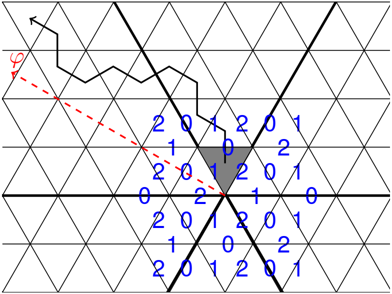

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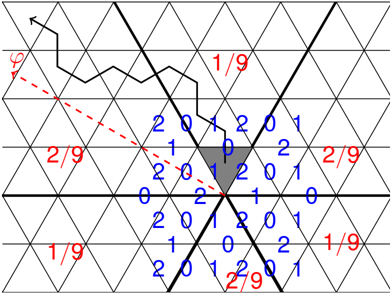

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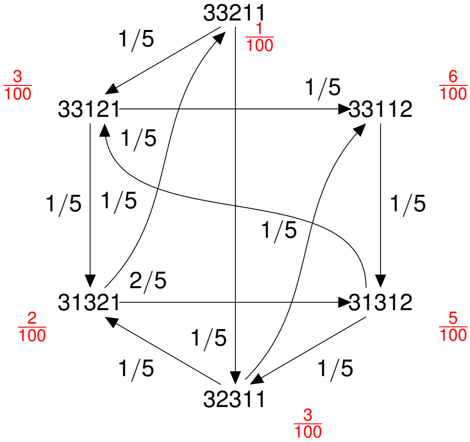

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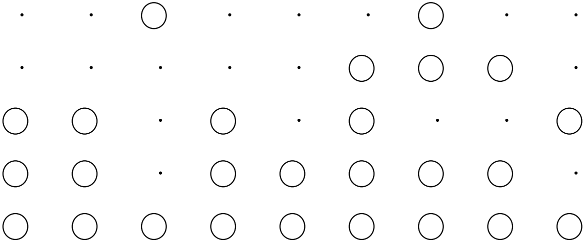

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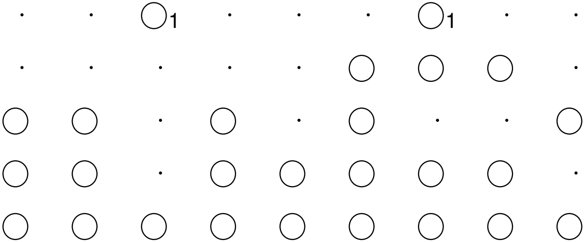

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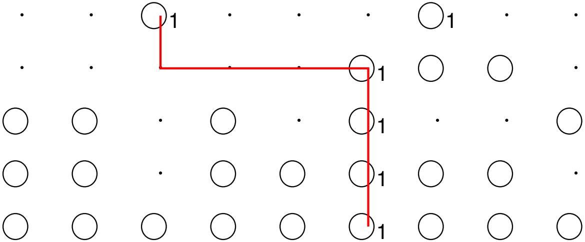

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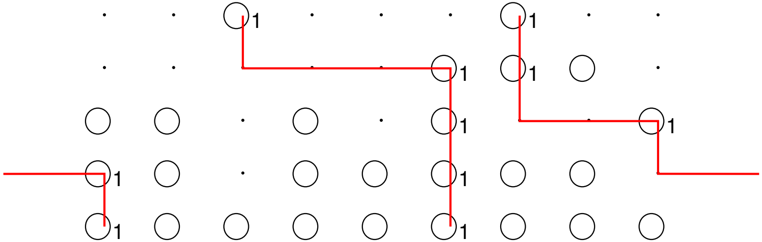

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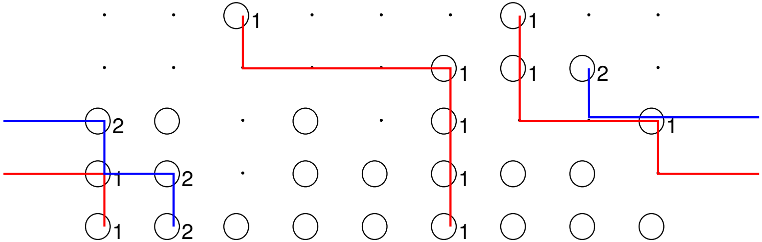

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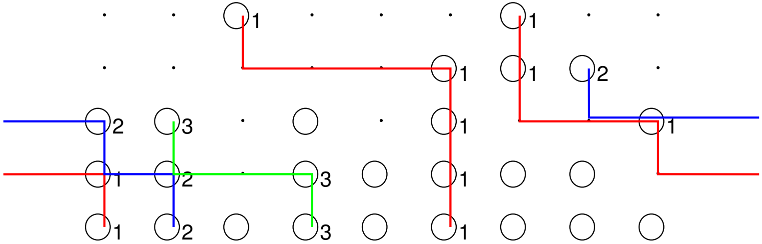

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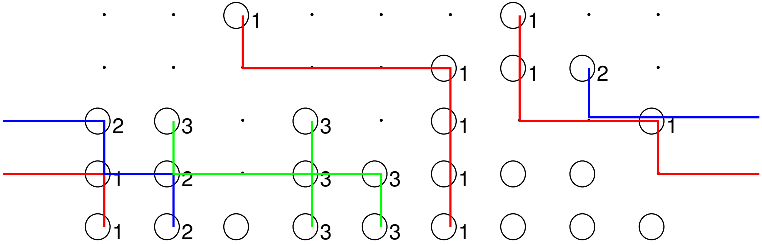

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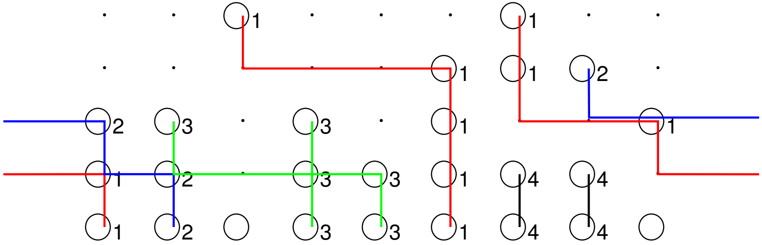

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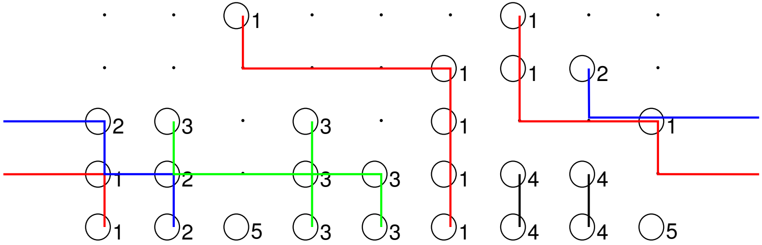

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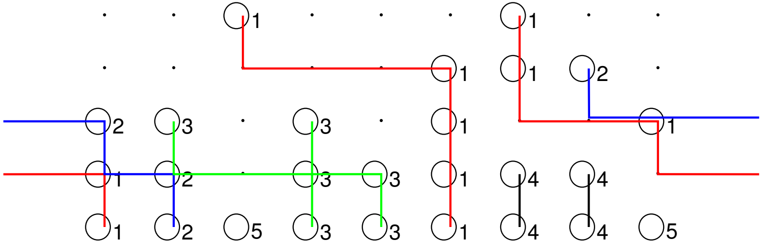

This gives a map $B : MLQ_{\mathbf{m}} \to$ Multipermutations with content $\mathbf{m}$.

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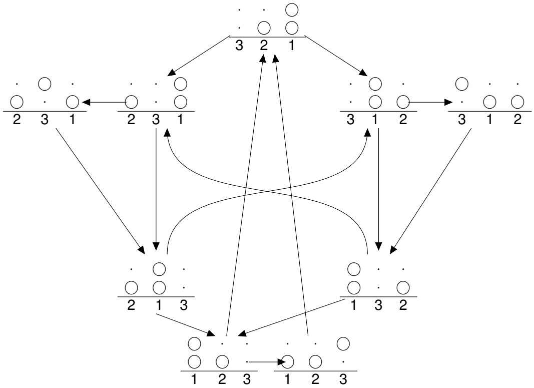
Corollary

For any permutation σ we have

$$p_\sigma = \frac{\#\{q : B(q) = \sigma\}}{Z_{\mathbf{m}}}$$

$\mathbf{m} = (1, 1, 1)$



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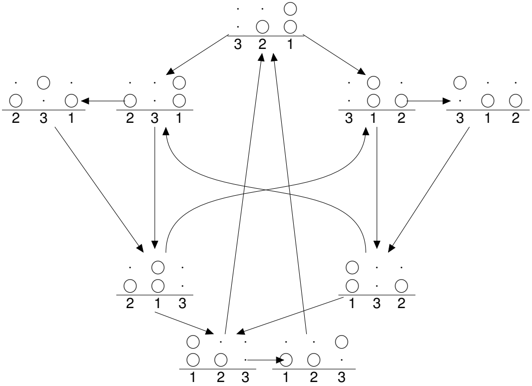
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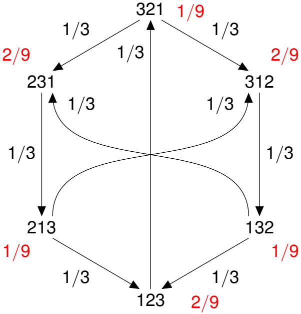
321
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231
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312
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213
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132
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123
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1/3
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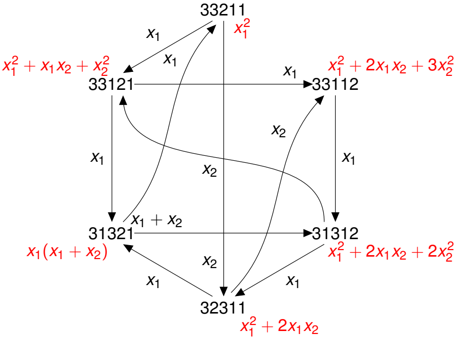

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The third statement is open.

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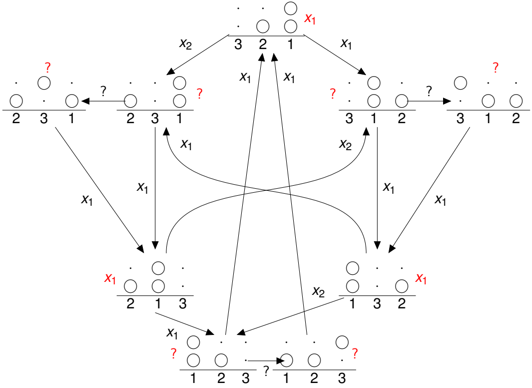

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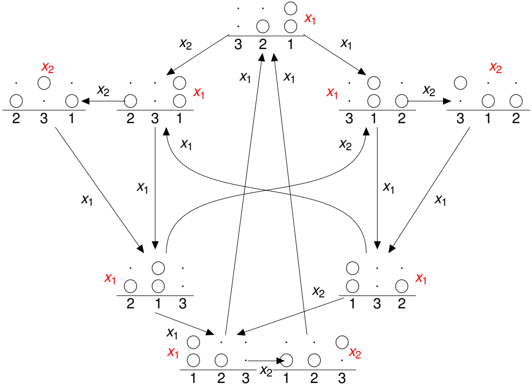

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OPEN: Find conceptual proof of the independence $1/n^2$.

Correlations

Return to case of permutations, all $x_i = 1/n$.
Let $c_{i,j} = Prob(\sigma = i\ j\ \ldots)$.

Theorem (Ayyer & Linusson '13+)

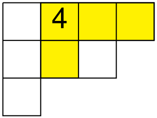
For any $1 \leq i, j \leq n$, *we have*

$$c_{i,j} = \begin{cases} \frac{i-j}{n\binom{n}{2}}, & \textit{if } i > j \\ 0, & \textit{if } i = j \\ \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)}, & \textit{if } i = j-1 \\ \frac{1}{n^2}, & \textit{if } i < j-1 \end{cases}$$

OPEN: Find conceptual proof of the independence $1/n^2$.
These correlations prove a conjecture of Lam about random n-cores.

n-cores

In an integer partition, the **hook length** is the number of squares to the left or below a given square.



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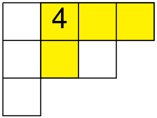

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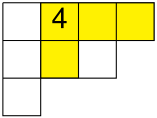

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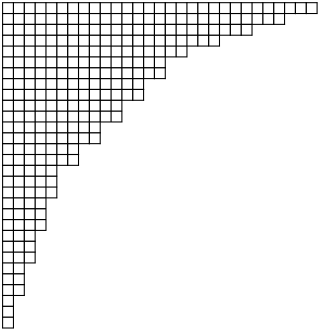
It follows from the work of Lam that determining the limit shape of a random *n*-core is the same as determining the direction φ of the reduced random walk in $\tilde{A}_{n-1}$.

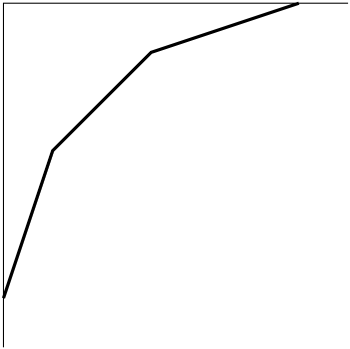
Theorem (Ayyer & L., Conjectured by Lam)

$\varphi =$ *the sum of all positive roots.*

n-cores

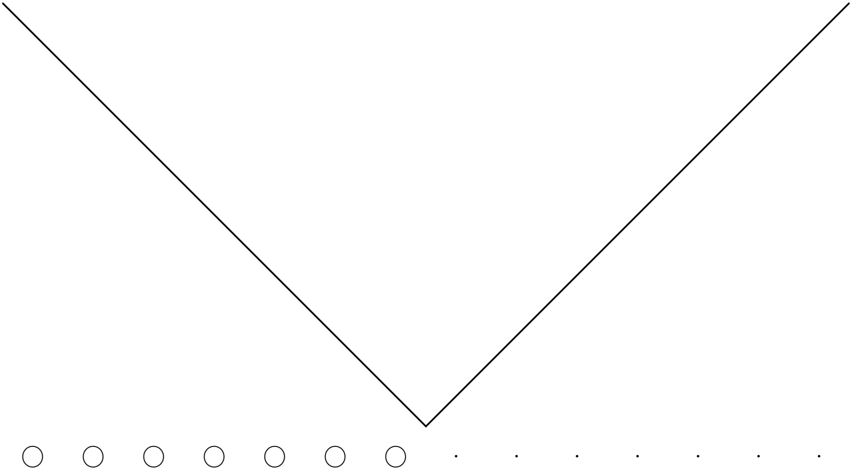
A large 4-core and the limiting shape.





(mod n) jump process

Example $n = 3$.



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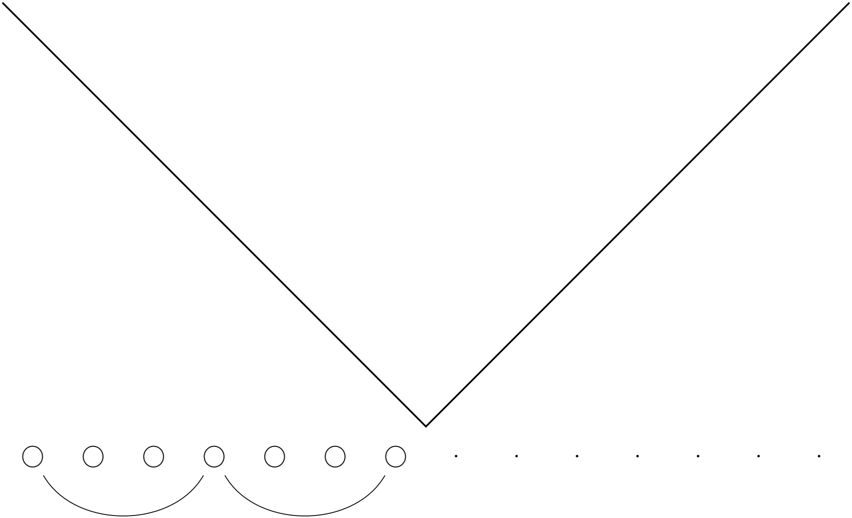

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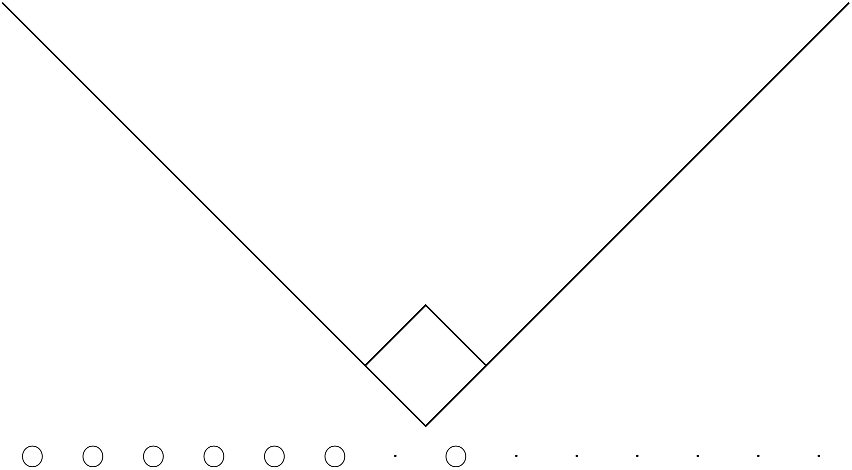

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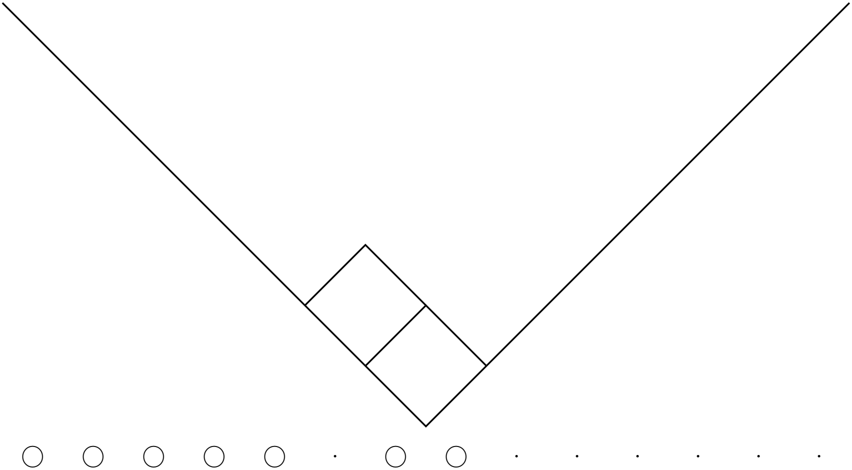

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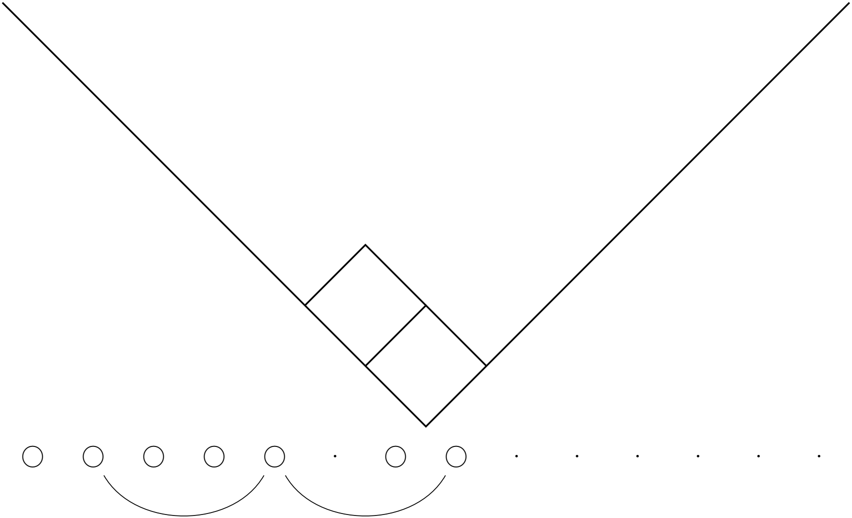

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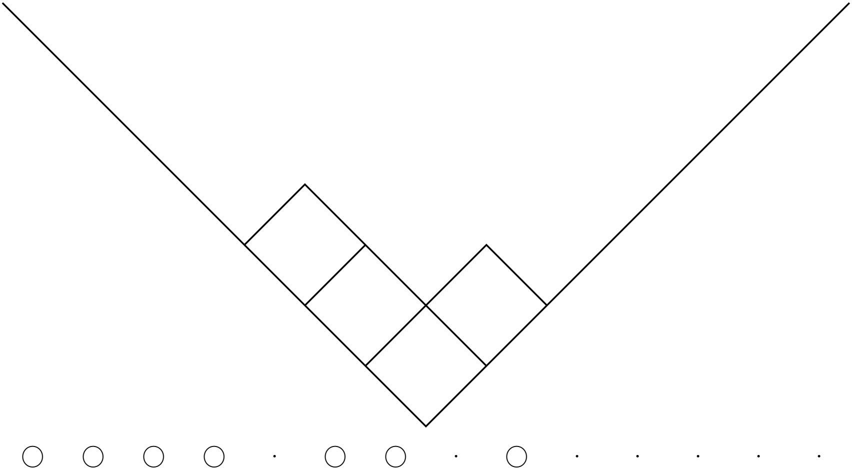

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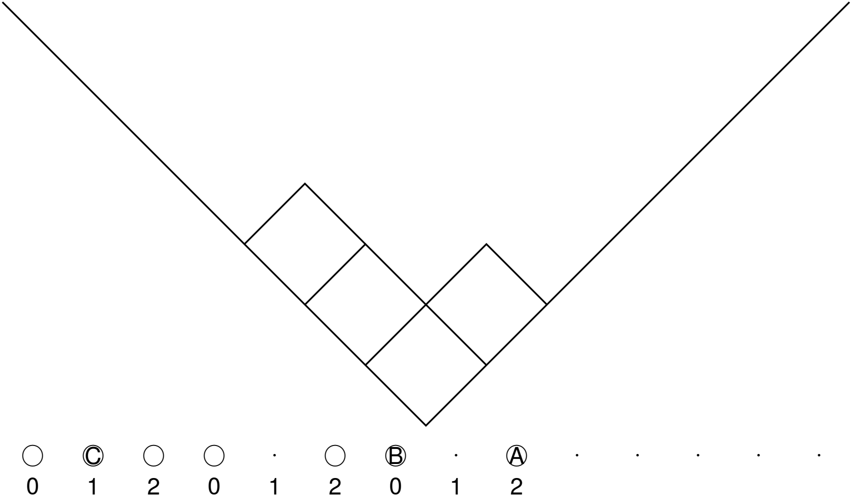

Long distance correlations

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Wanted: Conceptual proof.

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Conjecture (Ayyer & Linusson)

$$c_{i,j,k} = \frac{1}{n^3}, \quad \textit{if } i < j-1 < k-2.$$

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Merci à tous pour votre attention!