Overview

**Goal:** Generalize algebraic graph theory...
- definition and enumeration of spanning trees
- combinatorial Laplacian
- critical group
- chip-firing / sandpile model
- lattices of cuts and flows

...to higher-dimensional generalizations of graphs
(i.e., simplicial/cell complexes)

**Tools:** linear algebra, homological algebra, algebraic topology
Incidence and Laplacian Matrices

$G = (V, E)$: connected, loopless graph; $|V| = n$; edges oriented arbitrarily

(Signed) incidence matrix $\partial = [\partial_{ve}]_{v \in V, e \in E}$

\[
\partial_{ve} = \begin{cases} 
1 & \text{if } v = \text{head}(e) \\
-1 & \text{if } v = \text{tail}(e) \\
0 & \text{otherwise}
\end{cases}
\]

Laplacian matrix $L = \partial \partial^* = [\ell_{vw}]_{v, w \in V}$

\[
\ell_{vw} = \begin{cases} 
\deg(v) = \# \text{ incident edges} & \text{if } v = w \\
-(\# \text{ edges joining } v, w) & \text{if } v \neq w
\end{cases}
\]

Note: $\text{rank } \partial = \text{rank } L = n - 1$. 
The Critical Group

**Definition**

The critical group $K(G)$ is the torsion summand of $\text{coker } L (= \mathbb{Z}^n / \text{im } L)$.

Alternatively, if $L_i$ is the reduced Laplacian obtained from $L$ by deleting the $i^{th}$ row and column, then $K(G) = \text{coker } L_i$.

**Example:** $G = K_3$; $L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $L_i = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\text{coker } L = \mathbb{Z}^3 / \text{colspan}(L) \cong \mathbb{Z} \oplus \mathbb{Z} / 3\mathbb{Z}$$

**Matrix-Tree Theorem:** $|K(G)| = \det L_i = \# \text{ of spanning trees of } G$
The Chip-Firing Game (a.k.a. the Sandpile Model)

**Chip-firing game on G:**

- Choose one vertex \( q \) as the bank.
- Each vertex \( v \neq q \) starts with \( c_v \) dollars euros.
- If \( c_v \geq \deg(v) \), then \( v \) fires by transferring 1€ along each incident edge.
- When no non-bank vertices can fire, the configuration is stable. Then, and only then, the bank fires.
- Each starting configuration evolves to exactly one critical (\( \equiv \) stable and recurrent) configuration.

**Punchline:** The critical configurations correspond bijectively to the elements of the critical group \( K(G) \).
The Sandpile Model (a.k.a. the Chip-Firing Game)

The chip-firing game/sandpile model has many wonderful properties!


- Gen. func. for critical configs is a Tutte-Grothendieck invariant [Merino]

- Critical configurations are in bijection with G-parking functions and regions of the G-Shi hyperplane arrangement [Hopkins–Perkinson]


Cut and Flow Spaces

**Definition**

The cut space and flow space of $G$ are

$$\text{Cut}(G) = \text{im} \, \partial^* \subseteq \mathbb{R}^E,$$

$$\text{Flow}(G) = \ker \partial \subseteq \mathbb{R}^E.$$  

These spaces are orthogonal complements, and

$$\dim \text{Cut}(G) = |V| - 1, \quad \dim \text{Flow}(G) = |E| - |V| + 1.$$
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A cut vector

A flow vector
Proposition

Let $T$ be a spanning tree of $G$.

1. For each edge $e \in T$, the graph with edges $T \setminus e$ has two components. The corresponding cut vectors form a basis for $\text{Cut}(G)$.

2. For each edge $e \not\in T$, there is a unique cycle in $T \cup e$. The signed characteristic vectors of all such cycles form a basis for $\text{Flow}(G)$.

3. These are in fact $\mathbb{Z}$-module bases for the cut lattice $\mathcal{C}(G) = \text{Cut}(G) \cap \mathbb{Z}^E$ and the flow lattice $\mathcal{F}(G) = \text{Flow}(G) \cap \mathbb{Z}^E$.

(General matroid theory predicts bases of the forms (1) and (2), but not the combinatorial interpretation of their coefficients.)
Theorem (Bacher, de la Harpe, Nagnibeda)

For every graph $G$, there are isomorphisms

$$K(G) \cong \mathcal{F}^\# / \mathcal{F} \cong \mathcal{C}^\# / \mathcal{C} \cong \mathbb{Z}^E / (\mathcal{C} \oplus \mathcal{F}).$$

Here $\mathcal{L}^\#$ means the dual of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$:

$$\mathcal{L}^\# = \{ w \in \mathcal{L} \otimes \mathbb{R} \mid v \cdot w \in \mathbb{Z} \ \forall v \in \mathcal{L} \}$$
$$= \text{Hom}(\mathcal{L}, \mathbb{Z}) \quad \text{(via standard dot product)}$$

For instance, if $v = (1, 1, \ldots, 1) \in \mathbb{Z}^n$ then $(\mathbb{Z}v)^\# = \frac{1}{n} \mathbb{Z}v$. 
Example: $G = K_3$

Flow lattice

$F = \ker \partial = \langle (1, -1, 1) \rangle$

$F^\# = \langle (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}) \rangle$

Cut lattice

$C = \text{im } \partial^* = \langle (1, 0, -1), (0, 1, 1) \rangle$

$C^\# = \langle (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \rangle$

Here, $F^\# / F = C^\# / C = \mathbb{Z}^3 / (C \oplus F) = K(G) = \mathbb{Z}/3\mathbb{Z}$
**Central Problem:** What happens to the theory of cuts, flows, critical groups, sandpiles/chip-firing, ... when we replace the graph $G$ with something more general?

Topologically, a graph is a 1-dimensional simplicial (multi)complex — it consists of edges and vertices. Can we develop the theory for general combinatorial/topological spaces?
Cell Complexes

Cell complexes (= CW complexes) are higher-dimensional generalizations of graphs (like simplicial complexes, but even more general).

Examples: graphs, simplicial complexes, polytopes, polyhedral fans, . . .

Rough definition: A cell complex $X$ consists of cells (homeomorphic copies of $\mathbb{R}^k$ for various $k$) together with attaching maps

$$\partial_k(X) : C_k(X) \to C_{k-1}(X)$$

where $C_k(X) = \text{free } \mathbb{Z}-\text{module generated by } k\text{-dimensional cells.}$ (Note: $\partial_k \partial_{k+1} = 0$ for all $k$.) The integer $\partial_k(X)_{\rho,\sigma}$ specifies the multiplicity with which the $k$-cell $\sigma$ is attached to the $(k-1)$-cell $\rho$.

— Attaching maps can be topologically complicated, but the only data we need is the cellular chain complex $\cdots \to C_k(x) \to C_{k-1}(X) \to \cdots$
A cellular spanning tree (CST) of $X^d$ is a subcomplex $Y \subseteq X$ such that $Y \supseteq X_{(d-1)}$ and any of these two conditions hold:

- $\tilde{H}_d(Y, \mathbb{Q}) = 0$;
- $\tilde{H}_{d-1}(Y, \mathbb{Z})$ is finite;
- $|Y_d| = |X_d| - \tilde{\beta}_d(X) + \tilde{\beta}_{k-1}(X)$ (where $\beta_i(X) = \dim_{\mathbb{Q}} \tilde{H}_i(X, \mathbb{Q})$)

The “right” count of CSTs is

$$\tau(X) := \sum_{\text{CSTs } Y \subseteq X} \left|\tilde{H}_{d-1}(Y, \mathbb{Z})\right|^2$$

which can be obtained as a determinant of a reduced Laplacian [DKM ’09,’11, Lyons ’11, Catanzaro-Chernyak-Klein ’12]
The Cellular Critical Group

**Definition**
The **critical group** of a $d$-dimensional cell complex $X$ is

$$K(X) = \ker \partial_{d-1} / \text{im} \partial_d \partial_d^*.$$ 

**Fact:** $K(X)$ is finite abelian of order $\tau(X)$, and can also be expressed in terms of the reduced Laplacian [DKM ’13]

**Questions:**

- How can $K(X)$ be expressed in terms of cuts and flows?
- What are cellular cuts and flows in the first place?
- Is there a cellular chip-firing game for which elements of $K(X)$ correspond to critical states?
**Example of flow vector:** find a non-contractible $d$-sphere in $X^d$ and orient all its cells consistently

**Example of cut vector:** poke a line through $X^d$ and pick an orientation around the line

If $d = 1$, these pictures reduce to the usual cuts and flows in graphs.
Cellular Cuts and Flows

**Definition**

Let $X$ be a $d$-dimensional cell complex with $n$ facets (max-dim cells).

$$\text{Cut}(X) := \text{im} \partial_d^*(X) \subseteq \mathbb{R}^n$$

$$\text{Flow}(X) := \ker \partial_d(X) \subseteq \mathbb{R}^n$$

$$\mathcal{C}(X) := \text{Cut}(X) \cap \mathbb{Z}^n$$

$$\mathcal{F}(X) := \text{Flow}(X) \cap \mathbb{Z}^n$$

**Theorem (DKM ’13+)**

Fix a cellular spanning tree $Y \subset X$.

1. There are natural $\mathbb{R}$-bases of $\text{Cut}(X)$ and $\text{Flow}(X)$ indexed by the facets contained / not contained in $Y$.
2. The basis vector for each facet is supported on its fundamental cocircuit / circuit. Coeff’ts are sizes of certain homology groups.
3. Under certain conditions on $\tilde{H}_{d-1}(Y)$: $\mathbb{Z}$-bases for $\mathcal{C}(X), \mathcal{F}(X)$. 
Question

Do the Bacher-de la Harpe-Nagnibeda isomorphisms

\[ K(X) \cong \mathcal{F}^\# / \mathcal{F} \cong \mathcal{C}^\# / \mathcal{C} \cong \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \]

still hold if \( X \) is an arbitrary cell complex?

Answer: Not quite.
The Bacher–de la Harpe–Nagnibeda isomorphisms do not hold in general.
The Bacher–de la Harpe–Nagnibeda isomorphisms do not hold in general.

**Example:** \( X = \mathbb{R}P^2 \): cell complex with one vertex, one edge, and one 2-cell, and cellular chain complex

\[
C_2 = \mathbb{Z} \xrightarrow{\partial_2 = [2]} C_1 = \mathbb{Z} \xrightarrow{[\partial_1 = 0]} C_0 = \mathbb{Z}
\]

- \( C/C^\# \cong \mathbb{Z}/4\mathbb{Z} \) because \( C = \text{im} \partial_2^* = 2\mathbb{Z} \) and so \( C^\# = \frac{1}{2}\mathbb{Z} \).
- \( F^\#/F = 0 \) because \( F = \ker \partial_2 = 0 \).
- \( \mathbb{Z}/(C \oplus F) \cong \mathbb{Z}/2\mathbb{Z} \).

The culprit is probably torsion (note that \( \tilde{H}_1(X) = \mathbb{Z}/2\mathbb{Z} \)).

In fact \( K(G) \cong \mathbb{Z}/4\mathbb{Z} \). What is special about cuts?
The Critical Group via Cuts and Flows

**Theorem (DKM ’13+)**

For any cell complex $X$, there are short exact sequences

\[
0 \rightarrow \mathbb{Z}^n/(C \oplus F) \rightarrow K(X) \cong C^\# / C \rightarrow T(\tilde{H}_{d-1}(X)) \rightarrow 0
\]

and

\[
0 \rightarrow T(\tilde{H}_{d-1}(X)) \rightarrow \mathbb{Z}^n/(C \oplus F) \rightarrow K^*(X) \cong F^\# / F \rightarrow 0.
\]

- $T(A)$ means the torsion summand of $A$ (i.e., $T(A)$ is finite and $A = T(A) \oplus \mathbb{Z}^{\text{something}}$)
- “$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ short exact” means “$C \cong B / A$”
- For graphs, these exact sequences reduce to the Bacher-de la Harpe-Nagnibeda isomorphisms (because torsion terms are trivial)
The Cocritical Group

To define the cocritical group $K^*(X)$, first construct an acyclization $\Omega$ of $X$ by adjoining $(d + 1)$-cells so as to eliminate all $d$-homology.

Then, $K^*(X) = C_{d+1}(\Omega; \mathbb{Z})/\text{im } \partial_{d+1}^* \partial_{d+1}$

Compare $K(X) = \text{ker } \partial_{d-1}/\text{im } \partial_d \partial_d^*$

\[ = T(\text{coker } L_{d+1}^{du}(\Omega)). \]
\[ = T(\text{coker } L_{d-1}^{ud}(\Omega)). \]
\[ = T(\text{coker } L_{d-1}^{ud}(X)). \]
Open Problems

- **Chip-firing/sandpiles for cell complexes?**
  (We have some ideas. Big problems: (a) torsion and (b) no “conservation of matter” for arbitrary cell complexes.)

- **Riemann-Roch theory in higher dimension?**
  (Baker–Norine: graph-theoretic Riemann-Roch theorem in which $K(G)$ stands in for the Picard group of a Riemann surface.)

- **Combinatorial commutative algebra connection?**
  (Sandpile configurations = monomials; toppling = reduction modulo binomial Gröbner basis, in analogy to Cori–Rossin–Salvi)

- **Cellular max-flow/min-cut theorem?**
Thank you for listening!
Merci de votre attention!