Cuts and Flows in Cell Complexes

Art M. Duval (University of Texas, El Paso)
Caroline J. Klivans (Brown University)
Jeremy L. Martin (University of Kansas)

FPSAC/SFCA 25
Paris, France, June 28, 2013

Preprint: arXiv:1206.6157

Overview

Goal: Generalize algebraic graph theory...

- definition and enumeration of spanning trees
- combinatorial Laplacian
- critical group
- chip-firing / sandpile model
- lattices of cuts and flows

...to higher-dimensional generalizations of graphs
(i.e., simplicial/cell complexes)

Tools: linear algebra, homological algebra, algebraic topology

Incidence and Laplacian Matrices

$G = (V, E)$: connected, loopless graph; $|V| = n$; edges oriented arbitrarily

(Signed) incidence matrix $\partial = [\partial_{ve}]_{v \in V,\ e \in E}$

$$\partial_{ve} = \begin{cases} 1 & \text{if } v = \text{head}(e) \\ -1 & \text{if } v = \text{tail}(e) \\ 0 & \text{otherwise} \end{cases}$$

Laplacian matrix $L = \partial \partial^* = [\ell_{vw}]_{v,w \in V}$

$$\ell_{vw} = \begin{cases} \deg(v) = \#\text{ incident edges} & \text{if } v = w \\ -(\#\text{ edges joining } v, w) & \text{if } v \neq w \end{cases}$$

Note: $\operatorname{rank} \partial = \operatorname{rank} L = n - 1$.

The Critical Group

Definition

The critical group $K(G)$ is the torsion summand of coker L $(= \mathbb{Z}^n / \operatorname{im} L)$.

Alternatively, if L_i is the *reduced Laplacian* obtained from L by deleting the i^{th} row and column, then $K(G) = \operatorname{coker} L_i$.

Example: $G = K_3; \quad L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}; \quad L_i = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\operatorname{coker} L = \mathbb{Z}^3/\operatorname{colspan}(L) \cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/3\mathbb{Z}}_{K(G)}$$

Matrix-Tree Theorem: $|K(G)| = \det L_i = \#$ of spanning trees of G

The Chip-Firing Game (a.k.a. the Sandpile Model)

Chip-firing game on G:

- Choose one vertex q as the bank.
- Each vertex $v \neq q$ starts with c_v ~~dollars~~ euros
- If $c_v \geq \deg(v)$, then v fires by transferring 1€ along each incident edge
- When no non-bank vertices can fire, the configuration is stable. Then, and only then, the bank fires.
- Each starting configuration evolves to exactly one critical (= stable and recurrent) configuration.

Punchline: The critical configurations correspond bijectively to the elements of the critical group $K(G)$.

The Sandpile Model (a.k.a. the Chip-Firing Game)

The chip-firing game/sandpile model has many wonderful properties!

- Studied extensively in probability, statistical physics [Dhar, Bak–Tang–Wiesenfeld…; survey Levine–Propp, Notices AMS 2010]
- Gen. func. for critical configs is a Tutte-Grothendieck invariant [Merino]
- Critical configurations are in bijection with G-parking functions and regions of the G-Shi hyperplane arrangement [Hopkins–Perkinson]
- Gröbner bases, toric ideals [Cori–Rossin–Salvy, Perkinson–Wilmes, Dochtermann–Sanyal, Shokrieh–Mohammadi]
- Graph : Riemann surface :: Critical group : Picard group [Bacher–de la Harpe–Nagnibeda, Baker–Norine]

Cut and Flow Spaces

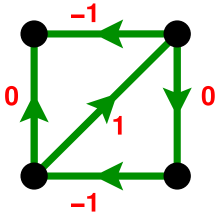
Definition

The cut space and flow space of G are

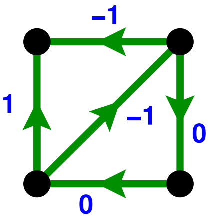
$$\mathrm{Cut}(G) = \operatorname{im} \partial^* \subseteq \mathbb{R}^E, \qquad \mathrm{Flow}(G) = \ker \partial \subseteq \mathbb{R}^E.$$

These space are orthogonal complements, and

$$\dim \mathrm{Cut}(G) = |V| - 1, \qquad \dim \mathrm{Flow}(G) = |E| - |V| + 1.$$



A cut vector



A flow vector

Cut and Flow Spaces

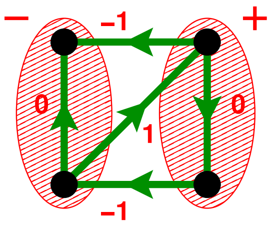
Definition

The cut space and flow space of G are

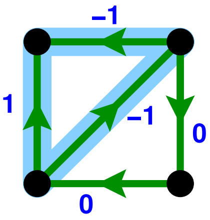
$$\mathrm{Cut}(G) = \mathrm{im}\,\partial^* \subseteq \mathbb{R}^E, \qquad \mathrm{Flow}(G) = \ker \partial \subseteq \mathbb{R}^E.$$

These space are orthogonal complements, and

$$\dim \mathrm{Cut}(G) = |V| - 1, \qquad \dim \mathrm{Flow}(G) = |E| - |V| + 1.$$



A cut vector



A flow vector

Bases of Cut and Flow Spaces

Proposition

Let T be a spanning tree of G.

1. *For each edge $e \in T$, the graph with edges $T \setminus e$ has two components. The corresponding cut vectors form a basis for* $\mathrm{Cut}(G)$.
2. *For each edge $e \notin T$, there is a unique cycle in $T \cup e$. The signed characteristic vectors of all such cycles form a basis for* $\mathrm{Flow}(G)$.
3. *These are in fact $\mathbb{Z}$-module bases for the cut lattice $\mathcal{C}(G) = \mathrm{Cut}(G) \cap \mathbb{Z}^E$ and the flow lattice $\mathcal{F}(G) = \mathrm{Flow}(G) \cap \mathbb{Z}^E$.*

(General matroid theory predicts bases of the forms (1) and (2), but not the combinatorial interpretation of their coefficients.)

Cuts, Flows and The Critical Group

Theorem (Bacher, de la Harpe, Nagnibeda)

For every graph G, there are isomorphisms

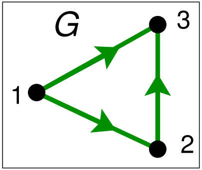
$$K(G) \;\cong\; \mathcal{F}^\sharp/\mathcal{F} \;\cong\; \mathcal{C}^\sharp/\mathcal{C} \;\cong\; \mathbb{Z}^E/(\mathcal{C} \oplus \mathcal{F}).$$

Here $\mathcal{L}^\sharp$ means the dual of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$:

$$\begin{aligned} \mathcal{L}^\sharp &= \{w \in \mathcal{L} \otimes \mathbb{R} \mid v \cdot w \in \mathbb{Z} \;\; \forall v \in \mathcal{L}\} \\ &= \operatorname{Hom}(\mathcal{L}, \mathbb{Z}) \qquad \text{(via standard dot product)} \end{aligned}$$

For instance, if $\mathbf{v} = (1, 1, \ldots, 1) \in \mathbb{Z}^n$ then $(\mathbb{Z}\mathbf{v})^\sharp = \frac{1}{n}\mathbb{Z}\mathbf{v}$.

Example: $G = K_3$



$$\partial = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \qquad L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}$$

Flow lattice

$$\mathcal{F} = \ker \partial = \langle (1, -1, 1) \rangle$$

$$\mathcal{F}^\sharp = \langle (\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}) \rangle$$

Cut lattice

$$\mathcal{C} = \operatorname{im} \partial^* = \langle (1, 0, -1),\ (0, 1, 1) \rangle$$

$$\mathcal{C}^\sharp = \langle (\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}),\ (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}) \rangle$$

Here, $\mathcal{F}^\sharp/\mathcal{F} = \mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}^3/(\mathcal{C} \oplus \mathcal{F}) = K(G) = \mathbb{Z}/3\mathbb{Z}$

Higher Dimension

Central Problem: What happens to the theory of cuts, flows, critical groups, sandpiles/chip-firing, . . . when we replace the graph G with something more general?

Topologically, a graph is a 1-dimensional simplicial (multi)complex — it consists of edges and vertices. Can we develop the theory for general combinatorial/topological spaces?

Cell Complexes

Cell complexes (= **CW complexes**) are higher-dimensional generalizations of graphs (like simplicial complexes, but even more general).

Examples: graphs, simplicial complexes, polytopes, polyhedral fans, ...

Rough definition: A cell complex X consists of cells (homeomorphic copies of $\mathbb{R}^k$ for various k) together with attaching maps

$$\partial_k(X) : C_k(X) \to C_{k-1}(X)$$

where $C_k(X)$ = free $\mathbb{Z}$-module generated by k-dimensional cells. (Note: $\partial_k \partial_{k+1} = 0$ for all k.) The integer $\partial_k(X)_{\rho,\sigma}$ specifies the multiplicity with which the k-cell σ is attached to the $(k-1)$-cell ρ.

— Attaching maps can be topologically complicated, but the only data we need is the cellular chain complex $\cdots \to C_k(x) \to C_{k-1}(X) \to \cdots$

Cellular Spanning Trees and Laplacians

Definition

A cellular spanning tree (CST) of X^d is a subcomplex $Y \subseteq X$ such that $Y \supseteq X_{(d-1)}$ and any of these two conditions hold:

- $\tilde{H}_d(Y, \mathbb{Q}) = 0$;
- $\tilde{H}_{d-1}(Y, \mathbb{Z})$ is finite;
- $|Y_d| = |X_d| - \tilde{\beta}_d(X) + \tilde{\beta}_{k-1}(X)$ (where $\beta_i(X) = \dim_{\mathbb{Q}} \tilde{H}_i(X, \mathbb{Q})$)

The "right" count of CSTs is

$$\tau(X) := \sum_{\text{CSTs } Y \subseteq X} |\tilde{H}_{d-1}(Y, \mathbb{Z})|^2$$

which can be obtained as a determinant of a reduced Laplacian [DKM '09,'11, Lyons '11, Catanzaro-Chernyak-Klein '12]

The Cellular Critical Group

Definition

The **critical group** of a d-dimensional cell complex X is

$$K(X) = \ker \partial_{d-1} / \operatorname{im} \partial_d \partial_d^*.$$

Fact: $K(X)$ is finite abelian of order $\tau(X)$, and can also be expressed in terms of the reduced Laplacian [DKM '13]

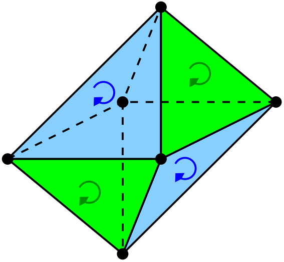
Questions:

- How can $K(X)$ be expressed in terms of cuts and flows?
- What are cellular cuts and flows in the first place?
- Is there a cellular chip-firing game for which elements of $K(X)$ correspond to critical states?

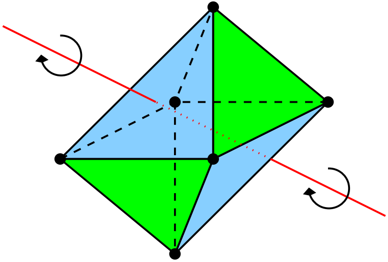
Cellular Cuts and Flows: Intuition

Example of flow vector: find a non-contractible d-sphere in X^d and orient all its cells consistently

Example of cut vector: poke a line through X^d and pick an orientation around the line



Flow



Cut

- If $d = 1$, these pictures reduce to the usual cuts and flows in graphs.

Cellular Cuts and Flows

Definition

Let X be a d-dimensional cell complex with n facets (max-dim cells).

$$\mathrm{Cut}(X) := \operatorname{im} \partial_d^*(X) \subseteq \mathbb{R}^n \qquad \mathcal{C}(X) := \mathrm{Cut}(X) \cap \mathbb{Z}^n$$
$$\mathrm{Flow}(X) := \ker \partial_d(X) \subseteq \mathbb{R}^n \qquad \mathcal{F}(X) := \mathrm{Flow}(X) \cap \mathbb{Z}^n$$

Theorem (DKM '13+)

Fix a cellular spanning tree $Y \subset X$.

1. *There are natural $\mathbb{R}$-bases of $\mathrm{Cut}(X)$ and $\mathrm{Flow}(X)$ indexed by the facets contained / not contained in Y.*
2. *The basis vector for each facet is supported on its fundamental cocircuit / circuit. Coeff'ts are sizes of certain homology groups.*
3. *Under certain conditions on $\tilde{H}_{d-1}(Y)$: $\mathbb{Z}$-bases for $\mathcal{C}(X)$, $\mathcal{F}(X)$.*

Cellular Cuts and Flows

Question

Do the Bacher-de la Harpe-Nagnibeda isomorphisms

$$K(X) \;\cong\; \mathcal{F}^\sharp/\mathcal{F} \;\cong\; \mathcal{C}^\sharp/\mathcal{C} \;\cong\; \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})$$

still hold if X is an arbitrary cell complex?

Answer: Not quite.

Cellular Cuts and Flows

The Bacher–de la Harpe–Nagnibeda isomorphisms do not hold in general.

The Bacher–de la Harpe–Nagnibeda isomorphisms do not hold in general.

Example: $X = \mathbb{R}P^2$: cell complex with one vertex, one edge, and one 2-cell, and cellular chain complex

$$C_2 = \mathbb{Z} \xrightarrow{\partial_2 = [2]} C_1 = \mathbb{Z} \xrightarrow{[\partial_1 = 0]} C_0 = \mathbb{Z}$$

- $\mathcal{C}/\mathcal{C}^\sharp \cong \mathbb{Z}/4\mathbb{Z}$ because $\mathcal{C} = \operatorname{im} \partial_2^* = 2\mathbb{Z}$ and so $\mathcal{C}^\sharp = \frac{1}{2}\mathbb{Z}$.
- $\mathcal{F}^\sharp/\mathcal{F} = 0$ because $\mathcal{F} = \ker \partial_2 = 0$.
- $\mathbb{Z}/(\mathcal{C} \oplus \mathcal{F}) \cong \mathbb{Z}/2\mathbb{Z}$.

- The culprit is probably torsion (note that $\tilde{H}_1(X) = \mathbb{Z}/2\mathbb{Z}$).
- In fact $K(G) \cong \mathbb{Z}/4\mathbb{Z}$. What is special about cuts?

The Critical Group via Cuts and Flows

Theorem (DKM '13+)

For any cell complex X, there are short exact sequences

$$0 \;\to\; \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \;\to\; K(X) \cong \mathcal{C}^\sharp/\mathcal{C} \;\to\; \mathbf{T}(\tilde{H}_{d-1}(X)) \;\to\; 0$$

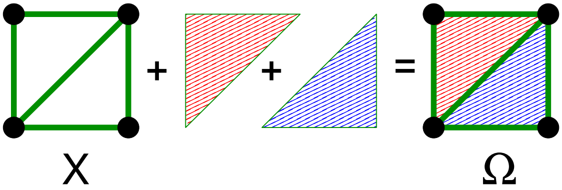
and

$$0 \;\to\; \mathbf{T}(\tilde{H}_{d-1}(X)) \;\to\; \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \;\to\; K^*(X) \cong \mathcal{F}^\sharp/\mathcal{F} \;\to\; 0.$$

- $\mathbf{T}(A)$ means the torsion summand of A (i.e., $\mathbf{T}(A)$ is finite and $A = \mathbf{T}(A) \oplus \mathbb{Z}^{\text{something}}$)
- "$0 \to A \to B \to C \to 0$ short exact" means "$C \cong B/A$"
- For graphs, these exact sequences reduce to the Bacher-de la Harpe-Nagnibeda isomorphisms (because torsion terms are trivial)

The Cocritical Group

To define the cocritical group $K^*(X)$, first construct an acyclization Ω of X by adjoining $(d+1)$-cells so as to eliminate all d-homology.



$$\begin{aligned} \text{Then,} \quad & K^*(X) = C_{d+1}(\Omega;\mathbb{Z})/\operatorname{im}\partial^*_{d+1}\partial_{d+1} && = \mathbf{T}(\operatorname{coker} L^{\mathrm{du}}_{d+1}(\Omega)). \\ \text{Compare} \quad & K(X) = \ker\partial_{d-1}/\operatorname{im}\partial_d\partial^*_d && = \mathbf{T}(\operatorname{coker} L^{\mathrm{ud}}_{d-1}(\Omega)) \\ & && = \mathbf{T}(\operatorname{coker} L^{\mathrm{ud}}_{d-1}(X)). \end{aligned}$$

Open Problems

- **Chip-firing/sandpiles for cell complexes?**

  (We have some ideas. Big problems: (a) torsion and (b) no "conservation of matter" for arbitrary cell complexes.)

- **Riemann-Roch theory in higher dimension?**

  (Baker–Norine: graph-theoretic Riemann-Roch theorem in which $K(G)$ stands in for the Picard group of a Riemann surface.)

- **Combinatorial commutative algebra connection?**

  (Sandpile configurations = monomials; toppling = reduction modulo binomial Gröbner basis, in analogy to Cori–Rossin–Salvi)

- **Cellular max-flow/min-cut theorem?**

Merci! Thanks!

Thank you for listening!
Merci de votre attention!