Matroids over a ring

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Introduction

A matroid captures the linear dependencies of a list of vectors.

The axioms can be given in several equivalent ways: for example, *independent sets*, or a *rank function*.

Many variants of matroids have arisen, each retaining some information about a vector configuration, which is *richer* than the purely linear algebraic information. For example "valuated matroids", and "arithmetic matroids": matroids decorated with extra data, defined by extra axioms.

We will try to unify some of these generalizations, by taking a new approach: a theory with only *one* simple, algebraic axiom.

When the ring R is a field we recover matroids, while when $R = \mathbb{Z}$ and $R = \mathbb{Z}_{(p)}$ we recover arithmetic matroids and valuated matroids respectively.

In general, a matroid over a ring R axiomatizes dependencies of elements in an R-module. So our theory generalizes matroid theory, in the same way as commutative algebra generalizes linear algebra.

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Definition

Let R be a commutative ring and E be a finite set.
A matroid over R on the *ground set* E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R-module $M(A)$ satisfying the following axiom:
For any $A \subseteq E$ and $b, c \in E \setminus A$,
there exist $x = x(b,c) \in M(A)$ and $y = y(b,c) \in M(A)$
such that there is a diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{/x} & M(A \cup \{b\}) \\ \downarrow{\scriptstyle /y} & \lrcorner & \downarrow{\scriptstyle /\overline{y}} \\ M(A \cup \{c\}) & \xrightarrow[/\overline{x}]{} & M(A \cup \{b, c\}). \end{array}$$

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Realizability

Fundamental example: "vector configurations" in an R-module.

Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, we have a matroid M_X associating to $A \subseteq X$ the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right).$$

For each $x_i \in X$ there is a quotient map

$$M_X(A) \xrightarrow{/\overline{x_i}} M_X(A \cup \{x_i\})$$

and this system of maps obviously satisfies the axiom.

We say that a matroid M over R is realizable if it actually comes from such a list.

Of course not all matroids over R are realizable!

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Of course not all matroids over R are realizable!

Classical matroids

We can, and will, assume that the module $M(E)$ has no nontrivial projective summands. This technical assumption makes many results simpler to state.

Recall that in a classical matroid the corank $\mathrm{cork}(A)$ of a set A is equal to $\mathrm{rk}(E) - \mathrm{rk}(A)$, where $\mathrm{rk}(E)$ is the rank of the matroid.

Proposition (Fink, M.)

Let $\mathbb{K}$ be a field. Matroids over $\mathbb{K}$ are equivalent to (classical) matroids. If M is a matroid over $\mathbb{K}$, then $\dim M(A)$ is the corank of A in the corresponding classical matroid.

Furthermore, a matroid over $\mathbb{K}$ is realizable if and only if, as a classical matroid, it is realizable over $\mathbb{K}$.

Idea of the proof: we can replace $M(A)$ by its $\mathbb{K}$-dimension without losing information.

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Sum, deletion, contraction, duality

Let M and M' be matroids over R on E and E'.
We define their direct sum $M \oplus M'$ on $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

For $i \in E$, we define two matroids over R on the ground set $E \setminus \{i\}$:
the deletion of i in M, denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the contraction of i in M, denoted $M \setminus i$, by

$$(M/i)(A) = M(A \cup \{i\}).$$

When R is a Dedekind domain, we can also define a dual matroid M^* having the expected properties (omitted).

If M is realizable, $M \setminus i$ and M/i can be realized in the usual way, while M^* can be realized by a generalization of Gale duality.

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Tensor product. Localizations and generic matroid

Let $R \to S$ be a map of rings. Then the tensor product $- \otimes_R S$ is a functor $R\text{-}\mathrm{Mod} \to S\text{-}\mathrm{Mod}$. If M is a matroid over R, then

$$(M \otimes_R S)(A) \doteq M(A) \otimes_R S.$$

defines a matroid over S.

Two special cases will be fundamental for us:

1. For every prime ideal $\mathfrak{m}$ of R, let $R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}$. We call $M \otimes_R R_{\mathfrak{m}}$ the localization of M at $\mathfrak{m}$.
2. If R is a domain, let $\mathrm{Frac}(R)$ be the fraction field of R. Then we call $M \otimes_R \mathrm{Frac}(R)$ the generic matroid of M.

Notice that every matroid over $R_{\mathfrak{m}}$ induces a matroid over the residue field $R_{\mathfrak{m}}/(\mathfrak{m})$.

We can study the matroid M via all these "classical" matroids.

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Dedekind rings and DVR

From now on, we will always assume R to be a Dedekind domain (i.e., an integral domain in which every nonzero proper ideal factors into a product of prime ideals).

The localization of a Dedekind domain at a prime ideal is a DVR (i.e. a Dedekind domain that is not a field and has a unique maximal ideal $\mathfrak{m}$).

Any indecomposible f.g. module over a DVR R is isomorphic to either R or $R/\mathfrak{m}^n$ for some integer $n \geq 1$.

So a f.g. R-module are parametrized by "partitions" that may have some infinitely long lines.

We denote by t_n the cardinality of the n-truncation of such a "partition".

Our first result is a combinatorial characterization of matroids over a DVR:

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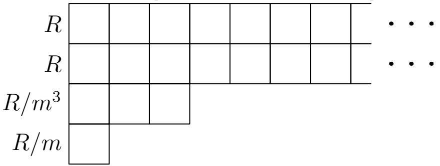

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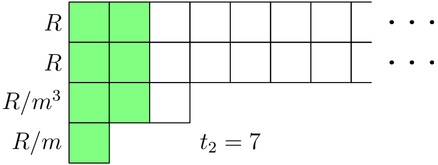

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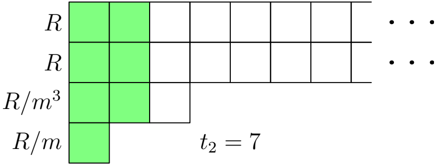

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Local theory: matroids over a DVR

Theorem (Fink, M.)

$M : 2^E \to \{$f. g. R-modules$\}$ is a matroid over R if and only if:
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The vector $\left(t_n(M(A)), |A| = k\right)$ defines a point on the Dressian $Dr(k, |E|)$*

In fact, we conjecture that in this way we get a point on the Dressian analogue of the full flag variety*. (* tropical varieties parametrizing tropical linear spaces, and full flags of t.l.s., respectively).

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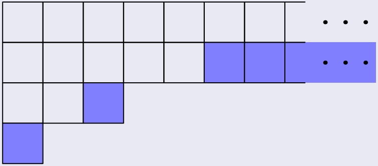

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As a consequence of the Proposition above, we get:

Corollary (Fink, M.)

Let M be a matroid over a DVR $(R, \mathfrak{m})$.
Then the function $\mathcal{V}(A) \doteq \dim_{R/\mathfrak{m}} M(A)$ makes the generic matroid of M into a valuated matroid.

A valuated matroid is defined as a matroid decorated with an integer valued function $\mathcal{V}$ on the set of the bases $\mathcal{B}$, satisfying a certain axiom [Dress and Wenzel].
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Global theory: matroids over a Dedekind domain

Let R be a Dedekind domain. Given an R-module N, let $N_{\mathrm{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors}.

There is a function det associating to every R-module an element of $\mathrm{Pic}(R)$, the Picard group of R.

By this function characterize matroids over a Dedekind domain R:

Theorem (Fink, M.)

$M : 2^E \to \{$f. g. R-modules$\}$ is a matroid over R if and only if every localization at a prime ideal $\mathfrak{m}$ is a matroid over $R_{\mathfrak{m}}$, and for every 1-element minor $N \to N'$ we have:

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Arithmetic matroids

If M is a matroid over $\mathbb{Z}$, we define the two functions

$$\mathrm{cork}(A) = \mathrm{rk}(M(A)_{\mathrm{proj}}) \text{ and } m(A) \doteq |M(A)_{\mathrm{tors}}|.$$

As a corollary of the previous theorem, we can prove that (E, cork, m) is a "quasi-arithmetic matroid", a structure closely related to the arithmetic matroids introduced by [D'Adderio-M].
(Arithmetic matroids also satisfy a further axiom (P), granting the positivity of the *arithmetic Tutte polynomial* [Brändén-M.]).

Notice that matroids over $\mathbb{Z}$ and quasi-arithmetic matroids and are *not* truly equivalent, since the information contained in the former is richer, since there are many groups with the same cardinality.

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Definition of the Tutte-Grothendieck group

Several invariants can be associated to a classical matroid; the universal deletion-contraction invariant is the well-known *Tutte polynomial*.
We will now define and compute the universal deletion-contraction invariant of matroids over any Dedekind domain R.

Essentially following Brylawski, define the Tutte-Grothendieck ring of matroids over R, $K(R\text{-Mat})$, to be the abelian group generated by a symbol $\mathbf{T}_M$ for each matroid M over R, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M \setminus a} + \mathbf{T}_{M/a}$$

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Theorem (Fink, M.)

The Tutte-Grothendieck ring $K(R\text{-Mat})$ is the subring of $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ generated by X^P and Y^P as P ranges over rank 1 projective modules, and $X^N Y^N$ as N ranges over torsion modules. The class of M is

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Classical Tutte polynomial and arithmetic Tutte polynomial

When R is a field, $\mathrm{Pic}(R)$ is trivial and there is no torsion, thus $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}] \simeq \mathbb{Z}[X, Y]$.

Then by the substitution $X = x - 1$ and $Y = y - 1$ we can see that $\mathbf{T}_M = \sum_{A \subseteq E} X^{M(A)} Y^{M^*(E \setminus A)}$ is simply the classical Tutte polynomial, since $\dim M(A)$ is the corank of A and $\dim M^*(E \setminus A)$ is its nullity.

When $R = \mathbb{Z}$, since there are nontrivial torsion modules, we get

$$\mathbf{T}_M = \sum_{A \subseteq E} X^{M(A)} Y^{M^*(E \setminus A)} X^{M(A)_{\mathrm{tors}}} Y^{M(A)_{\mathrm{tors}}}.$$

By evaluating $X^N Y^N$ to the cardinality of N for each torsion module N, we get the arithmetic Tutte polynomial. This polynomial proved to have several applications to toric arrangements, partition functions, Ehrhart polynomial of zonotopes, graphs, CW-complexes, . . .

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The Tutte quasi-polynomial

Another invariant that we can obtain from the Grothendieck-Tutte invariant $\mathbf{T}_M$ in the case $R = \mathbb{Z}$ is the Tutte quasi-polynomial

$$\mathbf{Q}_M(x, y) = \sum_{A \subseteq E} \frac{|M(A)_{\mathrm{tors}}|}{|q \cdot M(A)_{\mathrm{tors}}|} (x-1)^{\mathrm{rk}(E) - \mathrm{rk}(A)} (y-1)^{|A| - \mathrm{rk}(A)}.$$

where $q = (x-1)(y-1)$.

Since $(q + |G|)G = qG$ holds for any finite group G, $\mathbf{Q}_M(x, y)$ is a quasi-polynomial in q, interpolating between the classical and the arithmetic Tutte polynomials.

This polynomial was introduced in [Brändén- M.], and has application to generalized colorings and flows on graphs with labeled edges.

Notice that $\mathbf{Q}_M(x, y)$ is not an invariant of the arithmetic matroid, (as it depends on the groups $M(A)_{\mathrm{tors}}$ and not just on their cardinalities), but it is an invariant of the matroid over $\mathbb{Z}$.

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This polynomial was introduced in [Brändén- M.], and has application to generalized colorings and flows on graphs with labeled edges.
Notice that $\mathbf{Q}_M(x,y)$ is not an invariant of the arithmetic matroid, (as it depends on the groups $M(A)_{\mathrm{tors}}$ and not just on their cardinalities), but it is an invariant of the matroid over $\mathbb{Z}$.

Developments and applications

Future developments:

- study other examples, such as R coordinate ring of an algebraic curve (e.g. the affine line or an elliptic curve);
- provide cryptomorphic definitions (e.g. the base polytope; a semi-standard Young tableaux description);

Possible applications:

- combinatorial topology: [Bajo-Burdick-Chmutov], [Duval-Klivans-Martin], [Hughes-Swartz], [Cavazzani- M.];
- tropical geometry;
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