Fully Commutative Elements and Lattice Walks

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Joint work with Riccardo Biagioli and Frédéric Jouhet

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Fully commutative elements

- \((W, S)\) Coxeter group \(W\) given by Coxeter matrix \((m_{st})_{s,t \in S}\).

Relations:
\[
\begin{align*}
    s^2 &= 1 \\
    s t s \cdots &= t s t \cdots
\end{align*}
\]

\(\underbrace{m_{st}}_{m_{st}} \quad \underbrace{m_{st}}_{m_{st}}\) Braid relations
Fully commutative elements

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\]

- **Length** \(\ell(w)\) = minimal \(l\) such that \(w = s_1 s_2 \ldots s_l\).

The minimal words are the reduced decompositions of \(w\).
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The minimal words are the reduced decompositions of \(w\).

**Fundamental property** : Given any two reduced decompositions of \(w\), there is a sequence of braid relations which can be applied to transform one into the other.
An element $w$ is **fully commutative** if given two reduced decompositions of $w$, there is a sequence of **commutation relations** which can be applied to transform one into the other.

**Commutation class**: equivalence class of words under the commutation relations $st \equiv ts$ when $m_{st} = 2$.

So $w$ is fully commutative if its reduced decompositions form only one commutation class.
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So \( w \) is fully commutative if its reduced decompositions form only one commutation class.

**Proposition** [Stembridge ’96] A commutation class of reduced words corresponds to a FC element if and only no element in it contains a factor \( \underbrace{sts \cdots}_{m_{st}} \) for a \( m_{st} \geq 3 \).
Previous work

• The seminal papers are [Stembridge ’96,’98]:
  1. First properties;
  2. Classification of $W$ with a finite number of FC elements;
  3. Enumeration of these elements in each of these cases.
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- The seminal papers are [Stembridge ’96,’98]:
  1. First properties;
  2. Classification of $W$ with a finite number of FC elements;
  3. Enumeration of these elements in each of these cases.

- [Fan ’95] studies FC elements in the special case where $m_{st} \leq 3$ (the simply laced case).

- [Graham ’95] shows that FC elements in any Coxeter group $W$ naturally index a basis of the (generalized) Temperley-Lieb algebra of $W$.

- Subsequent works [Green,Shi,Cellini,Papi] relate FC elements (and some related elements) to Kazhdan-Lusztig cells.

- [Hanusa-Jones ’09] enumerates FC elements for the affine type $\widetilde{A}_n$ with respect to length.
Results

We consider FC elements in all affine Coxeter groups $W$, and study their enumeration with respect to length:

$$W^{FC}(q) := \sum_{w \text{ is FC}} q^{\ell(w)} = \sum_{\ell \geq 0} W_{\ell}^{FC} q^\ell$$
Results

We consider FC elements in all **affine Coxeter groups** $W$, and study their enumeration with respect to length:

\[
W^{FC}(q) := \sum_{w \text{ is FC}} q^{\ell(w)} = \sum_{\ell \geq 0} W^{FC}_\ell q^\ell
\]

**Main Results** [Biagioli-Jouhet-N. '12]

(i) **Characterization** of FC elements for any affine $W$;
(ii) **Computation of** $W^{FC}(q)$;
(iii) If $W$ irreducible, $(W^{FC}_\ell)_{\ell \geq 0}$ is **ultimately periodic**.

<table>
<thead>
<tr>
<th>AFFINE TYPE</th>
<th>$\tilde{A}_{n-1}$</th>
<th>$\tilde{C}_n$</th>
<th>$\tilde{B}_{n+1}$</th>
<th>$\tilde{D}_{n+2}$</th>
<th>$\tilde{E}_6$</th>
<th>$\tilde{E}_7$</th>
<th>$\tilde{G}_2$</th>
<th>$\tilde{F}_4, \tilde{E}_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PERIODICITY</td>
<td>$n$</td>
<td>$n + 1$</td>
<td>$(n + 1)(2n + 1)$</td>
<td>$n + 1$</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof is case by case: I will focus on type $\tilde{A}$ today.
1. **FC elements and Heaps**
Heaps

Given \((W, S)\), consider the Coxeter graph \(\Gamma\) with vertices \(S\) and edges \(\{s, t\}\) iff \(m_{s,t} \geq 3\).

No edge between \(s\) and \(t\) \(\iff\) \(s\) and \(t\) commute.
Heaps

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\[
\begin{array}{c}
s_0 & s_1 & s_2 \\
4 & s_2 & 5 \\
& s_3 \\
\end{array}
\]

No edge between \(s\) and \(t\) \(\iff\) \(s\) and \(t\) commute.

**Definition:** A \(\Gamma\)-heap \((H, \leq, \epsilon)\) is a poset \((H, \leq)\) together with a labeling function \(\epsilon : H \rightarrow S\) such that:
1. For each edge \(\{s, t\} \in \Gamma\), the poset \(H|\{s,t\}\) is a chain.
2. The poset \((H, \leq)\) is the transitive closure of these chains.
Heaps $=$ Commutation classes

**Theorem** [Viennot ’86] Bijection between:

(i) Commutation classes in $W$.

(ii) $\Gamma$-heaps.
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$\circ s_1 s_0 s_3 s_2 s_0 s_3 s_1 s_2 s_1$

$\bullet$

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$S_1S_0S_3S_2S_0S_3S_1S_2S_1$

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(a) No covering relation

(b) No **convex** chain of the form
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![Diagram](image)

**Summary**

<table>
<thead>
<tr>
<th>FC element $w$</th>
<th>Heap $H$ satisfying (a) and (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length $\ell(w)$</td>
<td>Number of elements $</td>
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</table>
1. FC elements in type $\tilde{A}$
Affine permutations

\[ \tilde{A}_{n-1} \]

\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \]

\[ s_i s_j = s_j s_i, \quad |j - i| > 1 \]
Affine permutations

\[ \tilde{A}_{n-1} \]

\[ s_0 \]

\[ s_1 \quad s_2 \quad \ldots \quad s_{n-1} \]

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Representation as the group of permutations \( \sigma \) of \( \mathbb{Z} \) such that:

(i) \( \forall i \in \mathbb{Z} \) \( \sigma(i + n) = \sigma(i) + n \), and

(ii) \( \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i \).

\[ \ldots, 13, -12, -14, -1, 17, -8, -10, 3, 21, -4, -6, 7, 25, 0, -2, 11, 29, 4, \ldots \]

\[ \sigma(1) \sigma(2) \sigma(3) \sigma(4) \]
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**Theorem** [Green '01] Fully commutative elements of type \( \tilde{A}_{n-1} \) correspond to 321-avoiding permutations.

This generalizes [Billey, Jockush, Stanley '93] for type \( A_{n-1} \), i.e. the symmetric group \( S_n \).
Periodicity

**Theorem** [Hanusa-Jones ’09] The sequence \( \tilde{A}_{n-1,l}^{FC} \) is ultimately periodic of period \( n \).

\[
\begin{align*}
\tilde{A}_2^{FC}(q) & = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + \cdots \\
\tilde{A}_3^{FC}(q) & = 1 + 4q + 10q^2 + 16q^3 + 18q^4 + 16q^5 + 18q^6 + \cdots \\
\tilde{A}_4^{FC}(q) & = 1 + 5q + 15q^2 + 30q^3 + 45q^4 \\
& + 50q^5 + 50q^6 + 50q^7 + 50q^8 + 50q^9 + \cdots \\
\tilde{A}_5^{FC}(q) & = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 \\
& + 150q^7 + 156q^8 + 152q^9 + 156q^{10} + 150q^{11} + 158q^{12} \\
& + 150q^{13} + 156q^{14} + 152q^{15} + 156q^{16} + 150q^{17} + 158q^{18} \\
& + \cdots
\end{align*}
\]

Proof uses affine permutations.
Periodicity

**Theorem [Hanusa-Jones ’09]** The sequence $(\tilde{A}_{n-1,l}^{FC})_{l \geq 0}$ is ultimately periodic of period $n$.

- The authors also:
  - Show that periodicity starts no later than $l = 2\lceil n/2 \rceil \lfloor n/2 \rfloor$;
  - Compute all series $\tilde{A}_{n-1}^{FC}(q)$. 
Periodicity

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- The authors also:
  - Show that periodicity starts no later than $l = 2\lceil n/2 \rceil \lfloor n/2 \rfloor$;
  - Compute all series $\tilde{A}_{n-1}^{FC}(q)$.

- We revisit the same problem using FC heaps.
  - Proof that periodicity starts precisely at $l = 1 + \lceil (n - 1)/2 \rceil \lfloor (n + 1)/2 \rfloor$ (conjectured by [H-J]);
  - In the process, we will get simpler rules to compute the generating functions $\tilde{A}_{n-1}^{FC}(q)$. 
FC heaps in type $\tilde{A}$

$\rightarrow$ FC heaps must avoid

- $s_i$
- $s_i s_{i+1} s_{i+2}$
- $s_i s_{i+1} s_{i+2}$
FC heaps in type $\tilde{A}$

\[ S_0 \]
\[ S_1 \quad S_2 \quad S_{n-1} \]

→ FC heaps must avoid

\[ \begin{align*}
  & s_i \\
  & s_i \quad s_{i+1} \quad s_{i+2} \\
  & s_i \quad s_{i+1} \quad s_{i+2}
\end{align*} \]

**Proposition** FC heaps are characterized by:
For all $i$, $H_{\{s_i, s_{i+1}\}}$ is a chain with alternating labels

FC Heap

\[ S_0 \quad S_1 \quad S_2 \]

\[ S_0 \]
No labels needed at height 0.
Bijection

Let $O_n^*$ be the set of length $n$ positive paths with starting and ending point at the same height. Horizontal steps at height $h > 0$ are labeled $L$ or $R$. 
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**Theorem** [BJN '12] This is a bijection between
1. FC elements of $\tilde{A}_{n-1}$ and
2. $O_n^*$
Bijection

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**Theorem**[BJN ’12] This is a bijection between
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The non-trivial part of the proof is to show surjectivity.
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The non-trivial part of the proof is to show surjectivity.

- Remark that the **length** of the word is sent to the **area** under the path.

**Corollary** $\tilde{A}^{FC}_{n-1}(q) = O^*_n(q) - \frac{2q^n}{1 - q^n}$
Enumerative results

• For $l$ large enough, the sequence $(O_{n,l}^*)_l$ becomes periodic with period $n$ (proof: just shift the paths up by 1 unit).
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“Large enough”? Shifting is not bijective if there exists a path of area $l$ with a horizontal step at height $h = 0$

$\rightarrow l \leq l_0 = \lceil (n - 1)/2 \rceil \lfloor (n + 1)/2 \rfloor$.

$\rightarrow$ Periodicity starts exactly at $l_0 + 1$
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- Finally, $\tilde{A}_{n-1}^{FC}(q) = \frac{q^n(X_n(q) - 2)}{1 - q^n} + X_n^\ast(q)$
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• Finally, $\tilde{A}_{n-1}^{FC}(q) = \frac{q^n(X_n(q) - 2)}{1 - q^n} + X_n^*(q)$

$$\sum_{n \geq 0} X_n(q)x^n = Y(x) \left(1 + qx^2 \frac{\partial(xY)}{\partial x}(xq)\right)$$

$$\sum_{n \geq 0} X_n^*(q)x^n = Y^*(x) \left(1 + qx^2 \frac{\partial(xY)}{\partial x}(xq)\right)$$

$Y^*(x) = 1 + xY^*(x) + qx(Y^*(x) - 1)Y^*(qx)$

$Y(x) = \frac{Y^*(x)}{1 - xY^*(x)}$
3. Other types
Other affine types

\[ \tilde{C}_n \]

\[ \tilde{D}_{n+2} \]

\[ \tilde{B}_{n+1} \]

\[ \tilde{E}_6 \]

\[ \tilde{E}_7 \]

\[ \tilde{G}_2 \]
Theorem [BJN '12] For each irreducible affine group $W$, the sequence $(W_l^{FC})_{l \geq 0}$ is ultimately periodic, with period recorded in the following table.

<table>
<thead>
<tr>
<th>Affine Type</th>
<th>$\tilde{A}_{n-1}$</th>
<th>$\tilde{C}_n$</th>
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<th>$\tilde{D}_{n+2}$</th>
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<tbody>
<tr>
<td>Periodicity</td>
<td>$n$</td>
<td>$n + 1$</td>
<td>$(n + 1)(2n + 1)$</td>
<td>$n + 1$</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Type $\tilde{C}$

We obtain here also certain heaps corresponding to paths, but there are in addition infinitely many exceptional FC heaps.

$$\tilde{C}_4^{FC}(q) = 1 + 5q + 14q^2 + 29q^3 + 47q^4 + 64q^5 + 76q^6 + 81q^7 + 80q^8 + 75q^9 + 68q^{10} + 63q^{11} + 61q^{12} + 59q^{13} + 59q^{14} + 60q^{15} + 59q^{16} + 59q^{17} + 59q^{18} + 59q^{19} + 60q^{20} + 59q^{21} + 59q^{22} + 59q^{23} + 59q^{24} + 60q^{25} + 59q^{26} + 59q^{27} + \cdots$$
Type $\widetilde{C}$

Two families of paths survive for large enough length:

1. Finite factors of

2. Path
Type $\tilde{B}$

$\tilde{B}_3^{FC}(q) = 1 + 4q + 9q^2 + 15q^3 + 19q^4 + 21q^5 + 21q^6 + 18q^7 + 17q^8 + 19q^9 + 18q^{10} + 17q^{11} + 19q^{12} + 17q^{13} + 17q^{14} + 20q^{15} + 17q^{16} + 17q^{17} + 19q^{18} + 17q^{19} + 18q^{20} + 19q^{21} + 17q^{22} + 17q^{23} + 19q^{24} + 18q^{25} + 17q^{26} + 19q^{27} + 17q^{28} + 17q^{29} + 20q^{30} + 17q^{31} + 17q^{32} + 19q^{33} + 17q^{34} + 18q^{35} + 19q^{36} + 17q^{37} + 17q^{38} + 19q^{39} + 18q^{40} + 17q^{41} + 19q^{42} + 17q^{43} + 17q^{44} + 20q^{45} + 17q^{46} + 17q^{47} + 19q^{48} + 17q^{49} + 18q^{50} + 19q^{51} + 17q^{52} + 17q^{53} + 19q^{54} + 18q^{55} + 17q^{56} + 19q^{57} + 17q^{58} + 17q^{59} + 20q^{60} + 17q^{61} + 17q^{62} + 19q^{63} + 17q^{64} + 18q^{65} + 19q^{66} + 17q^{67} + 17q^{68} + 19q^{69} + 18q^{70} + 17q^{71} + 19q^{72} + 17q^{73} + 17q^{74} + 20q^{75} + 17q^{76} + \cdots$

Period 15 corresponding to $(n + 1)(2n + 1)$ for $n = 2$. 
Exceptional types

\[\tilde{E}_6\quad \tilde{E}_7\quad \tilde{G}_2\]
Related Work

• Enumeration of finite Coxeter groups wrt to length.

• FC involutions correspond to “self-dual FC heaps”. Our methods can be easily applied, and similar results hold (periodicity, generating functions)
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**Theorem** [Jouhet, N. ’13]
For all affine groups $W$, we can determine the **minimal period**.
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- FC involutions correspond to “self-dual FC heaps”. Our methods can be easily applied, and similar results hold (periodicity, generating functions)

**Theorem** [Jouhet, N. ’13]
For all affine groups $W$, we can determine the **minimal period**.

**Theorem in progress** [N. ’13]
(i) For any Coxeter system $(W, S)$, the series $W^{FC}(q)$ is a **rational function**.
(ii) The sequence $(W_l^{FC})_{l \geq 0}$ is ultimately periodic if and only if $W$ is affine, $FC$-finite or is one of two exceptions.
Further questions

• Other statistics to consider, e.g. descent numbers.
• Formulas for our generating functions? (and not just functional equations/recurrences).
• Type-free proofs and formulas?
• Applications to Temperley-Lieb algebras?
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- Type-free proofs and formulas?
- Applications to Temperley-Lieb algebras?

THANK YOU
Type $\tilde{C}_2$
Type $\tilde{\mathcal{C}}$

Other families