Fully Commutative Elements and Lattice Walks

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Joint work with Riccardo Biagioli and Frédéric Jouhet

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Fully commutative elements

- (W, S) Coxeter group W given by Coxeter matrix $(m_{st})_{s,t \in S}$.

Relations: $\begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \end{cases}$ → Braid relations

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The minimal words are the reduced decompositions of w.

Fundamental property : Given any two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other.

Fully commutative elements

An element w is **fully commutative** if given two reduced decompositions of w, there is a sequence of commutation relations which can be applied to transform one into the other.

Commutation class: equivalence class of words under the commutation relations $st \equiv ts$ when $m_{st} = 2$.

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Proposition [Stembridge '96] A commutation class of reduced words corresponds to a FC element if and only no element in it contains a factor $\underbrace{sts\cdots}_{m_{st}}$ for a $m_{st} \geq 3$.

Previous work

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- [Fan '95] studies FC elements in the special case where $m_{st} \leq 3$ (*the simply laced case*).
- [Graham '95] shows that FC elements in any Coxeter group W naturally index a basis of the (generalized) Temperley-Lieb algebra of W.
- Subsequent works [Green,Shi,Cellini,Papi] relate FC elements (and some related elements) to Kazhdan-Lusztig cells.

- [Hanusa-Jones '09] enumerates FC elements for the affine type $\widetilde{A}_n$ with respect to length.

Results

We consider FC elements in all **affine Coxeter groups** W, and study their enumeration with respect to length:

$$W^{FC}(q) := \sum_{w \text{ is FC}} q^{\ell(w)} = \sum_{\ell \geq 0} W_\ell^{FC} q^\ell$$

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Main Results [Biagioli-Jouhet-N. '12]
(i) **Characterization** of FC elements for any affine W;
(ii) **Computation of** $W^{FC}(q)$;
(iii) If W irreducible, $(W_\ell^{FC})_{\ell \geq 0}$ is **ultimately periodic.**

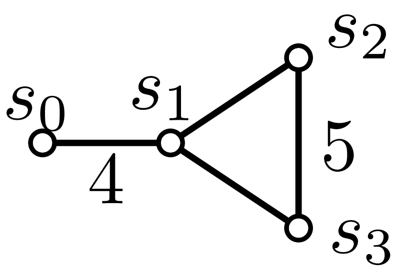
Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

Proof is case by case: I will focus on type $\widetilde{A}$ today.

1. FC elements and Heaps

Heaps

Given (W, S), consider the Coxeter graph Γ with vertices S and edges $\{s, t\}$ iff $m_{s,t} \geq 3$.



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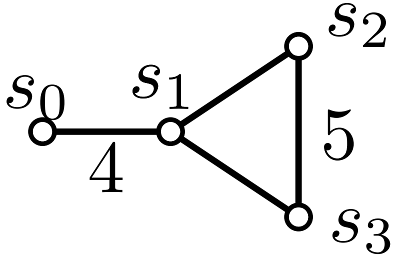

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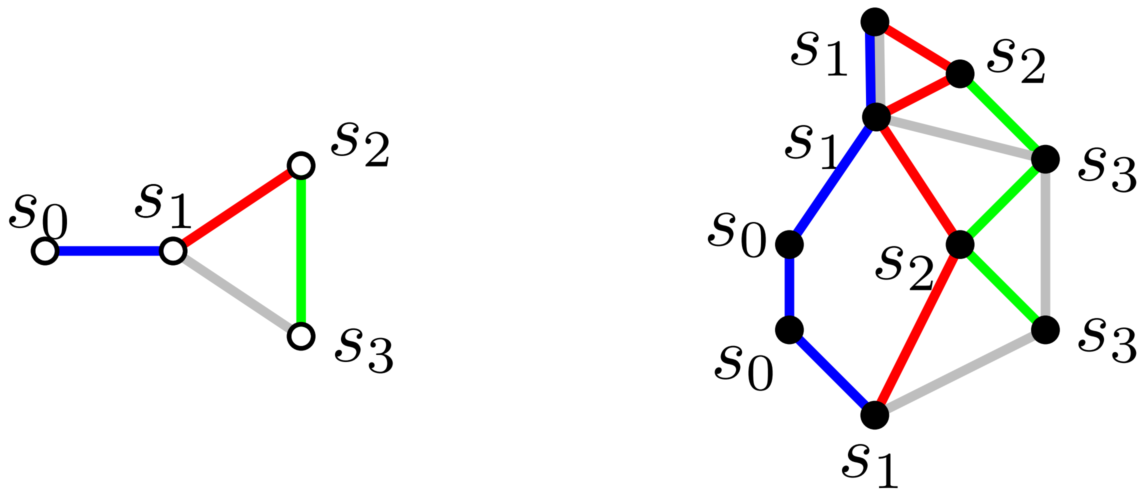
Definition: A Γ-**heap** $(H, \leq, \epsilon)$ is a poset $(H, \leq)$ together with a labeling function $\epsilon : H \to S$ such that:

1. For each edge $\{s, t\} \in \Gamma$, the poset $H_{|\{s,t\}}$ is a chain.
2. The poset $(H, \leq)$ is the transitive closure of these chains.



Heaps = Commutation classes

Theorem [Viennot '86] Bijection between:
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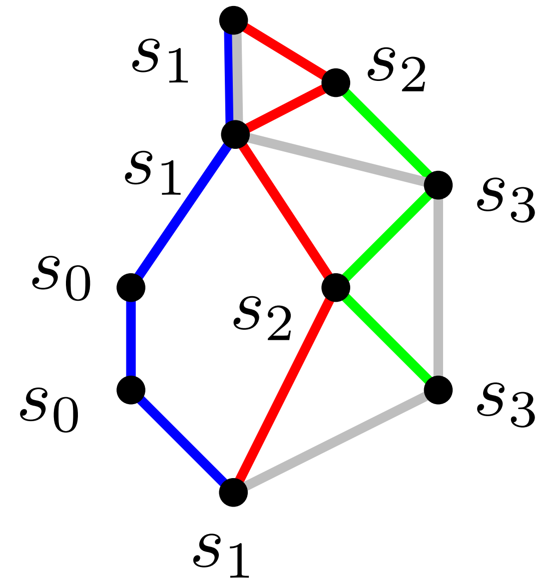

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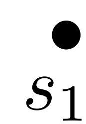

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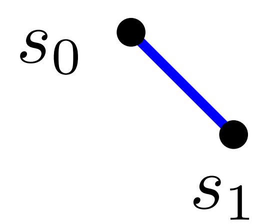

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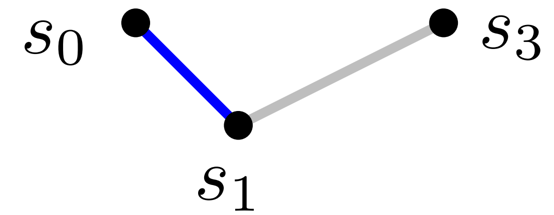

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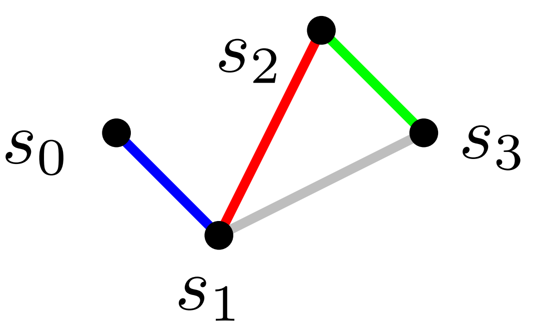

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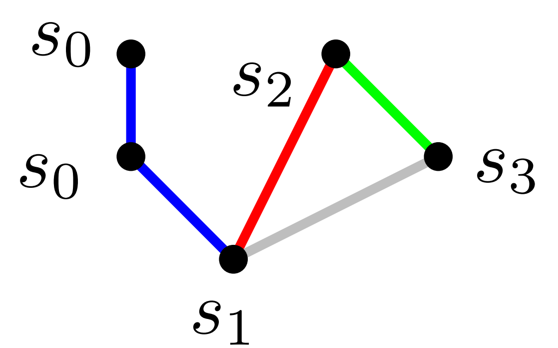

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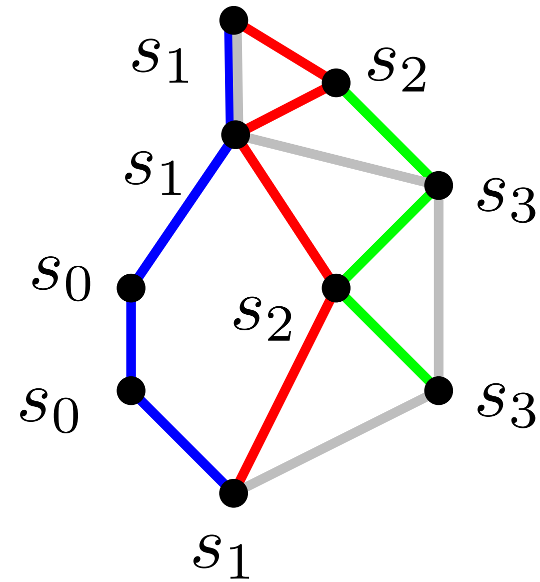

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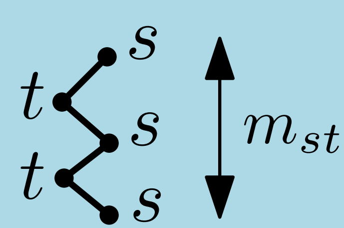
Proposition [Stembridge '95] FC heaps are characterized by the following two restrictions:

(a) No covering relation

(b) No convex chain of the form



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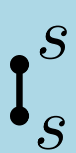

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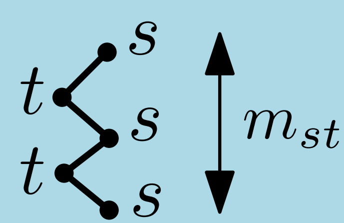
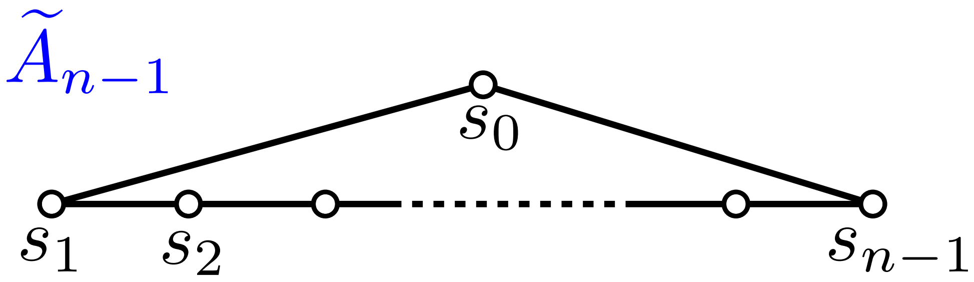


Summary

FC element w	$\longleftrightarrow$	Heap H satisfying (a) and (b)
Length $\ell(w)$	$\longleftrightarrow$	Number of elements $\lvert H\rvert$

1. FC elements in type $\widetilde{A}$

Affine permutations



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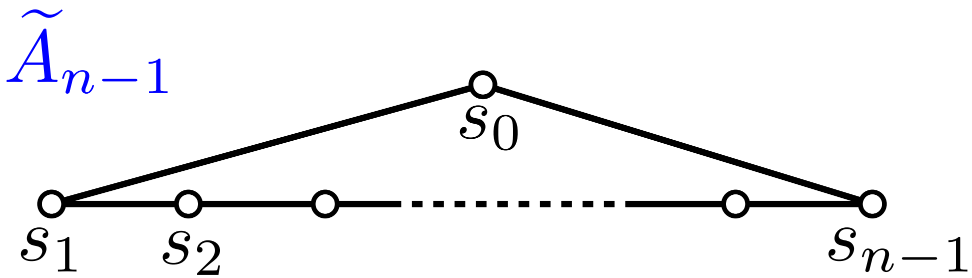

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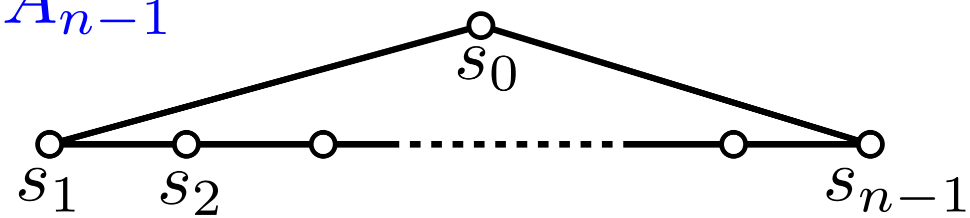
(i) $\forall i \in \mathbb{Z}\ \sigma(i+n) = \sigma(i) + n$, and

(ii) $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$.

$$\ldots, 13, -12, \,|\, -14, -1, 17, -8, \,|\, \underset{\sigma(1)}{-\mathbf{10}}, \underset{\sigma(2)}{\mathbf{3}}, \underset{\sigma(3)}{\mathbf{21}}, \underset{\sigma(4)}{-\mathbf{4}}, \,|\, -6, 7, 25, 0, \,|\, -2, 11, 29, 4, \ldots$$

Affine permutations

$\widetilde{A}_{n-1}$



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Theorem [Green '01] Fully commutative elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding permutations.

This generalizes [Billey,Jockush,Stanley '93] for type A_{n-1}, i.e. the symmetric group S_n.

Periodicity

Theorem [Hanusa-Jones '09] The sequence $(\widetilde{A}^{FC}_{n-1,l})_{l\geq 0}$ is ultimately periodic of period n.

$$\widetilde{A}_2^{FC}(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$\widetilde{A}_3^{FC}(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

$$\widetilde{A}_4^{FC}(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots$$

$$\widetilde{A}_5^{FC}(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} + \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \mathbf{156q^{16}} + \mathbf{150q^{17}} + \mathbf{158q^{18}} + \cdots$$

Proof uses affine permutations.

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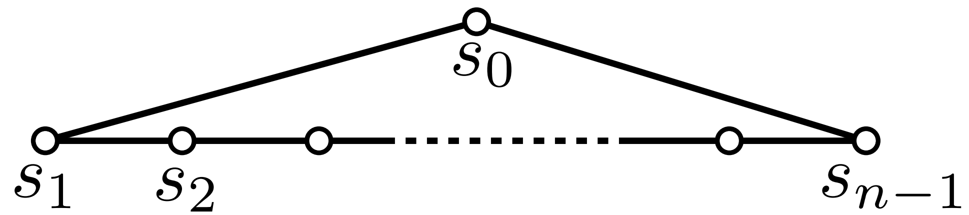
• We revisit the same problem using FC heaps.

- Proof that periodicity starts precisely at $l = 1 + \lceil (n-1)/2\rceil\lfloor (n+1)/2\rfloor$ (conjectured by [H-J]);

- In the process, we will get simpler rules to compute the generating functions $\widetilde{A}^{FC}_{n-1}(q)$.

FC heaps in type $\widetilde{A}$



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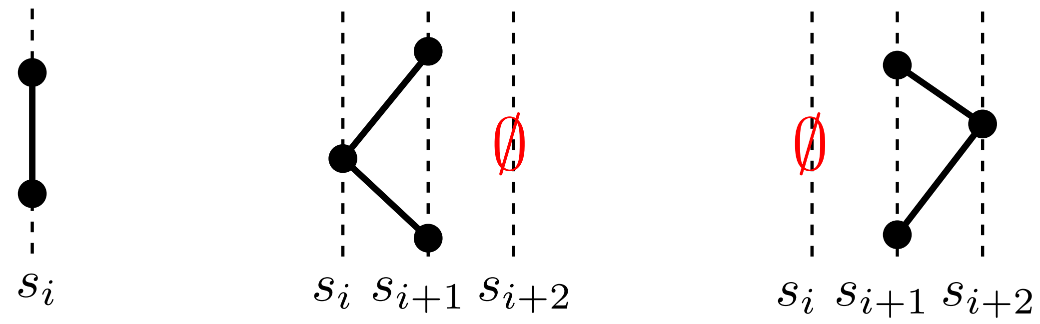

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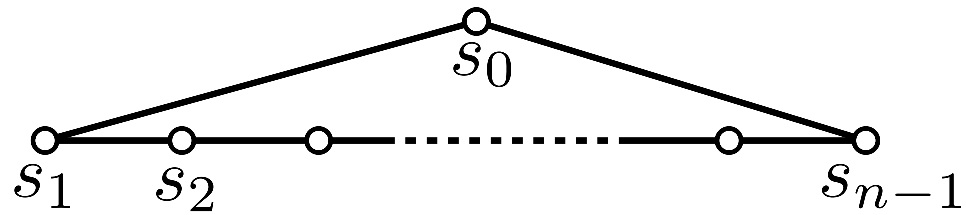

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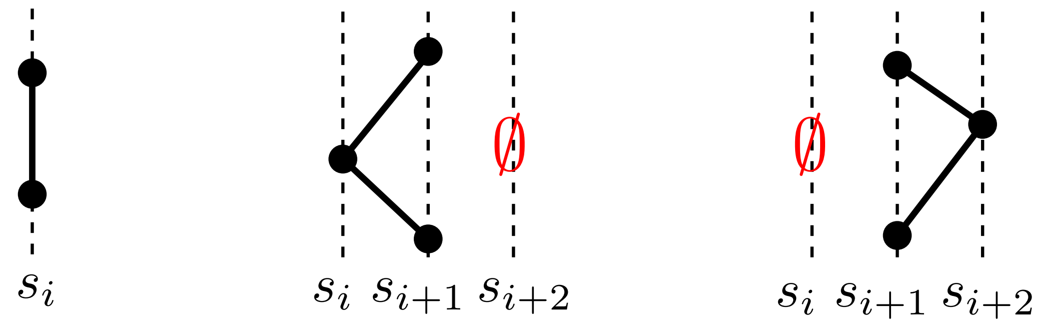
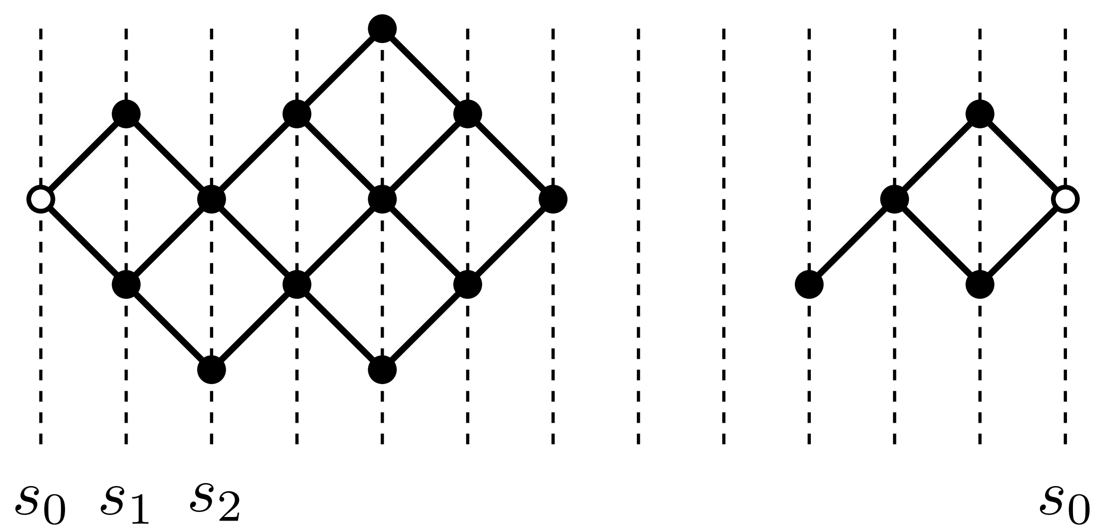
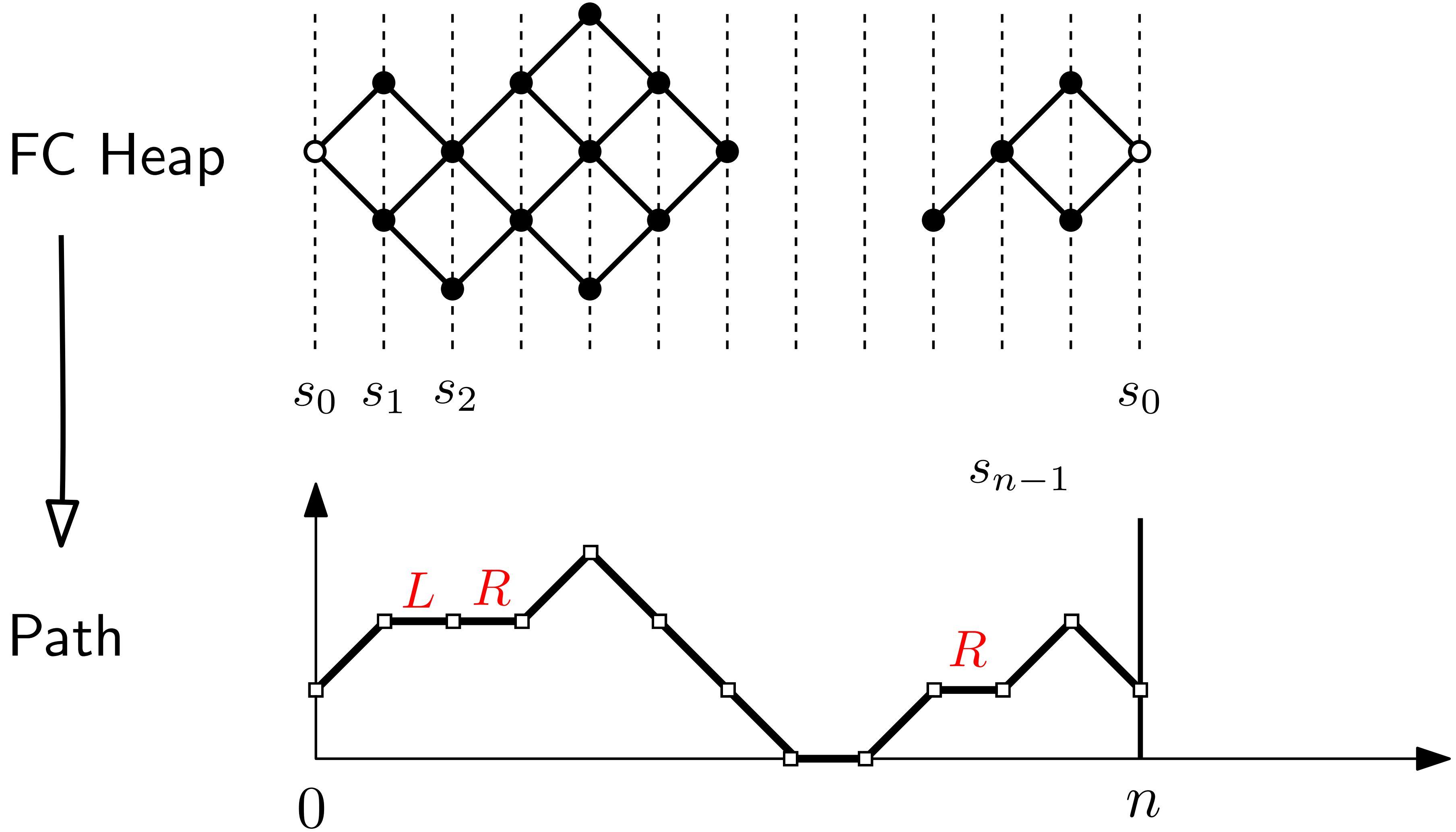


Proposition FC heaps are characterized by:
For all i, $H_{|\{s_i,s_{i+1}\}}$ is a chain with alternating labels

FC Heap



From heaps to paths



No labels needed at height 0.

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• Remark that the **length** of the word is sent to the **area** under the path.

Corollary $\widetilde{A}_{n-1}^{FC}(q) = \mathcal{O}_n^*(q) - \dfrac{2q^n}{1-q^n}$

Enumerative results

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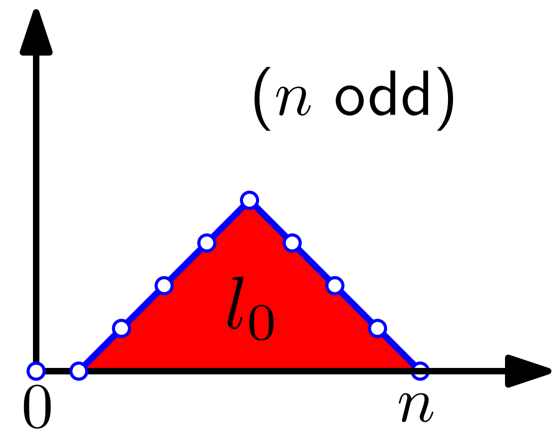

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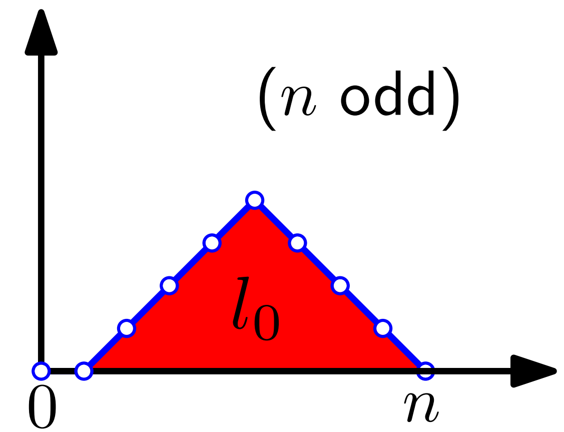

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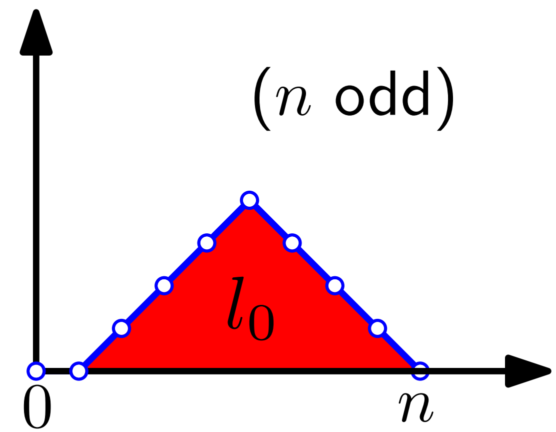

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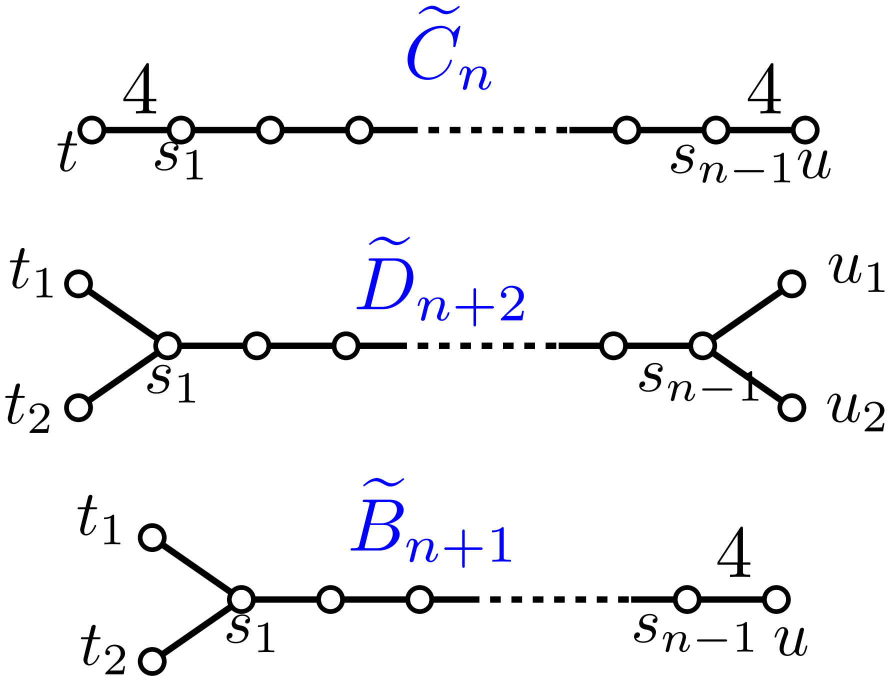
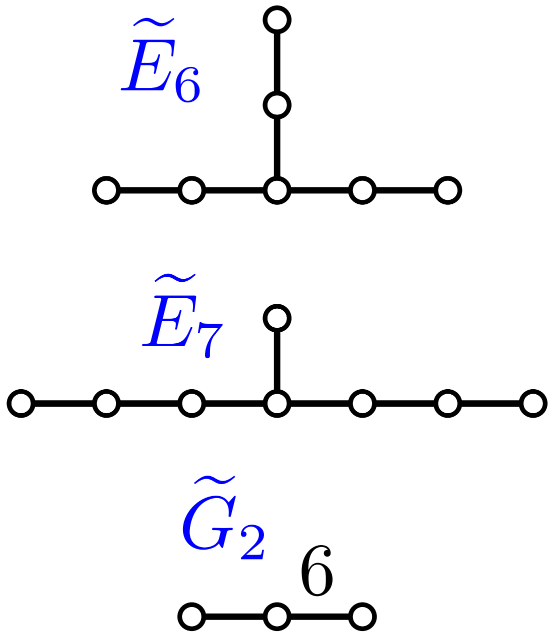
$\sum_{n\geq 0} X_n(q) x^n = Y(x) \left(1 + qx^2 \frac{\partial (xY)}{\partial x}(xq)\right)$	$Y^*(x) = 1 + xY^*(x) + qx(Y^*(x) - 1)Y^*(qx)$
$\sum_{n\geq 0} X^*_n(q) x^n = Y^*(x) \left(1 + qx^2 \frac{\partial (xY)}{\partial x}(xq)\right)$	$Y(x) = \frac{Y^*(x)}{1 - xY^*(x)}$

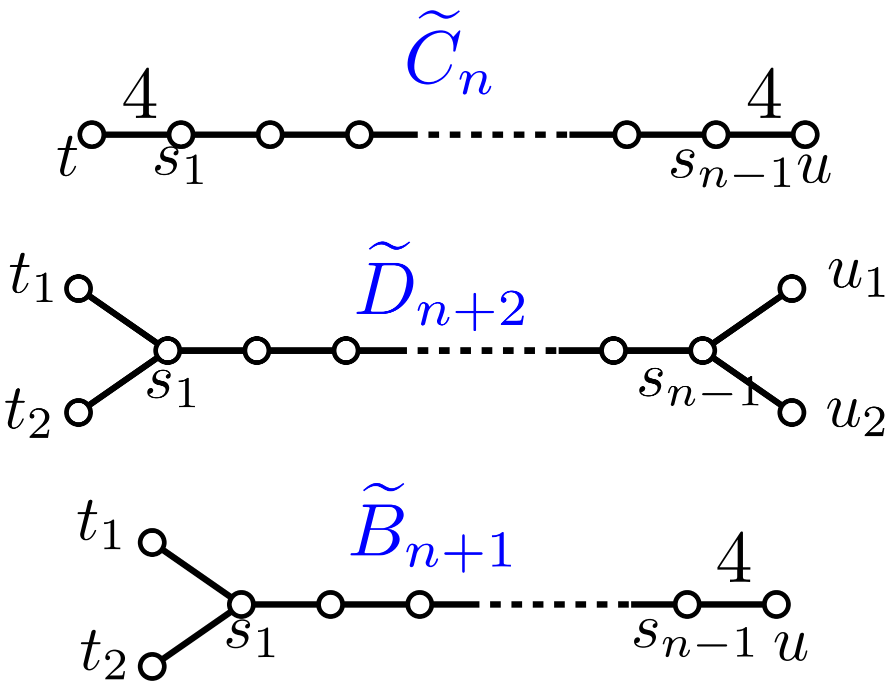
3. Other types

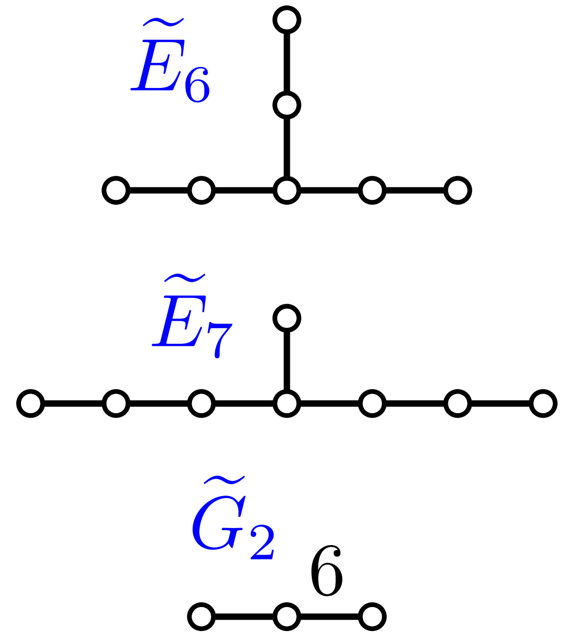
Other affine types





Other affine types

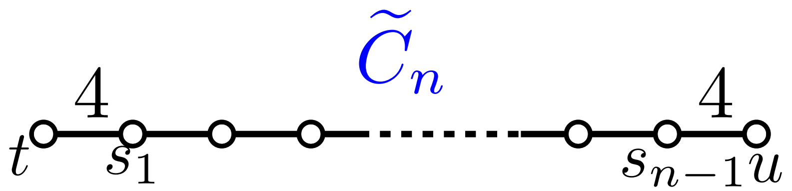




Theorem [BJN '12] For each irreducible affine group W, the sequence $(W_l^{FC})_{l\geq 0}$ is ultimately periodic, with period recorded in the following table.

Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

Type $\widetilde{C}$

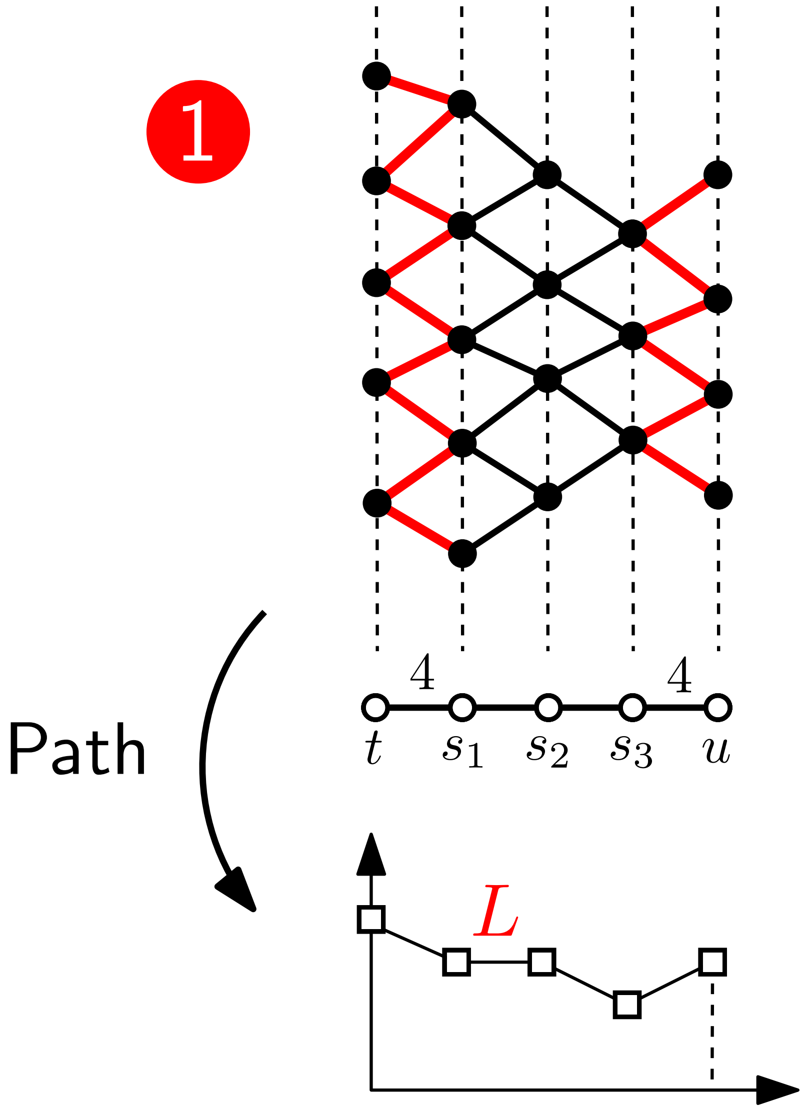


$$\begin{aligned}\widetilde{C}_4^{FC}(q) =& 1 + 5q + 14q^2 + 29q^3 + 47q^4 + 64q^5 + 76q^6 + 81q^7 \\ & + 80q^8 + 75q^9 + 68q^{10} + 63q^{11} + 61q^{12} \\ & + \mathbf{59q^{13}} + \mathbf{59q^{14}} + \mathbf{60q^{15}} + \mathbf{59q^{16}} + \mathbf{59q^{17}} \\ & + \mathbf{59q^{18}} + \mathbf{59q^{19}} + \mathbf{60q^{20}} + \mathbf{59q^{21}} + \mathbf{59q^{22}} \\ & + \mathbf{59q^{23}} + \mathbf{59q^{24}} + \mathbf{60q^{25}} + \mathbf{59q^{26}} + \mathbf{59q^{27}} \\ & + \cdots\end{aligned}$$

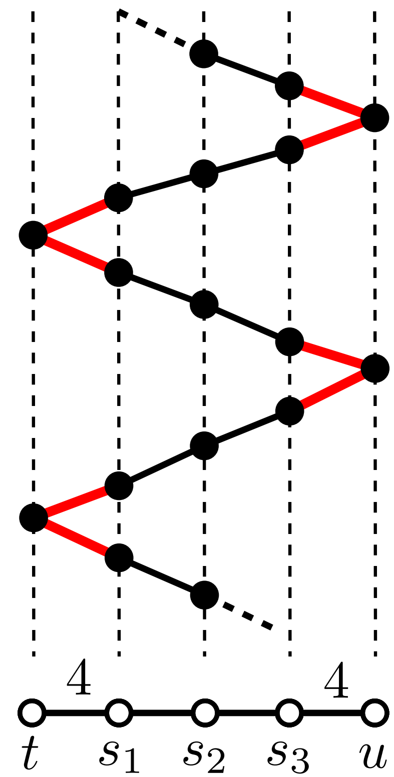
We obtain here also certain heaps corresponding to paths, but there are in addition infinitely many exceptional FC heaps.

Type $\widetilde{C}$

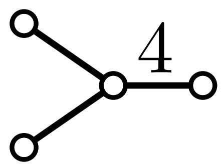
Two families of paths survive **for large enough length**:



2 Finite factors of



Type $\widetilde{B}$

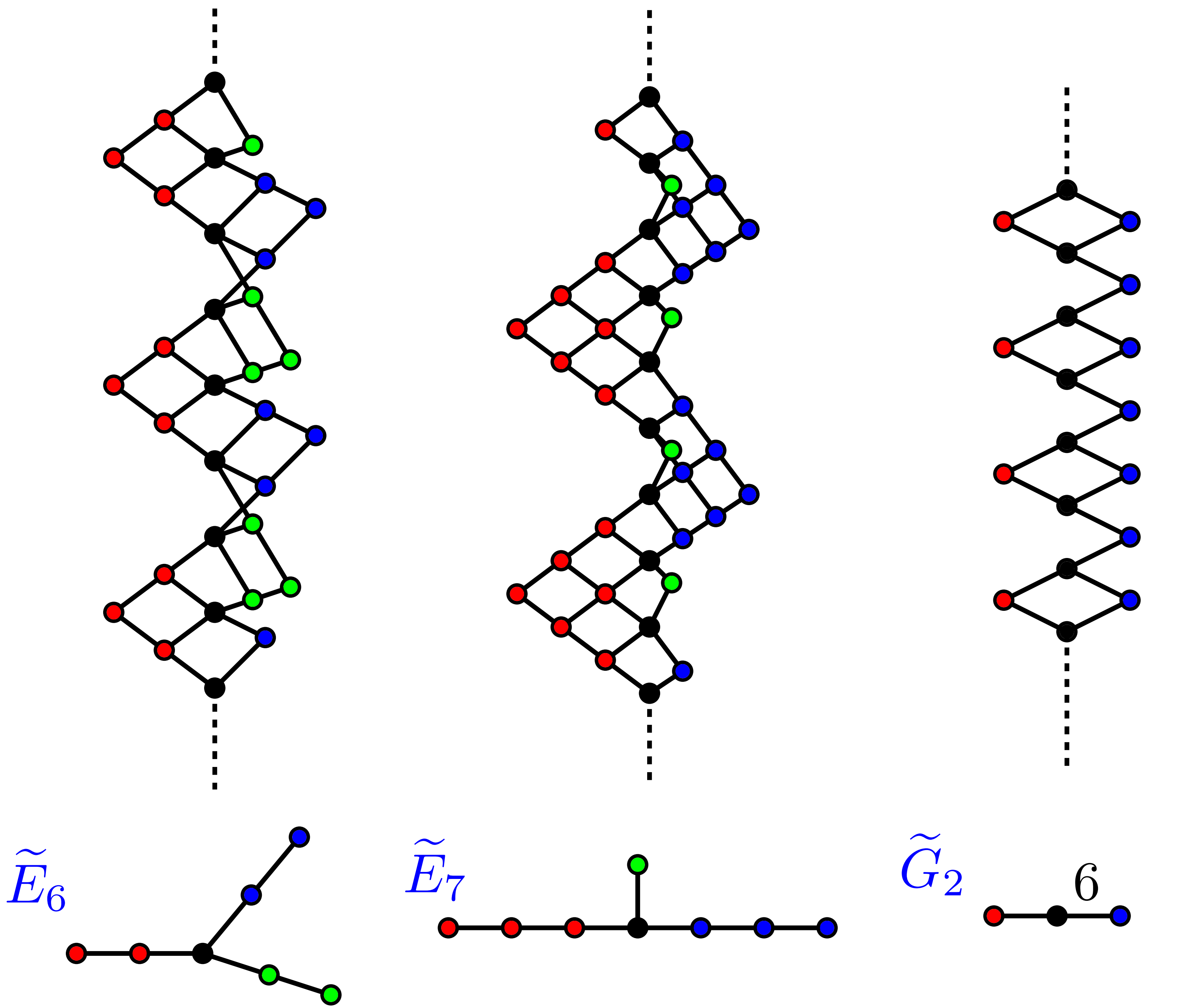


$\widetilde{B}_3$

$$\widetilde{B}_3^{FC}(q) = 1 + 4q + 9q^2 + 15q^3 + 19q^4 + 21q^5 + 21q^6 + 18q^7 + 17q^8 + 19q^9 + 18q^{10} + 17q^{11} + 19q^{12} + 17q^{13} + 17q^{14} + 20q^{15} + 17q^{16} + 17q^{17} + 19q^{18} + 17q^{19} + 18q^{20} + 19q^{21} + 17q^{22} + 17q^{23} + 19q^{24} + 18q^{25} + 17q^{26} + 19q^{27} + 17q^{28} + 17q^{29} + 20q^{30} + 17q^{31} + 17q^{32} + 19q^{33} + 17q^{34} + 18q^{35} + 19q^{36} + 17q^{37} + 17q^{38} + 19q^{39} + 18q^{40} + 17q^{41} + 19q^{42} + 17q^{43} + 17q^{44} + 20q^{45} + 17q^{46} + 17q^{47} + 19q^{48} + 17q^{49} + 18q^{50} + 19q^{51} + 17q^{52} + 17q^{53} + 19q^{54} + 18q^{55} + 17q^{56} + 19q^{57} + 17q^{58} + 17q^{59} + 20q^{60} + 17q^{61} + 17q^{62} + 19q^{63} + 17q^{64} + 18q^{65} + 19q^{66} + 17q^{67} + 17q^{68} + 19q^{69} + 18q^{70} + 17q^{71} + 19q^{72} + 17q^{73} + 17q^{74} + 20q^{75} + 17q^{76} + \cdots$$

Period 15 corresponding to $(n+1)(2n+1)$ for $n = 2$.

Exceptional types



Related Work

• Enumeration of finite Coxeter groups wrt to length.

• FC involutions correspond to “self-dual FC heaps”.
Our methods can be easily applied, and similar results hold (periodicity, generating functions)

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- **Theorem** [Jouhet, N. '13]

For all affine groups W, we can determine the **minimal period**.

- **Theorem in progress** [N. '13]

(i) For any Coxeter system (W, S), the series $W^{FC}(q)$ is a *rational function*.

(ii) The sequence $(W_l^{FC})_{l \geq 0}$ is ultimately periodic if and only if W is affine, FC-finite or is one of two exceptions.

Further questions

• Other statistics to consider, e.g. descent numbers.

• Formulas for our generating functions ? (and not just functional equations/recurrences).

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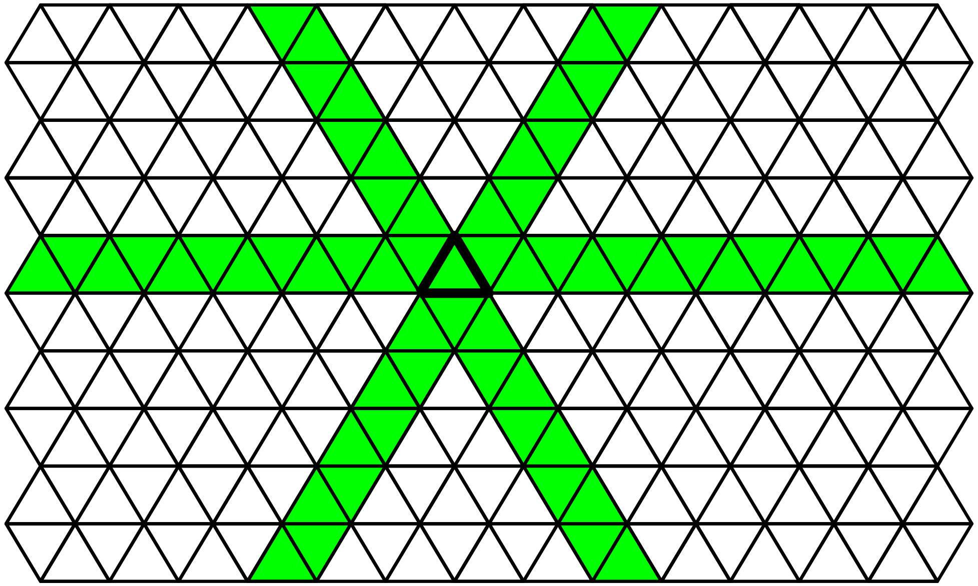
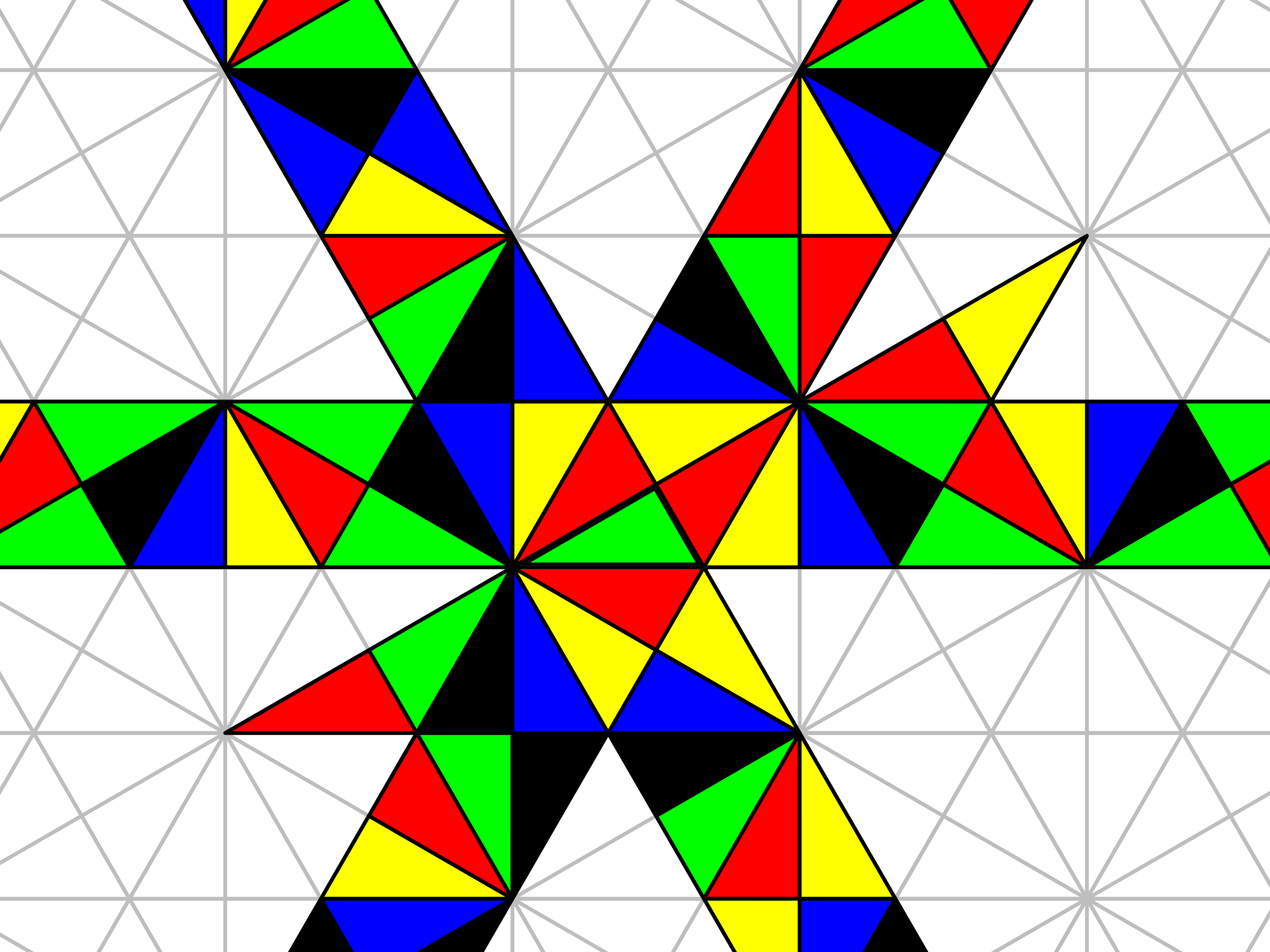
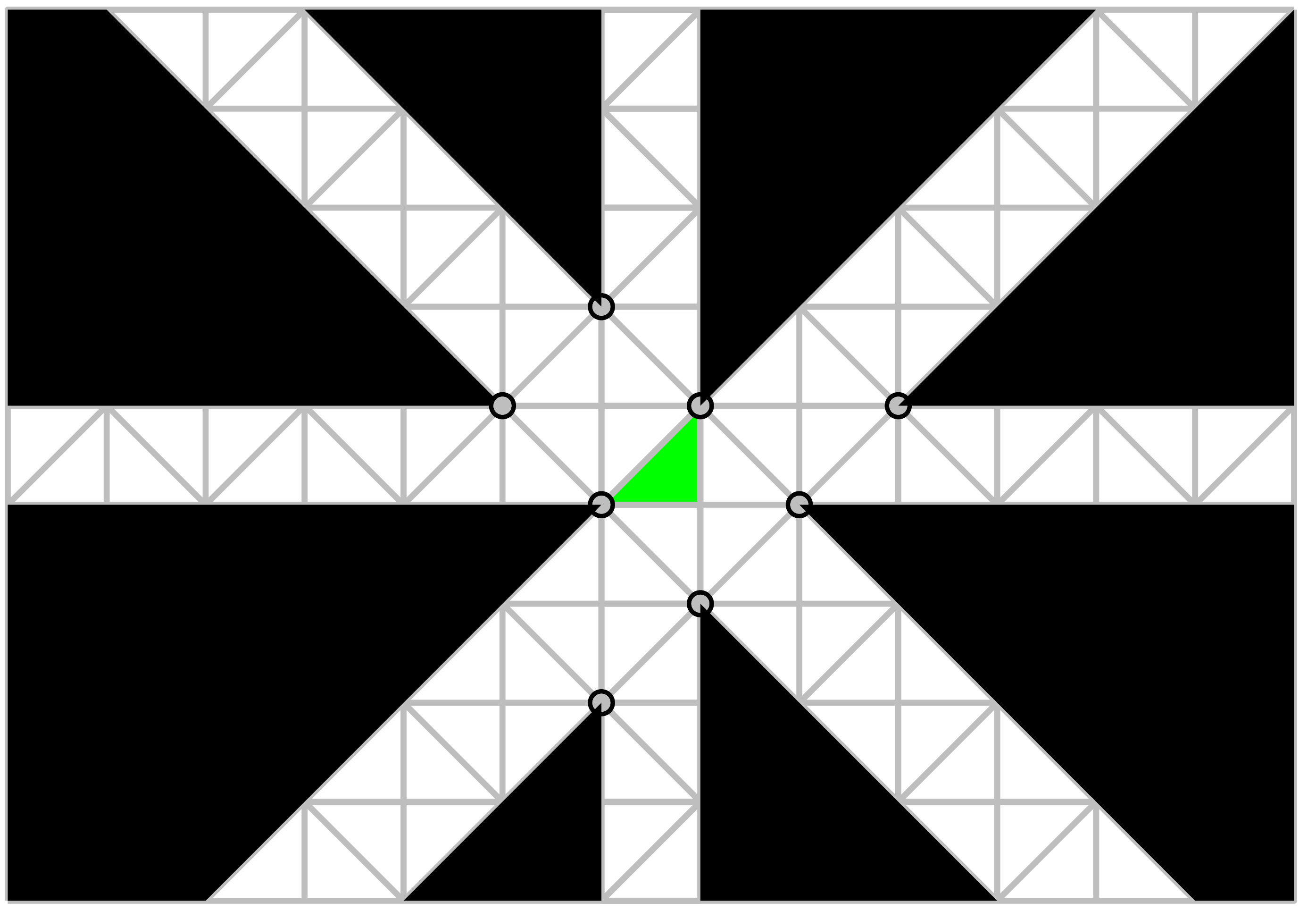
THANK YOU





Type $\widetilde{C}_2$



Type $\widetilde{C}$

Other families