ON r-STACKED TRIANGULATED MANIFOLDS

Satoshi Murai (Yamaguchi University)

and

Eran Nevo (Ben Gurion University of the Negev)

Stacked polytopes

$\Delta = \partial P$, P simplicial d-polytope.

- Recursive definition. $\Leftrightarrow$
- $\Delta = \partial \Delta'$, $||\Delta'|| = P$, $\Delta_{\leq d-2} = \Delta'_{\leq d-2}$. $\Rightarrow$
- f-vector $f(\Delta) =$ explicit $\mathrm{func(d, f_0)}$.

Denote: $\mathcal{P}_{st}(d, n) \subseteq \mathcal{P}(d, n)$ families of boundary complexes of stacked / all d-polytope on n vertices.

LBT (Barnette '71)

(1) $\Delta \in \mathcal{P}(d, n) \Rightarrow f(\Delta) \geq \mathrm{func(d, f_0)}$.

(2) Equality implies $\Delta \in \mathcal{P}_{st}(d, n)$.

Rem. Extended by Kalai to Δ any homology sphere.

$(r-1)$-stacked polytopes

• $\Delta = \partial P$ is $(r-1)$-*stacked* if exists

$\Delta = \partial \Delta', ||\Delta'|| = P, \Delta_{\leq d-r} = \Delta'_{\leq d-r}, \Rightarrow$

• $f(\Delta) =$ explicit $\mathrm{func}(\mathrm{d}, \mathrm{f}_0, ..., \mathrm{f}_{\mathrm{r}-1})$,

nicely expressed in terms of h-vectors:

f-vector:

$f_i(\Delta) =$ number of i-dimensional faces of Δ

$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \cdots, f_{d-1}(\Delta))$

h-vector:

$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$

$h(\Delta) = (h_0(\Delta), \cdots, h_d(\Delta))$

Example:

$\Delta = \partial(\mathrm{triangle}) \Rightarrow \ \mathrm{f}(\Delta) = (1, 3, 3), \ \ \mathrm{h}(\Delta) = (1, 1, 1)$

Dehn-Sommerville relations:

$\Delta = \partial P$, or Δ any homology $(d-1)$-sphere $\Rightarrow$

$h_i(\Delta) = h_{d-i}(\Delta)$ for any $0 \leq i \leq d$.

GLBC: (McMullen-Walkup,71)

$\Delta = \partial P$, P simplicial d-polytope, then $h(\Delta)$ satisfies

(1) $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}$

(2) TFAE:

(i) $h_{r-1} = h_r$ for some $1 \leq r \leq \lfloor \frac{d}{2} \rfloor$

(ii) Δ is $(r-1)$-stacked.

Proofs. (1) Stanley '79 using weak Lefschetz property (WLP),

(ii$\Rightarrow$i) McMullen-Walkup '71 combinatorially,

(i$\Rightarrow$ii) Murai-N. '12 geom.,topo., and mostly algeb.

$\mathbb{Q}$-homology manifolds

• M is d-dim *$\mathbb{Q}$-homology manifold WITH boundary* if M is pure and $\forall \emptyset \neq F \in M$, $\widetilde{H}_\bullet(lk_M(F);\mathbb{Q}) \cong$ either $\widetilde{H}_\bullet(S^{d-|F|};\mathbb{Q})$ or $\widetilde{H}_\bullet(B^{d-|F|};\mathbb{Q}) = 0$; and $\partial M := \{F \in M : \widetilde{H}_\bullet(lk_M(F);\mathbb{Q}) = 0\}$ is a $(d-1)$-dim *$\mathbb{Q}$-homology manifold W/O boundary*, i.e. ∂M is pure and $\forall \emptyset \neq F' \in M$, $\widetilde{H}_\bullet(lk_M(F');\mathbb{Q}) \cong \widetilde{H}_\bullet(S^{d-1-|F'|};\mathbb{Q})$.

Rem. Triangulated manifolds are $\mathbb{Q}$-homology manifolds.

• d-dim manifold M WITH boundary is *r-stacked* if it has NO *interior* faces of dim $< d-r$.

• Δ manifold W/O boundary is *r-stacked* if $\Delta = \partial M$ for some r-stacked manifold M.

r-stackedness depends only on (f, β)-vectors

h'-vector:

Let Δ be $(d-1)$-dim simplicial complex,

$h'_i(\Delta) = h_i(\Delta) - \binom{d}{i} \sum_{k=1}^{i-1} (-1)^{i-k} \beta_{k-1}(\Delta)$ for $0 \le i \le d$

h''-vector:

$h''_i(\Delta) = h'_i(\Delta) - \binom{d}{i} \beta_{i-1}(\Delta) =$

$h_i(\Delta) - \binom{d}{i} \sum_{k=1}^{i} (-1)^{i-k} \beta_{k-1}(\Delta)$ for $0 \le i \le d-1$, and

$h''_d(\Delta) = h'_i(\Delta) = \beta_{d-1}(\Delta)$.

Algebraic meaning - via Stanley-Reisner ring:

- *Face ring* of Δ on n vertices (over $\mathbb{Q}$) is

$\mathbb{Q}[\Delta] = \mathbb{Q}[x_1, \dots, x_n]/(x^F : F \in 2^{[n]} - \Delta)$.

Example: $\mathbb{Q}[\partial(\mathrm{triangle})] = \mathbb{Q}[\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]/(\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3)$.

- If Δ is $(d-1)$-manifold then $\mathbb{Q}[\Delta]$ is *Buchsbaum ring*.

Theorem: For $\Theta = (\theta_1, \dots, \theta_d)$ an l.s.o.p. of $\mathbb{Q}[\Delta]$:

(1) Schenzel '81: $\mathrm{hilb}(\mathbb{Q}[\Delta]/(\Theta)) = \mathrm{h}'(\Delta)$,

(2) Novik-Swartz '09: exists ideal $N \subseteq \mathrm{Soc}(\mathbb{Q}[\Delta]/(\Theta))$ with $\mathrm{hilb}((\mathbb{Q}[\Delta]/(\Theta))/\mathrm{N}) = \mathrm{h}''(\Delta)$.

Rem. In particular $h''(\Delta) \geq 0$.

$\tilde{g}$-vector:

$\tilde{g}_i(\Delta) = h''_i(\Delta) - h''_{i-1}(\Delta) - \binom{d}{i-1}\beta_{i-1}(\Delta) =$
$h_i(\Delta) - h_{i-1}(\Delta) - \binom{d+1}{i}\sum_{k=1}^{i}(-1)^{i-k}\beta_{k-1}(\Delta)$

for $0 \leq i \leq d$, and

$\tilde{g}(\Delta) = (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\lfloor \frac{d}{2} \rfloor})$.

The $\tilde{g}$-vector gives enumerative criterion for r-stackedness:

Main Theorem (enumerative criteria): Murai-N. '12

(1) For any $(d-1)$-dim homology manifold WITH boundary Δ and $1 \leq r \leq d$:

Δ is $(r-1)$-stacked $\Leftrightarrow h''_r(\Delta) = 0$.

(2) If $\Delta = \partial M$ for d-dim $(r-1)$-stacked homology manifold WITH boundary M and $1 \leq r \leq \frac{d}{2} \Rightarrow \tilde{g}_r(\Delta) = 0$.

(3) Let Δ be a connected $\mathbb{Q}$-orientable $(d-1)$-homology manifold W/O boundary and $1 \leq r < \frac{d}{2}$. Assume all vertex links have WLP (e.g. when all vertex links are polytopal). Then:

$\tilde{g}_r(\Delta) = 0 \Rightarrow \Delta$ is $(r-1)$-stacked.

Open problem:

Does (3) hold for $r = \frac{d}{2}$? Even the case $d = 4$ is open!

Conjecture (GLBC for manifolds):

Let Δ be a connected triangulated $(d-1)$-manifold W/O boundary.Then:

(1) Kalai '88: $\tilde{g}(\Delta) \geq 0$.

(2) Bagchi-Datta '12: If $\tilde{g}_r(\Delta) = 0$ for some $r < \frac{d}{2}$ then Δ is *locally* $(r-1)$*-stacked*.

Def. Δ is *locally* r*-stacked* if any vertex link is r-stacked.

Theorem (local–global): Bagchi-Datta '12, Murai-N. '12

Let Δ be a $(d-1)$-dim homology manifold W/O boundary and $1 \leq r < \frac{d}{2}$. Then:

Δ is $(r-1)$-stacked $\Leftrightarrow$ Δ is locally $(r-1)$-stacked.

Remarks: • $\Rightarrow$ is clear, for any r.

• $\Leftarrow$ is false for $r = \frac{d}{2}$, e.g. $\Delta = \partial\sigma^r * \partial\sigma^r \cong S^{2r-1}$.

Theorem (stackedness restricts topology): Murai-N. '12, also follows from Bagchi-Datta '12

(1) Δ $(r-1)$-stacked manifold WITH boundary $\Rightarrow \beta_k(\Delta) = 0$ for all $k \geq r$.

(2) Δ $(r-1)$-stacked $(d-1)$-manifold W/O boundary, $1 \leq r < \lfloor \frac{d}{2} \rfloor \Rightarrow \beta_k(\Delta) = 0$ for all $r \leq k \leq d-r-1$.

Question:

Are there more topological restrictions?

E.g. handle decompositions?

Theorem (uniqueness): Bagchi-Datta '12, Murai-N. '12

Δ $(r-1)$-stacked $(d-1)$-manifold W/O boundary, $1 \leq r \leq \lfloor \frac{d}{2} \rfloor$

$\Rightarrow$ $\exists!$ $(r-1)$-stacked d-manifold Δ' with $\partial\Delta' = \Delta$.

In fact, $\Delta' = \Delta(r) := \{F \subseteq \Delta_0 : \binom{F}{r+1} \subseteq \Delta\}$.

Rem.

- False for $r = \frac{d+1}{2}$.
- False for $\Delta(r-1)$, though true if $\Delta = \partial P$.

Theorem (upper bounds on $\tilde{g}$): (Murai-N.)

Let Δ be a connected orientable homology $(d-1)$-manifold W/O boundary. Assume all vertex links have WLP.

Then, $\exists$ l.s.o.p. Θ, a linear form ω, and an ideal J in $R = \mathbb{Q}[\Delta]/(\Theta + (\omega))$ s.t. $\dim_{\mathbb{Q}}(R/J)_i = \tilde{g}_i(\Delta)$, for all $i \leq \frac{d}{2}$.

In particular, $\tilde{g}(\Delta)$ is an M-sequence.

Rem. Conj. conclusion holds $\forall$ connected manifolds W/O ∂.

Conj. $\dim_{\mathbb{Q}}(Soc(R))_r \geq \binom{d+1}{r}\beta_{r-1}(\Delta)$ for all $r \leq \frac{d}{2}$.

Rem. • true for Δ $(r-1)$-stacked.

• Conj. $\Rightarrow$ Thm.: let $R' = \mathbb{Q}[\Delta]/(\Theta)$, then

$\dim_{\mathbb{Q}}(Soc(R))_r = \dim_{\mathbb{Q}} R'_r - \dim_{\mathbb{Q}}(R'/Soc(R'))_{r-1} =$

$h'_r - h''_{r-1} = \tilde{g}_r + \binom{d+1}{r}\beta_{r-1}(\Delta)$,

so can choose $J \subseteq Soc(R)$.

Thank you !