Boundaries of branching graphs

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Asymptotic theory of characters – 1

Consider a chain of growing compact or finite groups

$$G(1) \subset G(2) \subset \dots,$$

$$G(\infty) = \bigcup G(n) = \varinjlim G(n).$$

Two model examples are

$$U(\infty) := \varinjlim U(N), \qquad S_\infty := \varinjlim S_n, .$$

For a compact group G, $\widehat{G} := \{\text{irreducible characters}\}$.
Question: What is the dual object $\widehat{G(\infty)}$?
Conventional definition $\chi(g) = \operatorname{tr} \pi(g)$ is not applicable, because $\dim \pi = \infty \implies \chi(e) = \infty$.

Asymptotic theory of characters – 2

Idea: (Thoma 1964): replace irreducible characters $\chi \in \widehat{G}$ by *normalized* irreducible characters $\chi/\chi(e)$ and then generalize this notion.

Definition (Vershik-Kerov 1981): $\widehat{G(\infty)}$=functions $\psi : G(\infty) \to \mathbb{C}$ such that ψ=limit of normalized irreducible characters $\psi_N = \chi_N/\chi_N(e)$, where $\chi_N \in \widehat{G(N)}$, $N \to \infty$.

Here $\psi_n \to \psi$ means uniform convergence on each pre-limit subgroup:

$$\psi_n\big|_{G(k)} \to \psi\big|_{G(k)} \quad \forall k$$

The dual object to $U(\infty)$ – 1

Theorem: The functions $\psi \in \widehat{U(\infty)}$ depend on infinite collections of real nonnegative parameters $\omega = \{\{\alpha_i^\pm\}, \{\beta_i^\pm\}, \gamma^\pm\}$ such that

$$\sum \alpha_i^+ + \sum \alpha_i^- + \sum \beta_i^+ + \sum \beta_i^- < +\infty.$$

The corresponding function $\psi = \psi_\omega$ on $U(\infty)$ looks as follows:

• The collection ω of parameters determine an analytic function $\Phi(u;\omega)$ on the unit circle $|u| = 1$ in $\mathbb{C}$:

$$\Phi(u;\omega) := e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \frac{\prod(1+\beta_i^+(u-1))\prod(1+\beta_i^-(u^{-1}-1))}{\prod(1-\alpha_i^+(u-1))\prod(1-\alpha_i^-(u^{-1}-1))}$$

• The value of $\psi_\omega(U)$ at a given element $U \in U(\infty)$ equals

$$\Phi(u_1;\omega)\Phi(u_2;\omega)\dots,$$

where $u_1, u_2, \dots$ are the eigenvalues of U; note that only finitely many of them are distinct from 1, whereas $\Phi(1;\omega) = 1$, so the product is actually finite.

The dual object to $U(\infty)$ – 2

The theorem shows that $\widehat{U(\infty)}$ can be realized as a region in $\mathbb{R}_+^{4\infty+2}$. It is an infinite-dimensional locally compact space.

History:

Edrei 1953: Totally positive infinite Toepliz matrices.
Voiculescu 1976: Finite factor representations of $U(\infty)$.
Boyer 1983: Reduction to Edrei's theorem
Vershik and Kerov 1982: Asymptotic approach.
Okounkov and Olshanski 1998: Detailed proof based on V-K idea and a generalization.
Borodin and Olshanski 2012: New proof
Petrov 2013+: Alternative derivation.
Gorin and Panova 2013+: One more approach.

The Gelfand-Tsetlin graph $\mathbb{GT}$ – 1

Reformulation: There exists a bijective correspondence $\widehat{U(\infty)} \leftrightarrow \partial\mathbb{GT}$ (the Martin boundary of the Gelfand-Tsetlin graph $\mathbb{GT}$).

$\widehat{U(N)} \leftrightarrow \mathbb{GT}_N := \{\text{signatures of length } N\} = \{\text{highest weights}\}$.
A *signature* $\nu \in \mathbb{GT}_N$ is a vector $\nu = (\nu_1 \geq \cdots \geq \nu_N) \in \mathbb{Z}^N$.

Two signatures $\mu \in \mathbb{GT}_{N-1}$ and $\nu \in \mathbb{GT}_N$ *interlace*, $\mu \prec \nu$, if

$$\nu_1 \geq \mu_1 \geq \nu_2 \geq \cdots \geq \nu_{N-1} \geq \mu_{N-1} \geq \nu_N.$$

Definition: The Gelfand-Tsetlin graph is the graded graph with the vertex set $\bigsqcup_{N=1}^{\infty} \mathbb{GT}_N$ and the edges $\mu \prec \nu$.

It encodes the *Gelfand-Tsetlin branching rule* for the irreducible characters of the unitary groups.

The Gelfand-Tsetlin graph $\mathbb{GT}$ – 2

The GT branching rule:

$$\chi_\nu\big|_{U(N-1)} = \sum_{\mu:\, \mu \prec \nu} \chi_\mu.$$

Definition: A monotone path between $\varkappa \in \mathbb{GT}_K$ and $\nu \in \mathbb{GT}_N$, where $K < N$, is a sequence $\varkappa \prec \cdots \prec \nu$.

This is the same as a trapezoidal *GT pattern* or *scheme* with the top row ν and the bottom row $\varkappa$.

`Example of a trapezoidal GT pattern,` $K = 2,\ N = 5$

$$\begin{array}{ccccccccc}
\nu_1 & & \nu_2 & & \nu_3 & & \nu_4 & & \nu_5 \\
& * & & * & & * & & * & \\
& & * & & * & & * & & \\
& & & \varkappa_1 & & \varkappa_2 & & &
\end{array}$$

The Gelfand-Tsetlin graph $\mathbb{GT}$ – 3

If the initial vertex is on the first level and is not fixed, then we get triangular patterns with top row ν.

`Example of a triangular GT pattern,` $N = 4$

$$
\begin{array}{ccccccc}
\nu_1 & & \nu_2 & & \nu_3 & & \nu_4 \\
& * & & * & & * & \\
& & * & & * & & \\
& & & * & & &
\end{array}
$$

Notation: We set

$$\dim_{\mathbb{GT}} \nu = \#\{\text{triangular GT patterns with top row } \nu\} = \dim \chi_\nu$$

$$\dim_{\mathbb{GT}}(\varkappa, \nu) = \#\{\text{trapezoidal GT patterns with top row } \nu$$

$$\text{and bottom row } \varkappa\} = \left\{\text{the multiplicity of } \chi_\varkappa \text{ in } \chi_\nu\big|_{U(K)}\right\}$$

These quantities count integer points in some convex polytopes.

The Gelfand-Tsetlin graph $\mathbb{GT}$ – 4

Our object of study is the *relative dimension*

$$\frac{\dim_{\mathbb{GT}}(\varkappa, \nu)}{\dim_{\mathbb{GT}} \nu}, \qquad \varkappa \in \mathbb{GT}, \quad \nu \in \mathbb{GT}_N$$

We view it as a function in variable $\nu \in \mathbb{GT}$ indexed by $\varkappa \in \mathbb{GT}$. Its possible asymptotics as N goes to infinity and $\nu = \nu(N)$ varies together with N determines the Martin boundary $\partial\mathbb{GT}$.

Definition: More precisely, $\partial\mathbb{GT}$ consists on those functions $\varphi : \mathbb{GT} \to [0, 1]$ that can be obtained as limits

$$\varphi(\varkappa) = \lim_{N\to\infty} \frac{\dim_{\mathbb{GT}}(\varkappa, \nu(N))}{\dim_{\mathbb{GT}} \nu(N)} \qquad \forall \varkappa \in \mathbb{GT}.$$

Simple fact: There is a natural bijection $\widehat{U(\infty)} \leftrightarrow \partial\mathbb{GT}$

The Gelfand-Tsetlin graph $\mathbb{GT}$ – 5

Proof: We have

$$\left.\frac{\chi_\nu}{\chi_\nu(e)}\right|_{U(K)} = \sum_{\varkappa \in \mathbb{GT}_K} \frac{\dim_{\mathbb{GT}}(\varkappa, \nu)}{\dim_{\mathbb{GT}} \nu} \chi_\varkappa$$

Equivalently, in terms of the rational Schur functions (Laurent polynomials)

$$\frac{s_\nu(u_1, \dots, u_K, \overbrace{1, \dots, 1}^{N-K})}{s_\nu(\underbrace{1, \dots, 1}_{N})} = \sum_{\varkappa \in \mathbb{GT}_K} \frac{\dim_{\mathbb{GT}}(\varkappa, \nu)}{\dim_{\mathbb{GT}} \nu} s_\varkappa(u_1, \dots, u_K).$$

Then use the fact that uniform convergence of positive definite class functions on $U(K)$ is the same as simple convergence of their Fourier coefficients. (**Note:** Gorin and Panova 2013+ analyze convergence of LHS's directly.)

Cauchy identity for relative dimension – 1

Our aim is to reveal a combinatorial structure behind the relative dimension. We will see that the function

$$\nu \mapsto \frac{\dim_{\mathbb{GT}}(\varkappa, \nu)}{\dim_{\mathbb{GT}} \nu}$$

bears resemblance to the Schur symmetric function $s_\lambda \in \mathrm{Sym}$.

Recall the **classical Cauchy identity:**

$$H(y_1; X)H(y_2, X) \cdots = \sum_{\text{partitions } \lambda} s_\lambda(X) s_\lambda(Y).$$

Here $X = (x_i)$ and $Y = (y_j)$ are two collections of variables, and $H(y; X)$ is the generating series for the complete homogeneous symmetric functions in X:

$$H(y; X) = \prod_i \frac{1}{1 - x_i y}.$$

Cauchy identity for relative dimension – 2

Theorem (Borodin-Olshanski 2012): The following Cauchy-type identity holds: For $N > K$ and $\nu \in \mathbb{GT}_N$

$$F(t_1;\nu)\dots F(t_K;\nu) = \sum_{\varkappa\in\mathbb{GT}_K} \frac{\dim_{\mathbb{GT}}(\varkappa,\nu)}{\dim_{\mathbb{GT}}\nu} \mathfrak{S}_\varkappa(t_1,\dots,t_K).$$

Here $\mathfrak{S}_\varkappa$ (analog of $s_\lambda(Y)$) is given by a Schur-type formula,

$$\mathfrak{S}_\varkappa(t_1,\dots,t_K) = \mathrm{const}_{N,K}\, \frac{\det[g_{\varkappa_r+K-r}(t_j)]_{r,j=1}^K}{\prod_{i<j}(t_i - t_j)},$$

where $g_k(t)$ are certain rational fractions in variable t, indexed by $k \in \mathbb{Z}$. Next, $F(t;\nu)$ (analog of $H(y;X)$) is given by

$$F(t;\nu) := \prod_{i=1}^N \frac{t+i}{t+i-\nu_i}.$$

Note that if we neglect the shift by i and set $t = y^{-1}$ then F reduces to H.

Cauchy identity for relative dimension – 3

The difficulty in finding $\partial\mathbb{GT}$ comes from the numerator $\dim_{\mathbb{GT}}(\varkappa, \nu)$, the number of trapezoidal GT patterns: It can be expressed through a specialization of the skew Schur function, but the resulting expression is not suitable for the limit transition.

On the contrary, our Cauchy identity yields a contour integral representation for relative dimension, well suitable for the limit transition. This leads to the description of the boundary $\partial\mathbb{GT}$.

Idea: Let $N \to \infty$, $t_j \sim \frac{N}{u_j - 1}$, and let us assume that $\nu \in \mathbb{GT}_N$ varies together with N in a "regular" way meaning that some its parameters, after rescaling, converge to the coordinates of $\omega \in \partial\mathbb{GT}$. Then in this limit regime our Cauchy-type identity turns into the following relation:

$$\Phi(u_1;\omega)\dots\Phi(u_K;\omega) = \sum_{\varkappa \in \mathbb{GT}_K} \varphi(\varkappa;\omega) s_\varkappa(u_1,\dots,u_K), \quad K = 1, 2, \dots$$

The functions $\varkappa \to \varphi(\varkappa;\omega)$ are just the elements of $\partial\mathbb{GT}$.

Jacobi-Trudi-type formula for relative dimension

Theorem (Borodin-Olshanski, 2012) Let ν be a signature of length N and $\varkappa$ be a signature of length $K < N$. The relative $\mathbb{GT}$-dimension admits a Jacobi-Trudi-type representation

$$\frac{\dim_{\mathbb{GT}}(\varkappa,\nu)}{\dim_{\mathbb{GT}}\nu} = \det[\varphi^{(j)}_{\varkappa_i - i + j}(\nu)]^K_{i,j=1} \qquad (*)$$

where the functions $\varphi^{(j)}_k(\nu)$, $k \in \mathbb{Z}$, $j = 1, \dots, K$, are uniquely determined from the following expansion in rational fractions

$$F(t;\nu) := \frac{(t+1)\dots(t+N)}{(t+1-\nu_1)\dots(t+N-\nu_N)}$$
$$= \sum_{k\in\mathbb{Z}} \varphi^{(j)}_k(\nu) \frac{(t+j)(t+j+1)\dots(t+j+N-K)}{(t-k+j)(t-k+j+1)\dots(t-k+j+N-K)}$$

Remark: Formula $(*)$ is an approximative version of the limit formula $\varphi(\varkappa;\omega) = \det[\varphi_{\varkappa_i - i + j}(\omega)]$.

Comments

1) Modified Jacobi-Trudi already occurred in Macdonald's work (Schur Functions: Theme and Variations, 1992).

2) The theorem about $\partial\mathbb{GT}$ admits a q-version (Gorin 2012, Petrov 2013+).

3) I hope that the results on the $\mathbb{GT}$ relative dimension can be extended to the characters of other classical groups. This is work in progress.

4) Our approach yields new variations of Schur functions and symmetric functions in general. The conventional symmetric functions in finitely many variables are polynomials or Laurent polynomials, whereas we have to deal with symmetric functions which are not polynomials but *rational functions*. For instance, such are the functions $\mathfrak{S}_{\varkappa}(t_1, \dots, t_K)$.
Perhaps, one can go beyond the Schur-type functions framework and study more general symmetric functions with Jack or even Macdonald parameters.

The boundary of $\mathbb{Y}$ and Kerov's conjecture – 1

The branching graph related to $S_\infty = \varinjlim S_n$ is the Young graph $\mathbb{Y}$ (aka Young lattice). It encodes the Young branching rule for the symmetric group characters. The dual object $\widehat{S(\infty)}$ can be identified with the boundary $\partial\mathbb{Y}$. I will give an alternative definition of $\partial\mathbb{Y}$, based on an idea of Vershik-Kerov.

Consider the algebra Sym with its basis $\{s_\lambda\}$ formed by the Schur functions.

Definition: (Vershik-Kerov, 1986). The boundary $\partial\mathbb{Y}$ is the set of multiplicative linear functionals $\Psi : \mathrm{Sym} \to \mathbb{R}$ such that $\Psi(s_\lambda) \geq 0$ for every λ, and normalized by $\Psi(s_{(1)}) = 1$.

Theorem: The elements $\Psi \in \partial\mathbb{Y}$ are parametrized by the triples $(\{\alpha_i\}, \{\beta_i\}, \gamma) \in [0,1]^{2\infty+1}$ such that $\sum \alpha_i + \sum \beta_i + \gamma = 1$:

$$\Psi_{\alpha,\beta,\gamma}(1 + h_1 z + h_2 z^2 + \dots) = e^{\gamma z} \frac{\prod(1+\beta_i z)}{\prod(1-\alpha_i z)}.$$

The boundary of $\mathbb{Y}$ and Kerov's conjecture – 2

History: Edrei 1952, Thoma 1964, Vershik-Kerov 1981, Okounkov 1994, Kerov-Okounkov-Olshanski 1998.

Degeneration $\mathbb{GT} \to \mathbb{Y}$: Borodin-Olshanski 2013.

Generalizations: Replace $\{s_\lambda\}$ by the Jack symmetric functions (KOO 1998). In particular, Kingman's theorem (1978).

Kerov's conjecture 1992: predicts the form of the boundary when $\{s_\lambda\}$ is replaced with the Macdonald symmetric functions.

Obstacle: The approach of KOO 1998 is based on the study of the relative dimension which in turn relies on the theory of *shifted symmetric functions* (aka interpolation polynomials) (Okounkov-Olshanski 1997). Such a theory also exists in the case of Macdonald's (q, t) parameters (Knop, Sahi, Okounkov), but it seems that in this generality the connection with the relative dimension disappears, so new ideas are required.

The boundary of the graph $\mathbb{W}$ – 1

In another direction, the above definition of the boundary $\partial\mathbb{Y}$ as the set of positive multiplicative functionals on Sym can be extended to the *algebra of quasisymmetric functions*, QSym.

In QSym, there is an analog of the Schur functions — the *fundamental quasisymmetric functions* F_u; here u ranges over the set of *compositions*, which can be encoded by *binary words*.

The corresponding analog of $\mathbb{Y}$ is a graded graph $\mathbb{W}$, known under the name of *subword order*: $\mathbb{W} = \bigsqcup \mathbb{W}_n$, where $\mathbb{W}_n = \{\text{compositions of } n\} \leftrightarrow \{+,-\}^{n-1}$.

Teorem (Gnedin-Olshanski, 2006) The exists a bijection

$$\partial\mathbb{W} \leftrightarrow \{\text{colored open subsets } U \subset (0,1)\},$$

where a coloring of an open subset $U \subset (0,1)$ is an assignment to every interval of U one of the colors, black or white.

The boundary of the graph $\mathbb{W}$ – 2

Comments: 1) $\partial\mathbb{W}$ is a “suspension” over $\partial\mathbb{Y}$: the projection $U \mapsto (\alpha, \beta, \gamma)$ from $\partial\mathbb{W}$ to $\partial\mathbb{Y}$ turns the lengths of the plus/minus intervals to the alpha/beta parameters, and γ equals the mass of the complement $(0, 1) \setminus U$.

2) The projection comes from the embedding $\mathrm{Sym} \to \mathrm{QSym}$: Every Schur function expands in fundamental functions with *nonnegative* coefficients, so a positive homomorphism $\Psi : \mathrm{QSym} \to \mathbb{R}$ induces by restriction a positive homomorphism $\mathrm{Sym} \to \mathbb{R}$.

3) U can be viewed as triple (α, β, γ) together with a total ordering of the coordinates. The necessity of the ordering is related to the fact that the coproduct in QSym is noncommutative.

4) The origin of U: the black/white intervals come from the (scaled) $+/-$ clusters of a (very large) binary word $u \in \{+, -\}^{n-1}$.

The boundary of the graph $\mathbb{W}$ – 3

The relative dimension in $\mathbb{W}$ is given by

$$\frac{\dim_{\mathbb{W}}(u,v)}{\dim_{\mathbb{W}} v} = \frac{[F_{(1)}^{n-k} F_u : F_v]}{[F_{(1)}^n : F_v]}, \qquad u \in \mathbb{W}_k, \quad v \in \mathbb{W}_n,$$

where $[G : F_v]$ denotes the coefficient of F_v in the expansion of an element $G \in \mathrm{QSym}$ in the basis of fundamental quasisymmetric functions, and $F_{(1)} = s_{(1)} = x_1 + x_2 + \dots$.

Open question: Does there exist a good expression for the relative dimension, suitable for limit transition as n gets large?

The proof given in (Gnedin-Olshanski, 2006) avoided the study of relative dimension and used instead a probabilistic result due to Jacka and Warren 2007. It would be interesting to find a combinatorial approach.