Asymptotics of symmetric functions with applications to statistical mechanics and representation theory

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based on same-name paper arXiv:1301.0634 joined with Vadim Gorin

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Overview

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph

1	1	1	2	2	...
2	2	3	...		
...					

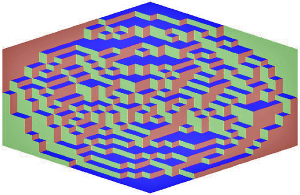
Alternating Sign Matrices (ASM)/ 6Vertex model:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

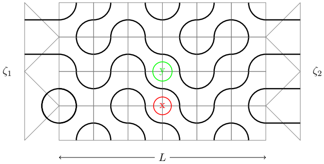
Normalized Schur functions:

$$S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1^{N-k})}{s_\lambda(1^N)}$$

Lozenge tilings:



Dense loop model:



Definitions and setup

In our context: Symmetric functions, Lie groups characters.

(mainly) **Schur functions:** $s_\lambda(x_1, \ldots, x_N)$ – characters of V_λ.

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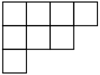
Irreducible (rational) representations V_λ of $GL(N)$ (or $U(N)$) are indexed by **dominant weights** (signatures/Young diagrams/integer partitions) λ:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,$$

where $\lambda_i \in \mathbb{Z}$, e.g. $\lambda = (4, 3, 1)$,



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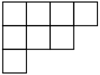


(mainly) **Schur functions:** $s_\lambda(x_1,\ldots,x_N)$ – characters of V_λ.

Weyl's determinantal formula:

$$s_\lambda(x_1,\ldots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{ij=1}^{N}}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux($\Leftrightarrow$ Gelfand-Tsetlin patterns) of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = s_{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}.$$

Definitions and setup

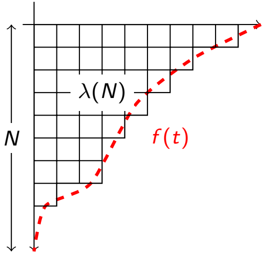
Object of study and main tool in the applications:

Normalized Schur functions:

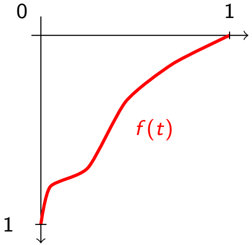
$$S_{\lambda(N)}(x_1, \ldots, x_k) = \frac{s_{\lambda(N)}(x_1, \ldots, x_k, \overbrace{1, \ldots, 1}^{N-k})}{s_{\lambda(N)}(\underbrace{1, \ldots, 1}_{N})}$$

Fix k, let $N \to \infty$ and let

$$\frac{\lambda(N)_i}{N} \to f\left(\frac{i}{N}\right)$$



Limit shape of $\lambda(N)$ is $f(t)$:



Integral formula, $k = 1$ asymptotics

Theorem (G–P)

For any signature $\lambda \in \mathbb{GT}_N$ and any $x \in \mathbb{C}$ other than 0 or 1 we have

$$S_\lambda(x; N, 1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi \mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz,$$

where the contour C includes all the poles of the integrand.

(Similar statements hold for a larger class of functions, e.g symplectic characters, Jacobi...also q–analogues; formula appears also in [Colomo,Pronko,Zinn-Justin])

Let $\dfrac{\lambda(N)_i}{N} \to f\left(\dfrac{i}{N}\right)$ under certain convergence conditions...

using the method of steepest descent we obtain various asymptotic formula:

Theorem (G–P)

Under [certain strong convergence conditions of] $\frac{\lambda(N)}{N}$ towards the limit shape f, as $N \to \infty$:

$$S_{\lambda(N)}(e^y; N, 1) = G(w_0, f) \frac{\exp(N(yw_0 - \mathcal{F}(w_0; f)))}{e^N (e^y - 1)^{N-1}} \Big(1 + o(1)\Big),$$

where $\mathcal{F}(w; f) = \int_0^1 \ln(w - f(t) - 1 + t) dt$, w_0 is the root of $\frac{\partial}{\partial w}\mathcal{F}(w; f) = y$ (inverse Hilbert transform) and G is a certain explicit function.

Integral formula, $k = 1$ asymptotics

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Let $\frac{\lambda(N)_i}{N} \to f\left(\frac{i}{N}\right)$ under certain convergence conditions...

using the method of steepest descent we obtain various asymptotic formula:

Theorem (G–P)

Under [some other convergence conditions of] $\frac{\lambda(N)}{N}$ towards the limit shape f, as $N \to \infty$

$$S_{\lambda(N)}(e^{h/\sqrt{N}}; N, 1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

where $E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$

From $k=1$ asymptotics to general k, multiplicativity

Theorem (G–P)

For any signature $\lambda \in \mathbb{GT}_N$ and any $k \le N$ we have

$$S_\lambda(x_1,\ldots,x_k;N) = \frac{s_\lambda(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} = \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times$$

$$\frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

where $D_{i,1} = x_i \frac{\partial}{\partial x_i}$ and $\Delta-$ Vandermonde determinant.

Similar theorems for symplectic characters, Jacobi; also q-analogues (replacing derivatives by q-shifts).
Note: appears in *[de Gier, Nienhuis, Ponsaing]* for symplectic characters.

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where $D_{i,1} = x_i \frac{\partial}{\partial x_i}$ and $\Delta-$ Vandermonde determinant.

Corollary (G–P)

Suppose that the sequence $\lambda(N)$ is such that

$$\lim_{N \to \infty} \frac{\ln \left(S_{\lambda(N)}(x; N, 1) \right)}{N} = \Psi(x)$$

uniformly on compact subsets of a region $M \subset \mathbb{C}$ (e.g. Theorem 2). Then

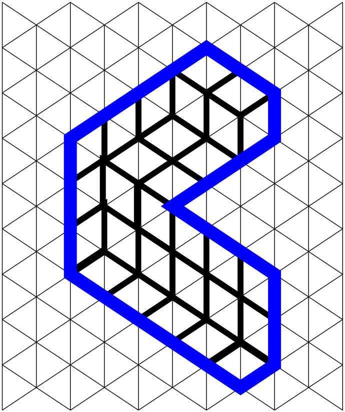
$$\lim_{N \to \infty} \frac{\ln \left(S_{\lambda(N)}(x_1, \ldots, x_k; N, 1) \right)}{N} = \Psi(x_1) + \cdots + \Psi(x_k)$$

for any k uniformly on compact subsets of M^k.

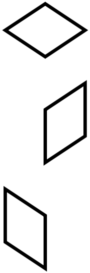
I.e., informally, under various regimes of convergence for $\lambda(N)$ we have

$$S_{\lambda(N)}(x_1, \ldots, x_k) \simeq S_{\lambda(N)}(x_1) \cdots S_{\lambda(N)}(x_k)$$

Lozenge tilings



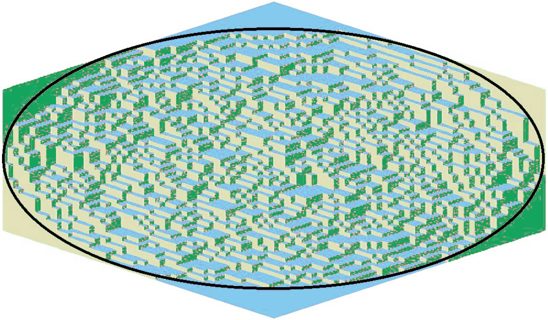
Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?

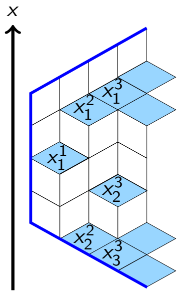
A well-known example: boxed plane partitions

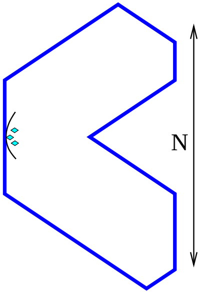
(Cohn–Larsen–Propp, 1998) Tiling is asymptotically *frozen* outside inscribed ellipse



(Kenyon–Okounkov, 2005) For general polygonal domain tiling is asymptotically frozen outside inscribed algebraic curve.

Behavior near the boundary, interlacing particles





Horizontal lozenges near a straight vertical segment of the boundary form an interlacing particle configuration $\leftrightarrow$ Gelfand-Tsetlin schemes.

$$\begin{array}{ccccc} & & x_1^1 & & \\ & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} & \\ & x_2^2 & & x_1^2 & \\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$}\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} \\ x_3^3 & & x_2^3 & & x_1^3 \end{array}$$

Question: What is the joint probability distribution of the positions of the horizontal lozenges near the boundary as $N \to \infty$ (scale $= \frac{1}{N}$)?

Conjecture ([Okounkov–Reshetikhin, 2006] with an explanation what the answer should be):
The joint distribution converges to a *GUE*-corners (aka *GUE*-minors [Johansson-Nordenstam]) process.

Gaussian Unitary Ensemble (GUE)

Gaussian Unitary Ensemble of rank N is the distribution on the set of $N \times N$ *Hermitian* matrices with density

$$\rho(X) \sim \exp\left(- \mathrm{Trace}(X^2)/2 \right).$$

Alternatively,
$\mathrm{Re}X_{ij}, \mathrm{Im}X_{ij}$ are i.i.d. with $\rho \sim \mathcal{N}(0, 1/2)$ for $i \neq j$ and
X_{ii} are i.i.d. with $\rho \sim \mathcal{N}(0, 1)$

The density of the eigenvalues of X, denoted $x_1^N, \ldots, x_N^N$, is (Weyl, 20-30s)

$$\rho(x_1^N, \ldots, x_N^N) \sim \prod_{i<j} (x_i^N - x_j^N)^2 \prod_{i=1}^{N} e^{-(x_i^N)^2/2}.$$

GUE–corners

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

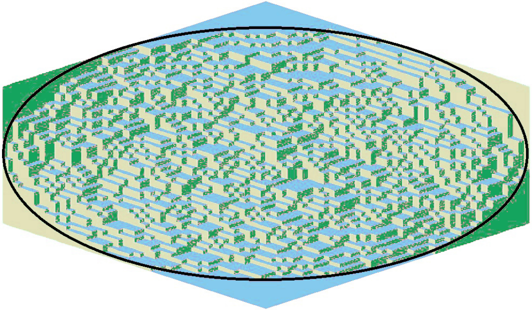
Let x_i^k be ith eigenvalue of top–left $k \times k$ corner of GUE. Interlacing condition: $x^j_{i-1} \leq x^{j-1}_{i-1} \leq x^j_i$

$$\begin{matrix} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \geq & & x_1^2 & & x_2^2 & & \leq \\ & & & x_1^1 & & & \end{matrix}$$

The joint distribution of x_i^j is known as *GUE–corners (also, GUE-minors) process*, denoted $\mathbb{GUE}_k$ for the top k corners.

Given $x_1^N, \ldots, x_N^N$, the distribution of x_i^j, $j < N$ is *uniform* on the polytope defined by interlacing conditions (Baryshnikov, 2001)

GUE in tilings: known cases



Theorem.[Johansson–Nordenstam, 2006; Nordenstam, 2009] For a hexagonal domain the fluctuations near the point where the inscribed ellipse touches the boundary are of order $\sqrt{N}$ and after rescaling the point process formed by the positions of one type of lozenges ("horizontal" for the vertical boundary) converges to GUE–minors process.

Method: Computation based on Lindström-Gessel-Viennot formula for the number of non-intersecting paths + certain determinant evaluations.

Other results: Okounkov–Reshetikhin, 2006, using determinantal point processes (in particular, the Schur process). Petrov, 2012, finite polygonal domains.

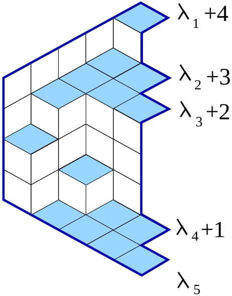
GUE in tilings: our results

GUE-minors convergence conjecture for a wide class of domains. Domain $\Omega_{N,\lambda(N)}$, parameterized by width N and the positions

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$

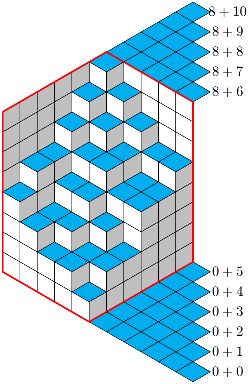
of its N horizontal lozenges at the right boundary.

Tiling Ω_λ = tiling certain *polygon*.



$(N = 5,\ \lambda(5) = (4, 3, 3, 0, 0))$

Note: $\frac{1}{N}\Omega_{N,\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$, e.g. $\lambda(N) = (N, N-1, \ldots, 2, 1)$



E.g. $\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$

$\leftrightarrow$ the hexagon with side lengths (a, b, c, a, b, c).

GUE in tilings: our results

Domain $\Omega_{N,\lambda(N)}$, parameterized by width N and the positions

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$

of its N horizontal lozenges at the right boundary.

Theorem (G–P)

Let $\lambda(N) = (\lambda_1(N) \geq \ldots \geq \lambda_N(N))$, $N = 1, 2, \ldots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

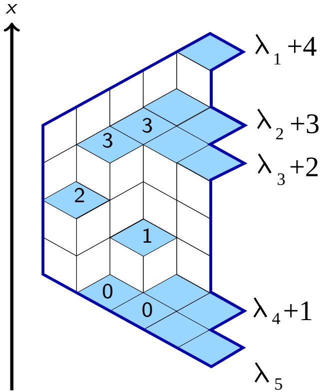
as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Let $\Upsilon(N)^k = \{x_i^j\}$ be the collection of the positions of the horizontal lozenges on lines $j = 1, \ldots, k$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k \text{ (GUE-corners process of rank } k\text{)}$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$$

GUE in tilings: our method, bijections



Tilings of domain $\Omega_{\lambda(N)}$

$\Leftrightarrow$ Gelfand-Tsetlin schemes with bottom row $\lambda(N)$

				2				
			0		3			
		0		1		3		
	0		0		3		3	
0		0		3		3		4

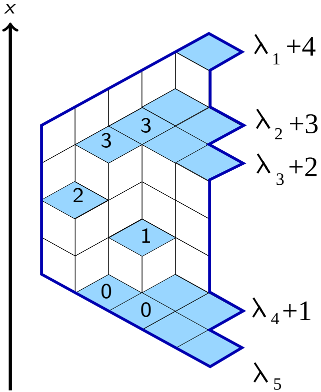
$\Leftrightarrow$ Semi-Standard Young Tableaux of shape $\lambda(N)$

$T=$

1	1	2	5
3	4	4	
5	5	5	

Positions of the horizontal lozenges on line j:
x^j –shape of subtableaux of T comprised of the entries $1,\ldots,j$.

GUE in tilings: our method, bijections



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$$\begin{array}{ccccccccc} &&&& 2 &&&& \\ &&& 0 && 3 &&& \\ && 0 && 1 && 3 && \\ & 0 && 0 && 3 && 3 & \\ 0 && 0 && 3 && 3 && 4 \end{array}$$

$\Leftrightarrow$ Semi-Standard Young Tableaux of shape $\lambda(N)$

$T=$

1	1	2	5
3	4	4	
5	5	5	

$x^3 = (3, 1, 0)$.

Positions of the horizontal lozenges on line j:
x^j –shape of subtableaux of T comprised of the entries $1, \ldots, j$.

GUE in tilings: our method, moment generating functions

Proposition

In a uniformly random tiling of Ω_λ the distribution of the positions of the horizontal lozenges on the kth line $x^k(\lambda)$ is given by:

$$\mathrm{Prob}\{x^k(\lambda)=\eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

where $s_{\lambda/\eta}$ is the skew Schur polynomial.

Proof: combinatorial definition of Schur functions as sums over SSYTs.

Proposition

Let ν^k be the positions of the horizontal lozenges on the kth vertical line in a uniformly random tiling of Ω_λ (where λ has length N).

$$\mathbb{E}\left(\frac{s_{\nu^k}(y_1,\ldots,y_k)}{s_{\nu^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} = S_\lambda(y_1,\ldots,y_k).$$

GUE in tilings: MGF and asymptotics

Proposition

$$\mathbb{E}B_k(x; \mathbb{GUE}_k) = \exp\left(\frac{1}{2}(x_1^2 + \cdots + x_k^2)\right),$$

where $B_k(x; y) = \dfrac{\det\left[\exp(x_i y_j)\right]_{i,j=1}^k}{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)} \prod_{i<j}(j - i)$ *, also*

$= \dfrac{s_{y-\delta_k}(x_1, \dots, x_k)}{s_{y-\delta_k}(\underbrace{1, \dots, 1}_{k})}$ *when* y *— strict partition.*

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$$= \frac{s_{y-\delta_k}(x_1,\ldots,x_k)}{s_{y-\delta_k}(\underbrace{1,\ldots,1}_{k})}$$ *when* y *— strict partition.*

Proposition (G–P)

For any k *reals* $h_1,\ldots,h_k$ *we have:*

$$\lim_{N\to\infty}\frac{s_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}},1^{N-k}\right)}{s_{\lambda(N)}(1^N)}\exp\left(-\frac{E(f)}{\sqrt{NS(f)}}(h_1+\cdots+h_k)\right)$$
$$= \exp\left(\frac{1}{2}(h_1^2+\cdots+h_k^2)\right).$$

GUE in tilings: MGF and asymptotics

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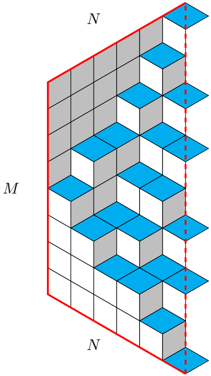
Theorem. $\dfrac{\Upsilon^k_{\lambda(N)}-NE(f)}{\sqrt{NS(f)}}\to\mathbb{GUE}_k$ (GUE-corners process of rank k). □

Free boundary



Limit shapes: [Di Francesco, Reshetikhin, 2009]
Let

$$T_f(N,M) := \cup_{\lambda \mid \ell(\lambda)=N, \lambda_1 \leq M} \text{tilings of } \Omega_\lambda,$$

i.e. the set of all tilings in an $N \times M \times N$ trapezoid with unrestricted positions of right horizontal lozenges.
$\Leftrightarrow$ Vertically symmetric tilings of the $N \times M \times N \times N \times M \times N$ hexagon.

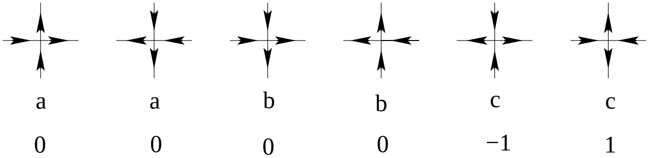
Theorem (P, –)

Let $\Upsilon^k_{N,M}$ denote the positions of the horizontal lozenges $\{x^i_j\}$ on the ith vertical line of a uniformly random tiling from $T_f(N,M)$. Then, as $N \to \infty$ and $\frac{M}{N} \to a$, where $0 < a < \infty$,

$$\frac{\Upsilon^k_{N,M} - M/2}{\sqrt{N(a^2+2a)/8}} \to \mathbb{GUE}_k.$$

6 Vertex model / ASM

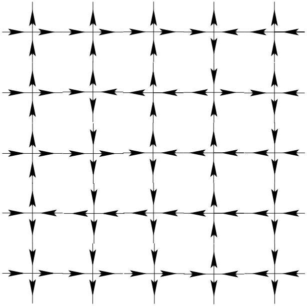
Six vertex types:



Alternating Sign Matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A 6 vertex model configuration:



Definitions and background on ASMs

Definition: An *Alternating Sign Matrix* of size n is an $n \times n$ matrix of 0s, 1s, -1s, such that the sum in each row or column is 1 and 1s and -1s alternate in each row or column. A *monotone triangle* is a Gelfand-Tsetlin pattern, s.t. the inequalities on each row are *strict*.

6 Vertex model $\leftrightarrow$ ASM $\leftrightarrow$ monotone triangles.

Uniform measure on ASMs $\leftrightarrow$ all vertices in 6V model have equal weight ("ice").

ASM: $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

positions of 1s $\Updownarrow$ in sum of first k rows

Monotone triangle:

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 2 & & 3 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Question: What does a uniformly random ASM look like as $n \to \infty$? What is the *distribution* of *the positions of the 1s and -1s near the boundary* of the ASM in the limit $\Leftrightarrow$ Distribution of *the numbers in the top k rows* of the monotone triangle?

Known results:
Limit behavior: [Behrend], [Colomo, Pronko, [[Zinn-Justin]], Di Francesco].
Free fermions point (weight 2 at 1,-1) $\leftrightarrow$ domino tilings, Aztec diamond.
Exact generating functions for certain statistics (e.g. positions of 1s on boundary, etc).

ASMs/6Vertex: new results

ASM A: $\quad \Psi_k(A) := \sum_{j=1:n\ ,\ A_{kj}=1} j - \sum_{j=1:n\ ,\ A_{kj}=-1} j$

Monotone triangle $M = [m^i_j]_{j\leq i}$: $\quad \Psi_k(M) = \sum_{j=1}^{k} m^k_j - \sum_{j=1}^{k-1} m^{k-1}_j$

$$k: \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{matrix} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 2 & & 3 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{matrix}$$

2 $\quad \Psi_2 = 2 + 5 - 4 = 3 \qquad \Psi_2 = (2+5) - (4) = 3$

3 $\quad \Psi_3 = 3 \qquad \Psi_3 = (2+3+5) - (2+5)$

ASMs/6Vertex: new results

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$\Psi_k(n)$ – the random variable $\Psi_k(A)$ as A is chosen uniformly random from ASMs of size n.

Theorem (G–P)

$\frac{\Psi_k(n)-n/2}{\sqrt{n}}$, $k = 1, 2, \ldots$ converge as $n \to \infty$ to the collection of i.i.d. Gaussian random variables, $N(0, \sqrt{3/8})$.

ASMs/6Vertex: new results

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Using this Theorem on $\Psi_k(n)$ and the Gibbs property:

Theorem (G, 2013; Conjecture in [G–P])

Fix any k*. As* $n \to \infty$ *the probability that the number of* (-1)*s in the first* k *rows of uniformly random ASM of size* n *is maximal tends to* 1*, and, thus,* 1*s in first* k *rows are interlacing. After centering and rescaling the distribution of the positions of* 1*s tends to GUE-corners process, i.e. top* k *rows of the monotone triangle* M *converge to the GUE-corners process:*

$$\sqrt{\frac{8}{3n}}\left([M]_{i=1:k} - \frac{n}{2}\right) \to \mathbb{GUE}_k.$$

6Vertex/ASMs: proofs

Vertex at position (i, j) and its weight (corresponding to the type):

$$a: \ q^{-1}u_i^2 - qv_j^2, \quad b: \ q^{-1}v_j^2 - qu_i^2, \quad c: \ (q^{-1} - q)u_i v_j$$

where $v_1, \ldots, v_N$, $u_1, \ldots, u_N$ are parameters, $q = \exp(\pi \mathbf{i}/3)$
Weight $W(\vartheta)$ of a configuration θ = product of weights of its vertices.
Set $\lambda(N) := (N-1, N-1, N-2, N-2, \ldots, 1, 1, 0, 0) \in \mathbb{GT}_{2N}$.

Proposition (Okada;Stroganov)

Let $\beth_N$ be the set of all 6Vertex configurations on an $N \times N$ grid.

$$\sum_{\vartheta \in \beth_N} W(\vartheta) = (-1)^{N(N-1)/2}(q^{-1}-q)^N \prod_{i=1}^{N} (v_i u_i)^{-1} s_{\lambda(N)}(u_1^2, \ldots, u_N^2, v_1^2, \ldots, v_N^2).$$

6Vertex/ASMs: proofs

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Set $\lambda(N) := (N-1,N-1,N-2,N-2,\ldots,1,1,0,0)\in\mathbb{GT}_{2N}$.

Proposition

Let $\widehat{x}_i$ be the number of vertices of type x on row i, then for any collection of rows $i_1,\ldots,i_m$ we have

$$\mathbb{E}_N\prod_{\ell=1}^{m}\left[\left(\frac{q^{-1}-qv_\ell^2}{q^{-1}-q}\right)^{\widehat{a}_{i_\ell}}\left(\frac{q^{-1}v_\ell^2-q}{q^{-1}-q}\right)^{\widehat{b}_{i_\ell}}(v_\ell)^{\widehat{c}_{j_\ell}}\right]$$

$$=\left(\prod_{\ell=1}^{n}v_\ell^{-1}\right)\frac{s_{\lambda(N)}(v_1,\ldots,v_m,1^{2N-m})}{s_{\lambda(N)}(1^{2N})}=\left(\prod_{\ell=1}^{n}v_\ell^{-1}\right)S_{\lambda(N)}(v_1,\ldots,v_m)$$

6Vertex/ASMs: proofs

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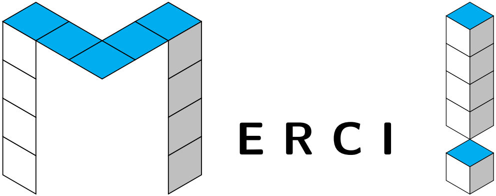
$$= \left(\prod_{\ell=1}^{n} v_\ell^{-1} \right) \frac{s_{\lambda(N)}(v_1, \dots, v_m, 1^{2N-m})}{s_{\lambda(N)}(1^{2N})} = \left(\prod_{\ell=1}^{n} v_\ell^{-1} \right) S_{\lambda(N)}(v_1, \dots, v_m)$$

Proof of Theorem: Use Proposition to derive the *moment generating function* as a Schur function. Choose parameters wisely to extract the main statistic and apply the asymptotics:

$$S_{\lambda(N)}(e^{y_1/\sqrt{n}}, \dots, e^{y_k/\sqrt{n}}) = \prod_{i=1}^{k} \exp\left[\sqrt{n}y_i + \frac{5}{12}y_i^2 + o(1) \right]$$


E R C I

Extreme Characters of $U(\infty)$

$U(N)$ – the group of $N \times N$ unitary matrices. $U(\infty) = \bigcup_{N=1}^{\infty} U(N)$. A (normalized) *character* of a group G is a continuous function $\chi(g)$, $g \in G$ s.t.:

1. $\chi(aba^{-1}) = \chi(b)$ for any $a, b \in G$,
2. χ is positive definite, i.e. the matrix $\left[\chi(g_i g_j^{-1})\right]_{i,j=1}^{k}$ is Hermitian non-negative definite, for any $\{g_1, \ldots, g_k\}$,
3. $\chi(e) = 1$.

An *extreme character* is an extreme point of the convex set of all characters. The normalized characters of $U(N)$ are the functions

$$\frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(1, \ldots, 1)}.$$

Extreme Characters of $U(\infty)$

$U(N)$ – the group of $N \times N$ unitary matrices. $U(\infty) = \bigcup_{N=1}^{\infty} U(N)$.

Theorem (Voiculescu-Edrei classification)

The extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spec}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}.$$

Extreme characters of $U(\infty)$

Proposition (Kerov-Vershik)

Every extreme normalized character χ of $U(\infty)$ is a uniform limit of extreme characters of $U(N)$. In other words, for every χ there exists a sequence $\lambda(N)\in\mathbb{GT}_N$ such that for every k

$$\chi(u_1,\ldots,u_k,1,\ldots)=\lim_{N\to\infty}S_\lambda(u_1,\ldots,u_k;N,1)$$

uniformly on the torus $(S_1)^k$.

Based on this fact we show which sequences approximate characters of $U(\infty)$:

For any λ set $p_i=\lambda_i-i+1/2,\quad q_i=\lambda'_i-i+1/2,\quad i=1,\ldots,d.$

$$\chi^{(\omega)}(u_1,u_2,\ldots)=\prod_j e^{\gamma^+(u_j-1)+\gamma^-(u_j^{-1}-1)}\prod_{i=1}^{\infty}\frac{1+\beta_i^+(u_j-1)}{1-\alpha_i^+(u_j-1)}\frac{1+\beta_i^-(u_j^{-1}-1)}{1-\alpha_i^-(u_j^{-1}-1)}.$$

Theorem (VK, OO, BO, P, Gorin-Panova)

Let $\omega=(\alpha^\pm,\beta^\pm,;\delta^\pm)$ and suppose that the sequence $\lambda(N)\in\mathbb{GT}_N$ is s.t.

$$p_i^+(N)/N\to\alpha_i^+,\quad p_i^-(N)/N\to\alpha_i^-,\quad q_i^+(N)/N\to\beta_i^+,\quad q_i^-(N)/N\to\beta_i^+,$$

$$|\lambda^+|/N\to\delta^+,\quad |\lambda^-|/N\to\delta^-.$$

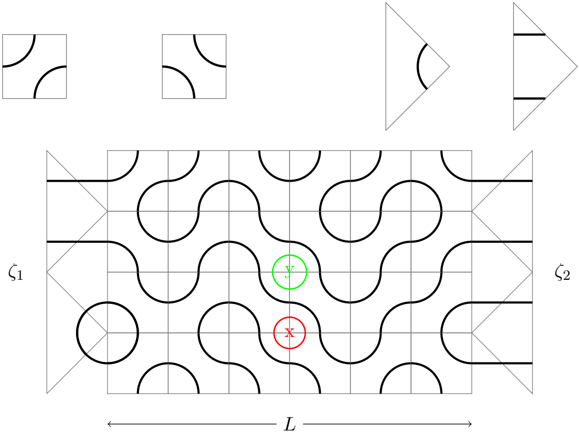
Then for every k

$$\chi(u_1,\ldots,u_k,1,\ldots)=\lim_{N\to\infty}S_{\lambda(N)}(u_1,\ldots,u_k;N,1)=\chi^\omega(u_1,\ldots,u_k,1,\ldots)\textit{(as defined above)}$$

uniformly on torus $(S_1)^k$.

The dense loop model

Given a finite grid (in this case, vertical strip of width L), each square is one of two kinds below, on the boundary – one of the triangles



The mean total current between two points x and y $F^{x,y}$ – the average number of paths connecting both boundaries and passing between x and y. Similar observables in the critical percolation model [Smirnov, 2009].

Dense loop model: the mean current

Let $\lambda^L = (\lfloor \frac{L-1}{2} \rfloor, \lfloor \frac{L-2}{2} \rfloor, \ldots, 1, 0, 0)$
Define:

$$u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L) = (-1)^L \imath \frac{\sqrt{3}}{2} \ln \left[\frac{\chi_{\lambda^{L+1}}(\zeta_1^2, z_1^2, \ldots, z_L^2)\chi_{\lambda^{L+1}}(\zeta_2^2, z_1^2, \ldots, z_L^2)}{\chi_{\lambda^L}(z_1^2, \ldots, z_L^2)\chi_{\lambda^{L+2}}(\zeta_1^2, \zeta_2^2, z_1^2, \ldots, z_L^2)} \right]$$

where χ_ν is the character for the irreducible representation of highest weight ν of the symplectic group $Sp(\mathbb{C})$.

$$X_L^{(j)} = z_j \frac{\partial}{\partial z_j} u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L)$$

$$Y_L = w \frac{\partial}{\partial w} u_{L+2}(\zeta_1, \zeta_2; z_1, \ldots, z_L, vq^{-1}, w)|_{v=w},$$

Proposition (De Gier, Nienhuis, Ponsaing)

Under certain assumptions the mean total current between two horizontally adjacent points is

$$X_L^{(j)} = F^{(j,i),(j+1,i)},$$

and Y is the mean total current between two vertically adjacent points in the strip of width L:

$$Y_L^{(j)} = F^{(j,i),(j,i+1)}.$$

Dense loop model: asymptotics of the mean current

Theorem

As $L \to \infty$ we have

$$X_L^{(j)}\Big|_{z_j=z;\, z_i=1,\ i\neq j} = \frac{\mathbf{i}\sqrt{3}}{4L}(z^3 - z^{-3}) + o\left(\frac{1}{L}\right)$$

and

$$Y_L\Big|_{z_i=1,\ i=1,\ldots,L} = \frac{\mathbf{i}\sqrt{3}}{4L}(w^3 - w^{-3}) + o\left(\frac{1}{L}\right)$$

Remark 1. When $z=1$, $X_L^{(j)}$ is identical zero and so is our asymptotics.
Remark 2. The fully homogeneous case corresponds to $w = \exp^{-\mathbf{i}\pi/6}$, $q = e^{2\pi\mathbf{i}/3}$. In this case

$$Y_L = \frac{\sqrt{3}}{2L} + o\left(\frac{1}{L}\right).$$

Proof: same type of asymptotic methods and results hold for symplectic characters + some tricks with the multivariate formula.