Asymptotics of symmetric functions with applications to statistical mechanics and representation theory

Greta Panova (UCLA)

Normalized Schur functions $S_{\lambda}$

Setup

Asymptotics of $S_{\lambda}(N)(x_1, \ldots, x_k)$

GUE in random lozenge tilings

Lozenge tilings

$N \to \infty$, behavior near boundary

GUE

GUE in tilings, results

ASM

GUE in ASMs

based on same-name paper arXiv:1301.0634 joined with Vadim Gorin

FPSAC 2013, Paris
Overview

Normalized Schur functions $S_\lambda$:

\[ S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1^{N-k})}{s_\lambda(1^N)} \]

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & \ldots \\
2 & 2 & 3 & \ldots & & \\
\ldots & & & & & \\
\end{array}
\]

Alternating Sign Matrices (ASM)/ 6Vertex model:

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Lozenge tilings:

Dense loop model:

ASM

GUE in ASM:

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Asymptotics of symmetric functions with applications to statistical mechanics and representation theory

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Definitions and setup

In our context: Symmetric functions, Lie groups characters.

(mainly) **Schur functions:** $s_{\lambda}(x_1, \ldots, x_N) = \text{characters of } V_{\lambda}$. 

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In our context: Symmetric functions, Lie groups characters.

Irreducible (rational) representations $V_\lambda$ of $GL(N)$ (or $U(N)$) are indexed by dominant weights (signatures/Young diagrams/integer partitions) $\lambda$:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,$$

where $\lambda_i \in \mathbb{Z}$, e.g. $\lambda = (4, 3, 1)$, 

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

(mainly) Schur functions: $s_\lambda(x_1, \ldots, x_N)$ – characters of $V_\lambda$. 

\[
\begin{array}{ccc}
& & \\
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where $\lambda_i \in \mathbb{Z}$, e.g. $\lambda = (4, 3, 1)$.

(maily) **Schur functions**: $s_\lambda(x_1, \ldots, x_N)$ – characters of $V_\lambda$.

Weyl’s determinantal formula:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[ x_i^{\lambda_j+N-j} \right]_i^j}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux ($\leftrightarrow$ Gelfand-Tsetlin patterns) of shape $\lambda$:

$$s_{(2,2)}(x_1, x_2, x_3) = s_{(1,1,1,1)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$
Definitions and setup

Object of study and main tool in the applications:

**Normalized Schur functions:**

\[ S_{\lambda(N)}(x_1, \ldots, x_k) = \frac{s_{\lambda(N)}(x_1, \ldots, x_k, 1, \ldots, 1)}{s_{\lambda(N)}(1, \ldots, 1)} \]

Fix \( k \), let \( N \to \infty \) and let

\[ \frac{\lambda(N)_i}{N} \to f \left( \frac{i}{N} \right) \]

Limit shape of \( \lambda(N) \) is \( f(t) \):
Integral formula, \( k = 1 \) asymptotics

**Theorem (G–P)**

*For any signature \( \lambda \in \mathbb{GT}_N \) and any \( x \in \mathbb{C} \) other than 0 or 1 we have*

\[
S_\lambda(x; N, 1) = \frac{(N - 1)!}{(x - 1)^{N-1}} \frac{1}{2\pi i} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz,
\]

*where the contour \( C \) includes all the poles of the integrand.*

(Similar statements hold for a larger class of functions, e.g symplectic characters, Jacobi...also \( q \)-analogues; formula appears also in [Colomo,Pronko,Zinn-Justin])

Let \( \frac{\lambda(N)i}{N} \rightarrow f \left( \frac{i}{N} \right) \) under certain convergence conditions...

using the method of steepest descent we obtain various asymptotic formula:

**Theorem (G–P)**

*Under [certain strong convergence conditions of] \( \frac{\lambda(N)}{N} \) towards the limit shape \( f \), as \( N \rightarrow \infty \):*

\[
S_{\lambda(N)}(e^y; N, 1) = G(w_0, f) \frac{\exp(N(yw_0 - F(w_0; f)))}{e^N(e^y - 1)^{N-1}} \left( 1 + o(1) \right),
\]

*where \( F(w; f) = \int_0^1 \ln(w - f(t) - 1 + t)dt \), \( w_0 \) is the root of \( \frac{\partial}{\partial w} F(w; f) = y \) (inverse Hilbert transform) and \( G \) is a certain explicit function.*
Integral formula, $k = 1$ asymptotics

**Theorem (G–P)**

*For any signature $\lambda \in GT_N$ and any $x \in \mathbb{C}$ other than 0 or 1 we have*

$$S_\lambda(x; N, 1) = \frac{(N - 1)!}{(x - 1)^{N-1}} \frac{1}{2\pi i} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz,$$

*where the contour $C$ includes all the poles of the integrand.*

(Similar statements hold for a larger class of functions, e.g symplectic characters, Jacobi...also $q$–analogues; formula appears also in [Colomo,Pronko,Zinn-Justin])

Let $\frac{\lambda(N)}{N} \to f \left( \frac{i}{N} \right)$ under certain convergence conditions...

using the method of steepest descent we obtain various asymptotic formula:

**Theorem (G–P)**

*Under [some other convergence conditions of] $\frac{\lambda(N)}{N}$ towards the limit shape $f$, as $N \to \infty$*

$$S_{\lambda(N)}(e^{h/\sqrt{N}}; N, 1) = \exp \left( \sqrt{N} E(f) h + \frac{1}{2} S(f) h^2 + o(1) \right),$$

*where $E(f) = \int_0^1 f(t) dt$, $S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1 - 2t) dt.$*
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From \( k = 1 \) asymptotics to general \( k \), multiplicativity

Theorem (G–P)

For any signature \( \lambda \in GT_N \) and any \( k \leq N \) we have

\[
S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1, \ldots, 1)}{s_\lambda(1, \ldots, 1)} = \prod_{i=1}^{N-k} \frac{(N-i)!}{(N-1)!(x_i - 1)^{N-k}} \times
\]

\[
\det \left[ D_{i,1}^{j-1} \right]_{i,j=1}^{k} \cdot \Delta(x_1, \ldots, x_k) \prod_{j=1}^{k} S_\lambda(x_j; N, 1)(x_j - 1)^{N-1}.
\]

where \( D_{i,1} = x_i \frac{\partial}{\partial x_i} \) and \( \Delta \) – Vandermonde determinant.

Similar theorems for symplectic characters, Jacobi; also \( q \)-analogues (replacing derivatives by \( q \)-shifts).

Note: appears in [de Gier, Nienhuis, Ponsaing] for symplectic characters.
From $k=1$ asymptotics to general $k$, multiplicativity

**Theorem (G–P)**

For any signature $\lambda \in \mathcal{GT}_N$ and any $k \leq N$ we have

$$S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1, \ldots, 1)}{s_\lambda(1, \ldots, 1)} = \prod_{i=1}^{N-k} \frac{(N-i)!}{(N-1)!(x_i - 1)^{N-k}} \times$$

$$\frac{\det \left[D_{j,1}^{i-1}\right]_{i,j=1}^{k}}{\Delta(x_1, \ldots, x_k)} \prod_{j=1}^{k} S_\lambda(x_j; N, 1)(x_j - 1)^{N-1}.$$

where $D_{i,1} = x_i \frac{\partial}{\partial x_i}$ and $\Delta$ – Vandermonde determinant.

**Corollary (G–P)**

Suppose that the sequence $\lambda(N)$ is such that

$$\lim_{N \to \infty} \frac{\ln \left(S_\lambda(N)(x; N, 1)\right)}{N} = \Psi(x)$$

uniformly on compact subsets of a region $M \subset \mathbb{C}$ (e.g. Theorem 2). Then

$$\lim_{N \to \infty} \frac{\ln \left(S_\lambda(N)(x_1, \ldots, x_k; N, 1)\right)}{N} = \Psi(x_1) + \cdots + \Psi(x_k)$$

for any $k$ uniformly on compact subsets of $M^k$.

I.e., informally, under various regimes of convergence for $\lambda(N)$ we have

$$S_\lambda(N)(x_1, \ldots, x_k) \simeq S_\lambda(N)(x_1) \cdots S_\lambda(N)(x_k)$$
Lozenge tilings

Tilings of a domain $\Omega$ (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").

**Question:** Fix $\Omega$ in the plane and let grid size $\to 0$, what are the properties of uniformly random tilings of $\Omega$?
A well-known example: boxed plane partitions

(Cohn–Larsen–Propp, 1998) Tiling is asymptotically frozen outside inscribed ellipse

(Kenyon–Okounkov, 2005) For general polygonal domain tiling is asymptotically frozen outside inscribed algebraic curve.
Behavior near the boundary, interlacing particles

Horizontal lozenges near a straight vertical segment of the boundary form an interlacing particle configuration $\Leftrightarrow$ Gelfand-Tsetlin schemes.

Question: What is the joint probability distribution of the positions of the horizontal lozenges near the boundary as $N \to \infty$ (scale $= \frac{1}{N}$)?

Conjecture ([Okounkov–Reshetikhin, 2006] with an explanation what the answer should be):
The joint distribution converges to a $GUE$-corners (aka $GUE$-minors [Johansson-Nordenstam]) process.
Gaussian Unitary Ensemble (GUE)

Gaussian Unitary Ensemble of rank $N$ is the distribution on the set of $N \times N$ Hermitian matrices with density

$$\rho(X) \sim \exp \left(-\text{Trace}(X^2)/2\right).$$

Alternatively, $\text{Re}X_{ij}, \text{Im}X_{ij}$ are i.i.d. with $\rho \sim \mathcal{N}(0, 1/2)$ for $i \neq j$ and $X_{ii}$ are i.i.d. with $\rho \sim \mathcal{N}(0, 1)$

The density of the eigenvalues of $X$, denoted $x_1^N, \ldots, x_N^N$, is (Weyl, 20-30s)

$$\rho(x_1^N, \ldots, x_N^N) \sim \prod_{i<j} (x_i^N - x_j^N)^2 \prod_{i=1}^N e^{-(x_i^N)^2/2}.$$
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GUE–corners

Let $x^k_i$ be $i$th eigenvalue of top–left $k \times k$ corner of GUE. Interlacing condition: $x^j_{i-1} \leq x^j_{i-1} \leq x^j_i$

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}
\]

The joint distribution of $x^j_i$ is known as GUE–corners (also, GUE-minors) process, denoted $\text{GUE}_k$ for the top $k$ corners.

Given $x^N_1, \ldots, x^N_N$, the distribution of $x^j_i, j < N$ is uniform on the polytope defined by interlacing conditions (Baryshnikov, 2001)
GUE in tilings: known cases

**Theorem.** [Johansson–Nordenstam, 2006; Nordenstam, 2009] For a hexagonal domain the fluctuations near the point where the inscribed ellipse touches the boundary are of order $\sqrt{N}$ and after rescaling the point process formed by the positions of one type of lozenges (“horizontal” for the vertical boundary) converges to $GUE$–minors process.

**Method:** Computation based on Lindström-Gessel-Viennot formula for the number of non-intersecting paths + certain determinant evaluations.

**Other results:** Okounkov–Reshetikhin, 2006, using determinantal point processes (in particular, the Schur process). Petrov, 2012, finite polygonal domains.
GUE in tilings: our results

GUE-minors convergence conjecture for a wide class of domains. Domain \( \Omega_{N,\lambda(N)} \), parameterized by width \( N \) and the positions

\[
\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N
\]

of its \( N \) horizontal lozenges at the right boundary.

Tiling \( \Omega_\lambda = \text{tiling certain polygon} \).

\[
\lambda_1 + 4 \\
\lambda_2 + 3 \\
\lambda_3 + 2 \\
\lambda_4 + 1 \\
\lambda_5
\]

(\( N = 5, \lambda(5) = (4, 3, 3, 0, 0) \))

Note: \( \frac{1}{N} \Omega_{N,\lambda(N)} \) is not necessarily a finite polygon as \( N \to \infty \), e.g.
\( \lambda(N) = (N, N - 1, \ldots, 2, 1) \)

E.g. \( \lambda = (a, \ldots, a, 0, \ldots, 0) \)

\( \leftrightarrow \) the hexagon with side lengths \((a, b, c, a, b, c)\).
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**GUE in tilings: our results**

Domain $\Omega_{N, \lambda(N)}$, parameterized by width $N$ and the positions

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$

of its $N$ horizontal lozenges at the right boundary.

**Theorem (G–P)**

Let $\lambda(N) = (\lambda_1(N) \geq \ldots \geq \lambda_N(N))$, $N = 1, 2, \ldots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Let $\gamma(N)^k = \{x^i_j\}$ be the collection of the positions of the horizontal lozenges on lines $j = 1, \ldots, k$. Then for every $k$ as $N \to \infty$ we have

$$\frac{\gamma^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \text{GUE}_k \ (\text{GUE-corners process of rank } k)$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t) dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t) dt.$$
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GUE in tilings: our method, bijections

Tilings of domain \( \Omega_{\lambda(N)} \)

\( \Leftrightarrow \) Gelfand-Tsetlin schemes with bottom row \( \lambda(N) \)

\[
\begin{array}{cccc}
2 & 3 \\
0 & 1 & 3 & 3 \\
0 & 0 & 3 & 3 & 4 \\
\end{array}
\]

\( \Leftrightarrow \) Semi-Standard Young Tableaux of shape \( \lambda(N) \)

\[
T = \begin{pmatrix}
1 & 1 & 2 & 5 \\
3 & 4 & 4 \\
5 & 5 & 5 \\
\end{pmatrix}
\]

Positions of the horizontal lozenges on line \( j \):

\( x^j \) – shape of subtableaux of \( T \)

comprised of the entries \( 1, \ldots, j \).
GUE in tilings: our method, bijections

Tilings of domain $\Omega_{\lambda(N)}$
$\Leftrightarrow$ Gelfand-Tsetlin schemes with bottom row $\lambda(N)$

$\begin{array}{cccc}
& & 3 & 3 \\
\lambda_1 & +4 & 2 & \\
& 3 & 3 & \\
\lambda_2 & +3 & 0 & 3 \\
& 2 & 0 & 1 \\
\lambda_3 & +2 & 0 & 0 \\
& 1 & 0 & 0 \\
\lambda_4 & +1 & & \\
& 0 & & \\
\lambda_5 & & & \\
\end{array}$

$\Leftrightarrow$ Semi-Standard Young Tableaux of shape $\lambda(N)$

$T= \begin{array}{cccc}
1 & 1 & 2 & 5 \\
3 & 4 & 4 & \\
5 & 5 & 5 & \\
\end{array}$

$x^3 = (3, 1, 0)$.

Positions of the horizontal lozenges on line $j$:
$x^j$ --shape of subtableaux of $T$ comprised of the entries $1, \ldots, j$. 
GUE in tilings: our method, moment generating functions

Proposition

In a uniformly random tiling of $\Omega_\lambda$ the distribution of the positions of the horizontal lozenges on the $k$th line $x^k(\lambda)$ is given by:

$$\text{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_\lambda/\eta(1^{N-k})}{s_\lambda(1^N)},$$

where $s_\lambda/\eta$ is the skew Schur polynomial.

Proof: combinatorial definition of Schur functions as sums over SSYTs.

Proposition

Let $\nu^k$ be the positions of the horizontal lozenges on the $k$th vertical line in a uniformly random tiling of $\Omega_\lambda$ (where $\lambda$ has length $N$).

$$\mathbb{E}\left( \frac{s_{\nu^k}(y_1, \ldots, y_k)}{s_{\nu^k}(1, \ldots, 1)} \right) = \frac{s_\lambda(y_1, \ldots, y_k, 1^N)}{s_\lambda(1^{N})} = S_\lambda(y_1, \ldots, y_k).$$
GUE in tilings: MGF and asymptotics

Proposition

$$\mathbb{E} B_k(x; \text{GUE}_k) = \exp \left( \frac{1}{2} (x_1^2 + \cdots + x_k^2) \right),$$

where $B_k(x; y) = \frac{\det \left[ \exp(x_i y_j) \right]^{k}_{i,j=1}}{\prod_{i<j} (x_i - x_j) \prod_{i<j} (y_i - y_j) \prod_{i<j} (j - i)}$, also

$$= \frac{s_{y-\delta_k}(x_1, \ldots, x_k)}{s_{y-\delta_k}(1, \ldots, 1)}$$

when $y$ — strict partition.

Theorem.

$$\Upsilon_k \left[ \mathcal{N} \left( f \right) \right] \overset{N \to \infty}{\to} \text{GUE}_k (\text{GUE-corners process of rank } k).$$
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**Proposition**

$$\mathbb{E} B_k(x; GUE_k) = \exp \left( \frac{1}{2} (x_1^2 + \cdots + x_k^2) \right),$$

where $B_k(x; y) = \frac{\det \left[ \exp(x_i y_j) \right]_{i,j=1}^k}{\prod_{i<j}(x_i - x_j) \prod_{i<j}(y_i - y_j) \prod_{i<j}(j - i)}$, also

$$B_{\lambda}(x_1, \ldots, x_k) = \frac{s_{y-\delta_k}(x_1, \ldots, x_k)}{s_{y-\delta_k}(1^{k})} \text{ when } y \text{ — strict partition.}$$

**Proposition (G–P)**

For any $k$ reals $h_1, \ldots, h_k$ we have:

$$\lim_{N \to \infty} \frac{\prod_{i=1}^k e^{\frac{h_i}{\sqrt{NS(f)}}} \cdot 1^{N-k}}{s_{\lambda(N)}(1^N)} \exp \left( - \frac{E(f)}{\sqrt{NS(f)}} (h_1 + \cdots + h_k) \right)$$

$$= \exp \left( \frac{1}{2} (h_1^2 + \cdots + h_k^2) \right).$$
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GUE in tilings: MGF and asymptotics

**Proposition**

$$\mathbb{E}B_k(x; \text{GUE}_k) = \exp \left( \frac{1}{2} (x_1^2 + \cdots + x_k^2) \right),$$

where $B_k(x; y) = \frac{\det [\exp(x_i y_j)]_{i,j=1}^k}{\prod_{i<j} (x_i - x_j) \prod_{i<j} (y_i - y_j) \prod_{i<j} (j - i)}$, also

$$= \frac{s_y - \delta_k(x_1, \ldots, x_k)}{s_y - \delta_k(1, \ldots, 1)} \quad \text{when } y \text{ — strict partition.}$$

**Proposition (G–P)**

For any $k$ reals $h_1, \ldots, h_k$ we have:

$$\lim_{N \to \infty} \frac{s_\lambda(N) \left( e^{\frac{h_1}{\sqrt{NS(f)}}}, \ldots, e^{\frac{h_k}{\sqrt{NS(f)}}}, 1^{N-k} \right)}{s_\lambda(N)(1^N)} \exp \left( - \frac{E(f)}{\sqrt{NS(f)}} (h_1 + \cdots + h_k) \right)$$

$$= \exp \left( \frac{1}{2} (h_1^2 + \cdots + h_k^2) \right).$$

**Theorem.** $\frac{\gamma^k_\lambda(N) - NE(f)}{\sqrt{NS(f)}} \to \text{GUE}_k \ (\text{GUE-corners process of rank } k).$
Free boundary

Limit shapes: [Di Francesco, Reshetikhin, 2009]
Let
\[ T_f(N, M) := \bigcup_{\lambda \mid \ell(\lambda) = N, \lambda_1 \leq M} \text{tilings of } \Omega_\lambda, \]
i.e. the set of all tilings in an \( N \times M \times N \) trapezoid with unrestricted positions of right horizontal lozenges.
⇔ Vertically symmetric tilings of the \( N \times M \times N \times N \times M \times N \) hexagon.

Theorem (P, –)
Let \( \Upsilon_{N,M}^k \) denote the positions of the horizontal lozenges \( \{x_i^j\} \) on the \( i \)th vertical line of a uniformly random tiling from \( T_f(N, M) \). Then, as \( N \to \infty \) and \( \frac{M}{N} \to a \), where \( 0 < a < \infty \),

\[
\frac{\Upsilon_{N,M}^k - M/2}{\sqrt{N(a^2 + 2a)/8}} \to \text{GUE}_k.
\]
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6 Vertex model / ASM

Six vertex types:

\[
\begin{array}{cccccc}
\text{a} & \text{a} & \text{b} & \text{b} & \text{c} & \text{c} \\
0 & 0 & 0 & 0 & -1 & 1 \\
\end{array}
\]

Alternating Sign Matrix:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

A 6 vertex model configuration:
Definitions and background on ASMs

**Definition:** An *Alternating Sign Matrix* of size $n$ is an $n \times n$ matrix of 0s, 1s, $-1$s, such that the sum in each row or column is 1 and 1s and $-1$s alternate in each row or column. A *monotone triangle* is a Gelfand-Tsetlin pattern, s.t. the inequalities on each row are strict.

**6 Vertex model $\leftrightarrow$ ASM $\leftrightarrow$ monotone triangles.**

Uniform measure on ASMs $\leftrightarrow$ all vertices in 6V model have equal weight ("ice").

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

ASM:

positions of $1$s $\iff$ in sum of first $k$ rows

**Monotone triangle:**

```
4
2 5
2 3 5
1 2 3 5
1 2 3 4 5
```

**Question:** What does a uniformly random ASM look like as $n \to \infty$? What is the *distribution* of the *positions of the $1$s and $-1$s near the boundary* of the ASM in the limit $\leftrightarrow$ Distribution of the *numbers in the top $k$ rows* of the monotone triangle?

**Known results:**

Limit behavior: [Behrend], [Colomo, Pronko, [Zinn-Justin]], Di Francesco.

Free fermions point (weight 2 at 1,-1) $\leftrightarrow$ domino tilings, Aztec diamond.

Exact generating functions for certain statistics (e.g. positions of $1$s on boundary, etc).
**ASM A:** \( \Psi_k(A) := \sum_{j=1:n, A_{kj}=1} j - \sum_{j=1:n, A_{kj}=-1} j \)

Monotone triangle \( M = [m^k_j]_{j \leq i} \): \( \Psi_k(M) = \sum_{j=1}^{k} m^k_j - \sum_{j=1}^{k-1} m^{k-1}_j \)

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

- \( k = 2 \): \( \Psi_2 = 2 + 5 - 4 = 3 \)
- \( k = 3 \): \( \Psi_3 = 3 \)

\[
\begin{pmatrix}
0 & 1 & 0 & -1 & 1 \\
0 & 1 & 2 & 3 & 5 \\
1 & 2 & 3 & 5
\end{pmatrix}
\]

- \( k = 2 \): \( \Psi_2 = (2 + 5) - (4) = 3 \)
- \( k = 3 \): \( \Psi_3 = (2 + 3 + 5) - (2 + 5) \)
ASM $A$: $\Psi_k(A) := \sum_{j=1:n, A_{kj}=1} j - \sum_{j=1:n, A_{kj}=-1} j$

Monotone triangle $M = [m^i_j]_{j \leq i}$: $\Psi_k(M) = \sum_{j=1}^{k} m^k_j - \sum_{j=1}^{k-1} m^{k-1}_j$

$\Psi_k(n)$ – the random variable $\Psi_k(A)$ as $A$ is chosen uniformly random from ASMs of size $n$.

**Theorem (G–P)**

$\frac{\Psi_k(n) - n/2}{\sqrt{n}}$, $k = 1, 2, \ldots$ converge as $n \to \infty$ to the collection of i.i.d. Gaussian random variables, $N(0, \sqrt{3/8})$. 

Asymptotics of symmetric functions with applications to statistical mechanics and representation theory

Greta Panova (UCLA)

Normalized Schur functions $S_{\lambda}$

**Setup**

Asymptotics of $S_{\lambda}(x_1, \ldots, x_k)$

**GUE in random lozenge tilings**

Lozenge tilings

$N \to \infty$, behavior near boundary

**GUE in tilings, results**

**ASM**

**GUE in ASMs**

**ASM** $A$:

$$
\Psi_k(A) := \sum_{j=1:n, A_{kj}=1} j - \sum_{j=1:n, A_{kj}=-1} j
$$

Monotone triangle $M = [m^i_j]_{j \leq i}$:

$$
\Psi_k(M) = \sum_{j=1}^k m^k_j - \sum_{j=1}^{k-1} m^{k-1}_j
$$

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$$

Using this Theorem on $\Psi_k(n)$ and the Gibbs property:

**Theorem (G, 2013; Conjecture in [G–P])**

Fix any $k$. As $n \to \infty$ the probability that the number of $(-1)$s in the first $k$ rows of uniformly random ASM of size $n$ is maximal tends to 1, and, thus, $1$s in first $k$ rows are interlacing. After centering and rescaling the distribution of the positions of $1$s tends to GUE-corners process, i.e. top $k$ rows of the monotone triangle $M$ converge to the GUE-corners process:

$$
\sqrt{\frac{8}{3n}} \left( [M]_{i=1:k} - \frac{n}{2} \right) \to \text{GUE}_k.
$$
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### 6Vertex/ASMs: proofs

Vertex at position $(i,j)$ and its weight (corresponding to the type):

$$ a : q^{-1}u_i^2 - qv_j^2, \quad b : q^{-1}v_j^2 - qu_i^2, \quad c : (q^{-1} - q)u_i v_j $$

where $v_1,\ldots,v_N$, $u_1,\ldots,u_N$ are parameters, $q = \exp(\pi i / 3)$

Weight $W(\vartheta)$ of a configuration $\theta = \text{product of weights of its vertices}$. Set $\lambda(N) := (N-1, N-1, N-2, N-2, \ldots, 1, 1, 0, 0) \in GT_{2N}$.

**Proposition (Okada;Stroganov)**

Let $\mathcal{I}_N$ be the set of all 6Vertex configurations on an $N \times N$ grid.

$$ \sum_{\vartheta \in \mathcal{I}_N} W(\vartheta) = (-1)^{N(N-1)/2}(q^{-1} - q)^N \prod_{i=1}^{N} (v_i u_i)^{-1} s_{\lambda(N)}(u_1^2,\ldots,u_N^2,v_1^2,\ldots,v_N^2). $$
6Vertex/ASMs: proofs

Vertex at position \((i, j)\) and its weight (corresponding to the type):

\[
a : q^{-1} u_i^2 - q v_j^2, \quad b : q^{-1} v_j^2 - q u_i^2, \quad c : (q^{-1} - q) u_i v_j
\]

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Weight \(W(\theta)\) of a configuration \(\theta = \) product of weights of its vertices.

Set \(\lambda(N) := (N - 1, N - 1, N - 2, N - 2, \ldots, 1, 1, 0, 0) \in \mathcal{G}_T 2N\).

Proposition

Let \(\hat{x}_i\) be the number of vertices of type \(x\) on row \(i\), then for any collection of rows \(i_1, \ldots, i_m\) we have

\[
\mathbb{E}_{N} \prod_{\ell=1}^{m} \left[ \left( \frac{q^{-1} - q \nu_\ell^2}{q^{-1} - q} \right)^{\hat{a}_{i_\ell}} \left( \frac{q^{-1} \nu_\ell^2 - q}{q^{-1} - q} \right)^{\hat{b}_{i_\ell}} \left( \nu_\ell \right)^{\hat{c}_{i_\ell}} \right] = \left( \prod_{\ell=1}^{n} \nu_\ell^{-1} \right) \frac{s_{\lambda(N)}(\nu_1, \ldots, \nu_m, 1^{2N-m})}{s_{\lambda(N)}(1^{2N})} = \left( \prod_{\ell=1}^{n} \nu_\ell^{-1} \right) S_{\lambda(N)}(\nu_1, \ldots, \nu_m)
\]
6Vertex/ASMs: proofs

Vertex at position \((i, j)\) and its weight (corresponding to the type):
\[
\begin{align*}
a : & \quad q^{-1}u_i^2 - qv_j^2, \\
b : & \quad q^{-1}v_j^2 - qu_i^2, \\
c : & \quad (q^{-1} - q)u_iv_j
\end{align*}
\]
where \(v_1, \ldots, v_N, u_1, \ldots, u_N\) are parameters, \(q = \exp(\pi i/3)\)

Weight \(W(\theta)\) of a configuration \(\theta = \text{product of weights of its vertices.}\)
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\]
\[
= \left( \prod_{\ell=1}^{n} v_\ell^{-1} \right) \frac{s_\lambda(N)(v_1, \ldots, v_m, 1^{2N-m})}{s_\lambda(N)(1^{2N})} \left( \prod_{\ell=1}^{n} v_\ell^{-1} \right) S_\lambda(N)(v_1, \ldots, v_m)
\]

**Proof of Theorem:** Use Proposition to derive the *moment generating function* as a Schur function. Choose parameters wisely to extract the main statistic and apply the asymptotics:
\[
S_\lambda(N)(e^{y_1}/\sqrt{n}, \ldots, e^{y_k}/\sqrt{n}) = \prod_{i=1}^{k} \exp \left[ \sqrt{n}y_i + \frac{5}{12} y_i^2 + o(1) \right]
\]
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Extreme Characters of $U(\infty)$

$U(N)$ – the group of $N \times N$ unitary matrices. $U(\infty) = \bigcup_{N=1}^{\infty} U(N)$. A (normalized) character of a group $G$ is a continuous function $\chi(g), g \in G$ s.t.:

1. $\chi(aba^{-1}) = \chi(b)$ for any $a, b \in G$,

2. $\chi$ is positive definite, i.e. the matrix $\left[\chi(g_i g_j^{-1})\right]_{i,j=1}^{k}$ is Hermitian non-negative definite, for any $\{g_1, \ldots, g_k\}$,

3. $\chi(e) = 1$.

An extreme character is an extreme point of the convex set of all characters. The normalized characters of $U(N)$ are the functions

$$
\frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(1, \ldots, 1)}.
$$
Extreme Characters of $U(\infty)$

$U(N)$ – the group of $N \times N$ unitary matrices. $U(\infty) = \bigcup_{N=1}^{\infty} U(N)$.

Theorem (Voiculescu-Edrei classification)

The extreme characters of $U(\infty)$ are parameterized by the points $\omega$ of the infinite-dimensional domain

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where $\Omega$ is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \text{Spec}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} \frac{1 + \beta_i^-(u^{-1}-1)}{1 - \alpha_i^-(u^{-1}-1)}.$$
Extreme characters of $U(\infty)$

**Proposition (Kerov-Vershik)**

Every extreme normalized character $\chi$ of $U(\infty)$ is a uniform limit of extreme characters of $U(N)$. In other words, for every $\chi$ there exists a sequence $\lambda(N) \in \mathcal{GT}_N$ such that for every $k$

$$\chi(u_1, \ldots, u_k, 1, \ldots) = \lim_{N \to \infty} S_\lambda(u_1, \ldots, u_k; N, 1)$$

uniformly on the torus $(S_1)^k$.

Based on this fact we show which sequences approximate characters of $U(\infty)$:

For any $\lambda$ set $p_i = \lambda_i - i + 1/2$, $q_i = \lambda'_i - i + 1/2$, $i = 1, \ldots, d$.

$$\chi^{(\omega)}(u_1, u_2, \ldots) = \prod_j e^{\gamma^+(w_j-1) + \gamma^-(w_j^{-1}-1)} \prod_{i=1}^\infty \frac{1 + \beta_i^+(u_j - 1)}{1 - \alpha_i^+(u_j - 1)} \frac{1 + \beta_i^-(u_j^{-1} - 1)}{1 - \alpha_i^-(u_j^{-1} - 1)}.$$  

**Theorem (VK, OO, BO, P, Gorin-Panova)**

Let $\omega = (\alpha^\pm, \beta^\pm, \delta^\pm)$ and suppose that the sequence $\lambda(N) \in \mathcal{GT}_N$ is s.t.

$$p_i^+(N)/N \to \alpha_i^+, \quad p_i^-(N)/N \to \alpha_i^-, \quad q_i^+(N)/N \to \beta_i^+, \quad q_i^-(N)/N \to \beta_i^-,$$

$$|\lambda^+|/N \to \delta^+, \quad |\lambda^-|/N \to \delta^-.$$

Then for every $k$

$$\chi(u_1, \ldots, u_k, 1, \ldots) = \lim_{N \to \infty} S_{\lambda(N)}(u_1, \ldots, u_k; N, 1) = \chi^{\omega}(u_1, \ldots, u_k, 1, \ldots) \text{(as defined above)}$$

uniformly on torus $(S_1)^k$.  

---

**Normalized Schur functions $S_\lambda$**

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The dense loop model

Given a finite grid (in this case, vertical strip of width $L$), each square is one of two kinds below, on the boundary – one of the triangles

The mean total current between two points $x$ and $y$ $F^x,y$ – the average number of paths connecting both boundaries and passing between $x$ and $y$. Similar observables in the critical percolation model [Smirnov, 2009].
Dense loop model: the mean current

Let $\lambda^L = ([L-1]/2, [L-2]/2, \ldots, 1, 0, 0)$

Define:

$$u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L) = (-1)^L \frac{\sqrt{3}}{2} \ln \left[ \frac{\chi_{\lambda^L+1}(\zeta_1^2, z_1^2, \ldots, z_L^2) \chi_{\lambda^L+1}(\zeta_2^2, z_1^2, \ldots, z_L^2)}{\chi_{\lambda^L}(z_1^2, \ldots, z_L^2) \chi_{\lambda^L+2}(\zeta_1^2, \zeta_2^2, z_1^2, \ldots, z_L^2)} \right]$$

where $\chi_{\nu}$ is the character for the irreducible representation of highest weight $\nu$ of the symplectic group $Sp(\mathbb{C})$.

$$X^{(j)} = z_j \frac{\partial}{\partial z_j} u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L)$$

$$Y_L = w \frac{\partial}{\partial w} u_{L+2}(\zeta_1, \zeta_2; z_1, \ldots, z_L, vq^{-1}, w)|_{v=w},$$

Proposition (De Gier, Nienhuis, Ponsaing)

Under certain assumptions the mean total current between two horizontally adjacent points is

$$X^{(j)} = F(j,i), (j+1,i),$$

and $Y$ is the mean total current between two vertically adjacent points in the strip of width $L$:

$$Y^{(j)} = F(j,i), (j,i+1).$$
Dense loop model: asymptotics of the mean current

**Theorem**

As \( L \to \infty \) we have

\[
X_L^{(j)} \bigg|_{z_j=z; \ z_i=1, \ i \neq j} = \frac{i\sqrt{3}}{4L} (z^3 - z^{-3}) + o \left( \frac{1}{L} \right)
\]

and

\[
Y_L \bigg|_{z_i=1, \ i=1,...,L} = \frac{i\sqrt{3}}{4L} (w^3 - w^{-3}) + o \left( \frac{1}{L} \right)
\]

**Remark 1.** When \( z = 1 \), \( X_L^{(j)} \) is identical zero and so is our asymptotics.

**Remark 2.** The fully homogeneous case corresponds to \( w = \exp^{-i\pi/6} \), \( q = e^{2\pi i/3} \). In this case

\[
Y_L = \frac{\sqrt{3}}{2L} + o \left( \frac{1}{L} \right).
\]

**Proof:** same type of asymptotic methods and results hold for symplectic characters + some tricks with the multivariate formula.