Permutation patterns, Stanley symmetric functions, and the Edelman-Greene correspondence

Sara Billey and Brendan Pawlowski

University of Washington

June 26, 2013

Reduced words

A *reduced word* for $w \in S_n$ is a minimal length sequence $a = a_1 \cdots a_\ell$ such that $w = t_{a_1} \cdots t_{a_\ell}$ where $t_i = (i \ i+1)$.

Write $\mathrm{Red}(w) = \{\text{reduced words of } w\}$.

Example

$\mathrm{Red}(2143) = \{31, 13\}$ since $2143 = (3\,4)(1\,2) = (1\,2)(3\,4)$.

Given $a \in \mathrm{Red}(w)$ define the set $C(a) = \{1 \leq i_1 \leq \cdots \leq i_{\ell(w)} : \text{if } a_j < a_{j+1} \text{ then } i_j < i_{j+1}\}$; here $\ell(w) = \#$ of inversions of w.

Example

$C(31) = \{1 \leq i_1 \leq i_2\}$ $\qquad$ $C(13) = \{1 \leq i_1 < i_2\}$

Stanley symmetric functions

Definition (Stanley '84)

The *Stanley symmetric function* of a permutation w is the degree $\ell(w)$ symmetric (!) function

$$F_w = \sum_{a \in \mathrm{Red}(w)} \sum_{i \in C(a)} x_{i_1} \cdots x_{i_{\ell(w)}},$$

where $C(a) = \{1 \leq i_1 \leq \cdots \leq i_{\ell(w)} : \text{if } a_j < a_{j+1} \text{ then } i_j < i_{j+1}\}$.

Example

$w = 2143$, $\mathrm{Red}(w) = \{31, 13\}$.
$C(31) = \{1 \leq i_1 \leq i_2\}$ $\qquad$ $C(13) = \{1 \leq i_1 < i_2\}$

So $F_{2143} = \sum_{i_1 \leq i_2} x_{i_1} x_{i_2} + \sum_{i_1 < i_2} x_{i_1} x_{i_2} = s_2 + s_{11}$.

Stanley symmetric functions

- Enumeration of reduced words: if $F_w = \sum_\lambda a_\lambda s_\lambda$ then $|\,\mathrm{Red}(w)| = \sum_\lambda a_\lambda \#\{\text{standard tableaux of shape } \lambda\}$.
- Schubert calculus: $F_w = \lim_{m\to\infty} \mathfrak{S}_{1^m \times w}$ where $1^m \times w = 12\cdots m(w_1+m)\cdots(w_n+m)$ (Lascoux-Schützenberger).
- S_ℓ representation theory: the F_w appear as Frobenius characteristics of some natural generalizations of Specht modules.

Patterns and F_w

Theorem (Edelman-Greene '87)

F_w is Schur-positive.

$M(w) =$ multiset of partitions such that $\sum_{\lambda \in M(w)} s_\lambda = F_w$.

Theorem (Billey-Pawlowski)

If v is contained as a pattern in w, there is an injection $\iota : M(v) \hookrightarrow M(w)$ such that $\lambda \subseteq \iota(\lambda)$ for $\lambda \in M(v)$. Moreover, ι maps multiple copies of $\lambda \in M(v)$ to multiple copies of one partition in $M(w)$.

Example

Since $M(2143) = \{(1,1),(2)\}$, if w contains 2143 then $|M(w)| \geq 2$: F_w isn't a single Schur function.

Patterns and F_w

Example

If $v = w_1 \cdots \widehat{w_j} \cdots w_n$ (flattened to a member of S_{n-1}) and

$$\begin{aligned} a &= \#\{p < j : w_p > w_j\} \\ b &= \#\{p > j : w_p < w_j\}, \end{aligned}$$

can take $\iota : M(v) \hookrightarrow M(w)$ to be $\lambda \mapsto (\lambda + (1^a)) \cup (b)$.

Take $w = 4317256$.

$$\begin{array}{rccccccccc} & F_w & = & s_{332} & + & s_{422} & + & s_{431} & + & s_{521} \\ j=1, a=0, b=3: & F_v & = & s_{32} & + & & & s_{41} & & \\ j=2, a=1, b=2: & F_v & = & & & s_{32} & & & + & s_{41} \\ j=5, a=3, b=0: & F_v & = & s_{221} & + & s_{311} & + & s_{32} & + & s_{41} \end{array}$$

k-vexillary and multiplicity-free permutations

Say w is *k-vexillary* if F_w has at most k Schur function terms. Say w is *multiplicity-free* if every Schur term of F_w has multiplicity 1.

Corollary

Suppose v is contained in w. If w is k-vexillary, so is v. If w is multiplicity-free, so is v.

Theorem (Billey-Pawlowski)

For any k the property of being k-vexillary is characterized by avoiding a finite set of patterns. One can take the patterns to have at most $4k$ letters.

Conjecture

The property of being multiplicity-free is characterized by avoiding a set of 454 patterns of at most 11 letters (checked through S_{12}).

k-vexillary permutations

- w is 1-vexillary iff it avoids 2143 (Stanley '84, Lascoux-Schützenberger '82).
- w is 2-vexillary iff it avoids 35 patterns in $S_5 \cup S_6 \cup S_7 \cup S_8$:

21543	231564	315264	5271436	26487153	54726183	64821537
32154	241365	426153	5276143	26581437	54762183	64872153
214365	241635	2547163	5472163	26587143	61832547	65821437
214635	312645	4265173	25476183	51736284	61837254	65827143
215364	314265	5173264	26481537	51763284	61873254	65872143

- w is 3-vexillary iff it avoids a list of 91 patterns in $S_6 \cup S_7 \cup S_8$.
- w is 4-vexillary iff it avoids a list of 2346 patterns in $S_6 \cup \cdots \cup S_{12}$ (thanks to Michael Albert for this verification).

Generalized Specht modules

Definition

A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$. Its elements are *cells*.

Example

$$D = \begin{array}{ccc} \circ & \circ & \cdot \\ \circ & \cdot & \cdot \\ \cdot & \circ & \circ \end{array} = \{(1,1),(1,2),(2,1),(3,2),(3,3)\}$$

Definition (James-Peel '79)

The *Specht module* of a diagram D with ℓ cells is the (complex) S_ℓ-module

$$S^D = \mathbb{C}[S_\ell]c_D,$$

with $c_D \in \mathbb{C}[S_\ell]$ the Young symmetrizer of a bijective filling of D.

Generalized Specht modules

Example

- When D is the Ferrers diagram of a partition λ, S^D is the usual irreducible Specht module S^λ.
- When D is a skew shape λ/μ, S^D is the skew representation $S^{\lambda/\mu}$.
- If D, D' differ by a permutation of rows and columns, then $S^D \simeq S^{D'}$.

In general no combinatorial algorithm for decomposing S^D into irreducibles is known. (Special cases: Littlewood-Richardson rule, Reiner-Shimozono, Liu).

The *Schur function* s_D of D is the Frobenius characteristic of S^D: if $S^D \simeq \bigoplus_\lambda a_\lambda S^\lambda$ then $s_D = \sum_\lambda a_\lambda s_\lambda$.

Permutation diagrams

Definition

The *diagram* $D(w)$ of a permutation $w \in S_n$ is the set of cells in $[n] \times [n]$ not lying (weakly) below or right of any $(i, w(i))$ (marked with $\times$ below).

Example

$$D(32154) = \begin{array}{ccccc} \circ & \circ & \times & \cdot & \cdot \\ \circ & \times & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \circ & \times \\ \cdot & \cdot & \cdot & \times & \cdot \end{array} \simeq \begin{array}{ccc} \circ & \circ & \cdot \\ \circ & \cdot & \cdot \\ \cdot & \cdot & \circ \end{array} = (2,1) \cdot (1)$$

Theorem (Reiner-Shimozono '94)

For any permutation w, $F_w = s_{D(w)}$.

Example

$$F_{32154} = s_{(2,1)\cdot(1)} = s_{21} s_1 = s_{31} + s_{22} + s_{211}.$$

James-Peel moves

Given a diagram D, define $R_{a\to b}D$ as the diagram obtained by moving cells from row a to row b if possible (if the corresponding position is empty). Likewise define $C_{c\to d}D$ on columns.

These operators are *James-Peel moves*.

Example

$$D = \begin{matrix} \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot \end{matrix} \qquad R_{2\to 1}D = \begin{matrix} \circ & \circ & \circ & \cdot \\ \circ & \cdot & \cdot & \cdot \end{matrix}$$

$$C_{3\to 2}D = \begin{matrix} \circ & \circ & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot \end{matrix}$$

Subdiagram Pieri rule

Theorem (Billey-Pawlowski)

Suppose $D \cap ([n] \times [n]) = (n-1, n-2, \ldots, 1) \cdot (1)$. *Define* $D_i = R_{n \to i} C_{n \to n-i+1} D$ *for* $i = 1, \ldots, n$. *Then* $S^{D_1} \oplus \cdots \oplus S^{D_n} \hookrightarrow S^D$.

(The case $n = 1$ is a theorem of James and Peel.)

Example

$$D = \begin{array}{cccc} \circ & \circ & \cdot & \circ \\ \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \circ \\ \cdot & \circ & \cdot & \cdot \end{array} \qquad D_2 = R_{3 \to 2} C_{3 \to 2} D = \begin{array}{cccc} \circ & \circ & \cdot & \circ \\ \circ & \circ & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot \end{array}$$

$$D_1 = R_{3 \to 1} D = \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \circ \\ \cdot & \circ & \cdot & \cdot \end{array} \qquad D_3 = C_{3 \to 1} D = \begin{array}{cccc} \circ & \circ & \cdot & \circ \\ \circ & \cdot & \cdot & \cdot \\ \circ & \cdot & \cdot & \circ \\ \cdot & \circ & \cdot & \cdot \end{array}$$

(so e.g. looking at D_2, $S^{331} \hookrightarrow S^D$)

Proof sketch of main theorem

$M(w)$ = multiset of partitions such that $\sum_{\lambda \in M(w)} s_\lambda = F_w$.

Theorem: If v is contained in w as a pattern, there's an injection $\iota : M(v) \hookrightarrow M(w)$ with $\lambda \subseteq \iota(\lambda)$.

Proof sketch.

- Lascoux-Schützenberger transition formula and $F_v = \lim_{m\to\infty} \mathfrak{S}_{1^m \times v} \Rightarrow$ recurrence $F_v = \sum_u F_u$ (L-S tree)
- Translated to permutation diagrams, this recurrence is the subdiagram Pieri rule starting from $D(v)$!
- v contained in $w \Rightarrow D(v) \subseteq D(w)$: apply subdiagram Pieri rule to $D(w)$ instead of $D(v)$.

□

Further work

- Stanley symmetric functions in other types, affine Stanley symmetric functions (cf. Yoo-Yun's affine permutation diagrams, balanced labellings)
- Better characterization of patterns characterizing k-vexillary permutations?
- By Edelman-Greene, an injection $M(v) \hookrightarrow M(w)$ gives an injection $\mathrm{Red}(v) \hookrightarrow \mathrm{Red}(w)$ respecting Coxeter-Knuth classes. Can this be made explicit?
- Analogues for (stable) Grothendieck polynomials? (Lascoux)

Fin

Thank you for listening!