The module of affine descent classes

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The module of affine descents

Descents

$W = \langle s_\alpha : \alpha \in \Delta \rangle$ finite Coxeter group

$\Delta = \{\alpha_1, \ldots, \alpha_n\}$ (nodes of Dynkin diagram)

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Type A is "cyclic" descents:

w	$D(w)$	$\widetilde{D}(w)$
123	$\emptyset$	$\{0\}$
132	$\{2\}$	$\{0,2\}$
213	$\{1\}$	$\{0,1\}$
231	$\{2\}$	$\{2\}$
312	$\{1\}$	$\{1\}$
321	$\{1,2\}$	$\{1,2\}$

Descent algebras

$$X_J := \sum_{D(w) \subseteq J} w \qquad \text{(in } \mathbb{Z}[W])$$

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Cellini (1995): $\mathrm{Span}\{\widetilde{x}_k : k \in [n]\}$ is a commutative subalgebra
$\mathrm{Span}\{\widetilde{X}_J\}$ is NOT a subalgebra. However...

Main result

$$\widetilde{\mathsf{Sol}}(W) := \mathsf{Span}\left\{ \widetilde{X}_J = \sum_{\widetilde{D}(w)\subseteq J} w : J \subseteq [0,n] \right\}$$

Theorem (Aguiar-P.)

$\widetilde{\mathsf{Sol}}(W)$ *is a left module over Solomon's descent algebra*

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$\widetilde{\mathrm{Sol}}(W)$ *is a left module over Solomon's descent algebra*

- We call $\widetilde{\mathrm{Sol}}(W)$ the "module of affine descents"
- Equivalent to one of Moszkowski's "reflection modules"

The geometric approach

Tits (1976), Bidigare (1997):

$$\text{Coxeter complex} \longleftrightarrow \mathsf{Sol}(W)$$

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Our approach:

$$\text{Steinberg torus} \longleftrightarrow \widetilde{\mathsf{Sol}}(W)$$

The module of affine descents

Some background

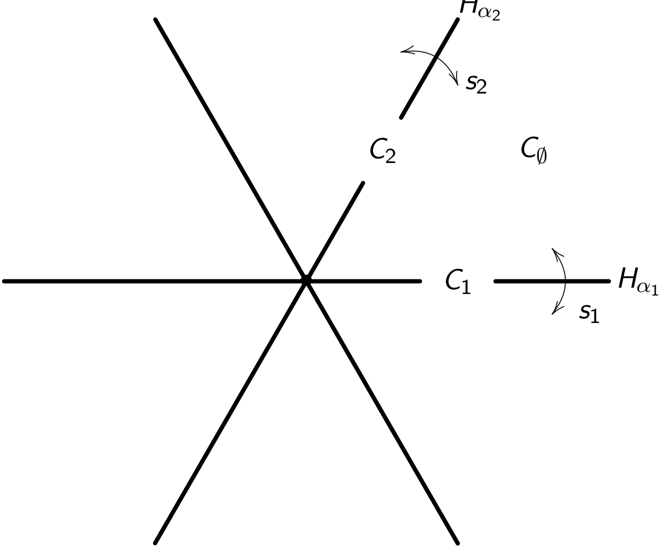
Solomon's descent algebra

Affine structures

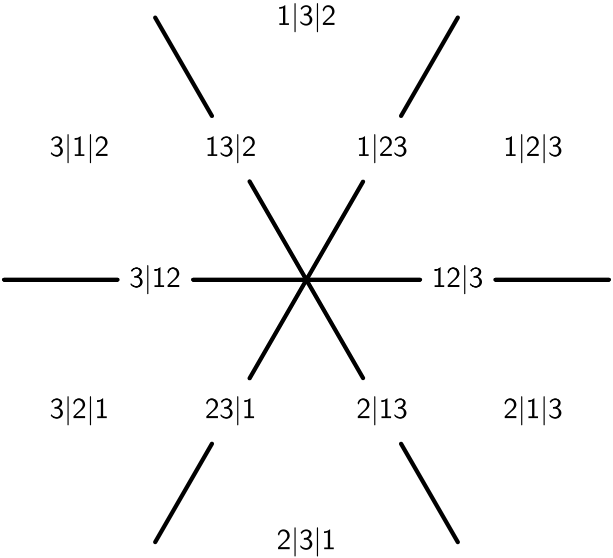
The affine descent module

The Coxeter arrangement

$\Sigma = \{H_\beta : \beta \in \Phi\}$, the *Coxeter arrangement*



Semigroup of faces



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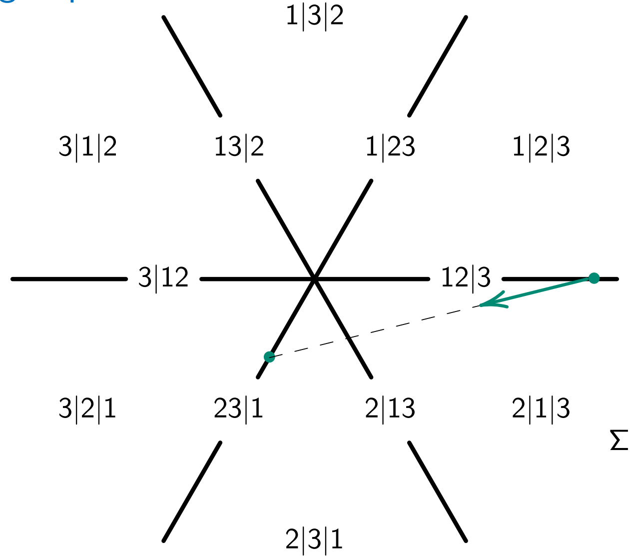
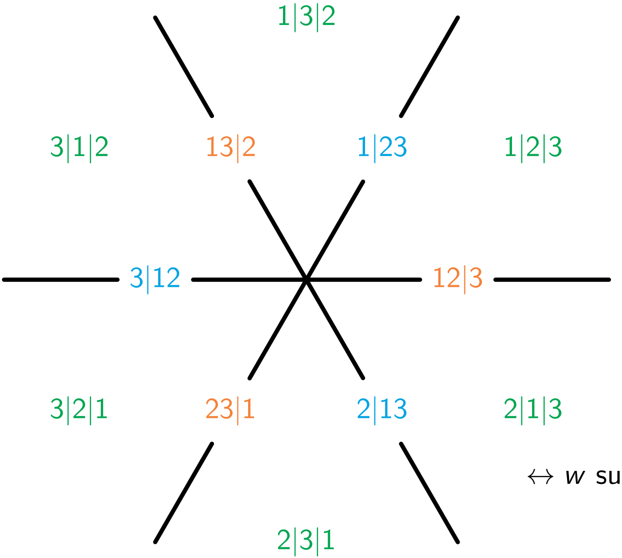


Σ is a semigroup

$$12|3 \cdot 23|1 = 2|1|3$$

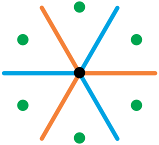
W-orbits



faces of color J

$\leftrightarrow w$ such that $D(w) \subseteq J$

The algebra Σ^W of invariants



$\sigma_\emptyset = 123$ (origin)
$\sigma_1 = 1|23 + 2|13 + 3|12$
$\sigma_2 = 12|3 + 13|2 + 23|1$
$\sigma_{12} = 1|2|3 + 1|3|2 + 2|1|3 + 2|3|1 + 3|1|2 + 3|2|1$

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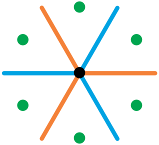

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W-orbits closed under products of faces:

$$\sigma_1\sigma_2 = \begin{array}{l} 1|2|3 + 1|3|2 + 1|23 + \\ 2|1|3 + 2|3|1 + 2|13 + \\ 3|1|2 + 3|2|1 + 3|12 \end{array}$$

$$= \sigma_{12} + \sigma_1$$

Solomon's descent algebra

Theorem (Bidigare, Tits, - see also Brown)

Σ^W *is (anti-)isomorphic to Solomon's descent algebra via:*

$$\sigma_J \leftrightarrow X_J := \sum_{D(w) \subseteq J} w \qquad (J \subseteq [n])$$

$\sigma_\emptyset = 123$	$X_\emptyset = 123$
$\sigma_1 = 1\vert 23 + 2\vert 13 + 3\vert 12$	$X_1 = 123 + 213 + 312$
$\sigma_2 = 12\vert 3 + 13\vert 2 + 23\vert 1$	$X_2 = 123 + 132 + 231$
$\sigma_{12} = 1\vert 2\vert 3 + 1\vert 3\vert 2 + 2\vert 1\vert 3 + 2\vert 3\vert 1 + 3\vert 1\vert 2 + 3\vert 2\vert 1$	$X_{12} = 123 + 132 + 213 + 231 + 312 + 321$

Solomon's descent algebra

$$\sigma_J \sigma_K = \sum_L c_{J,K}^L \sigma_L \qquad \sigma_1\sigma_2 = \begin{array}{l} 1|2|3 + 1|3|2 + 1|23 + \\ 2|1|3 + 2|3|1 + 2|13 + \\ 3|1|2 + 3|2|1 + 3|12 \end{array}$$

$$= \sigma_{12} + \sigma_1$$

$$X_K X_J = \sum_L c_{J,K}^L X_L \qquad X_2 X_1 = \begin{array}{l} 123 + 132 + 231 + \\ 213 + 312 + 321 + \\ 312 + 213 + 123 \end{array}$$

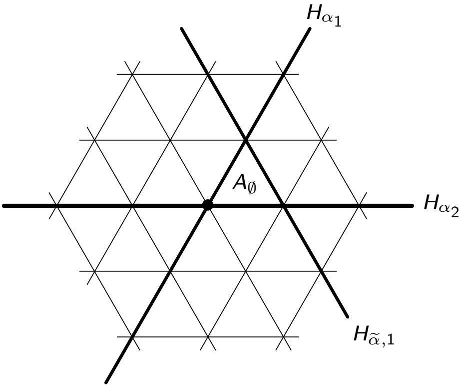
$$= X_{12} + X_1$$

The module of affine descents

The affine arrangement

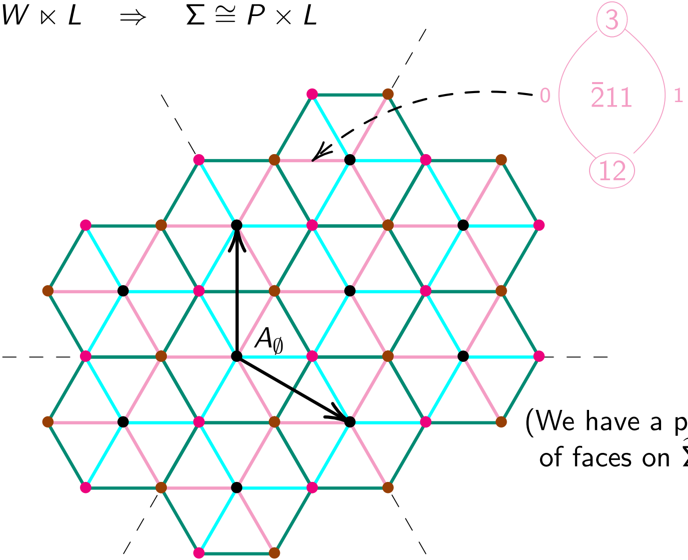
$\widetilde{W} = \langle s_\alpha : \alpha \in \Delta_0 \rangle$

$\widetilde{\Sigma}(A_2)$:



The affine arrangement

$$\widetilde{W} \cong W \ltimes L \quad \Rightarrow \quad \widetilde{\Sigma} \cong P \times L$$

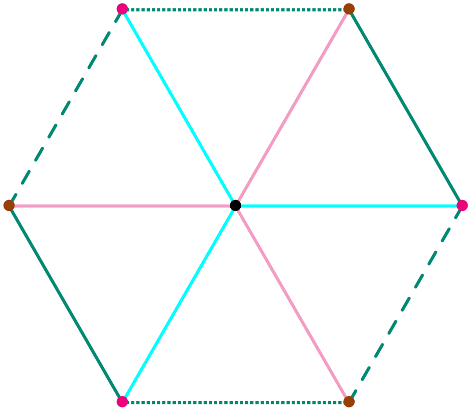


(We have a product of faces on $\widetilde{\Sigma}$)

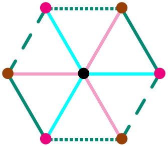
The Steinberg torus

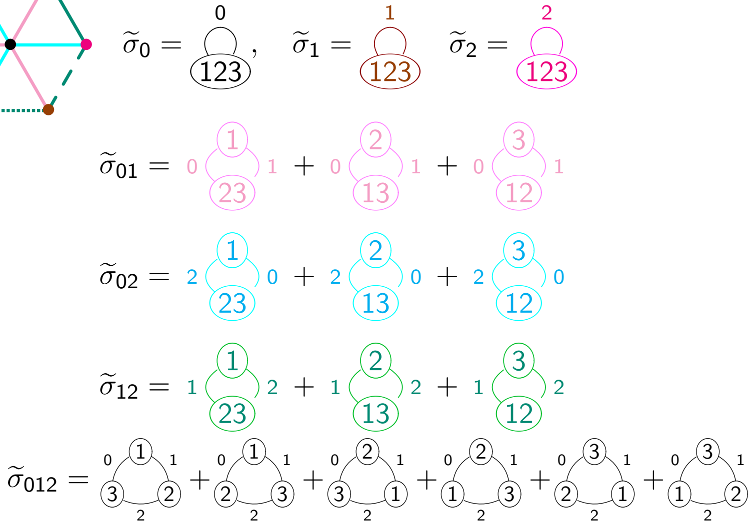
$\Sigma_T \cong \widetilde{\Sigma}/L$

(Product is ill-defined)



W-orbits in the Steinberg torus





The Steinberg torus

Theorem (Dilks-P.-Stembridge (2009))

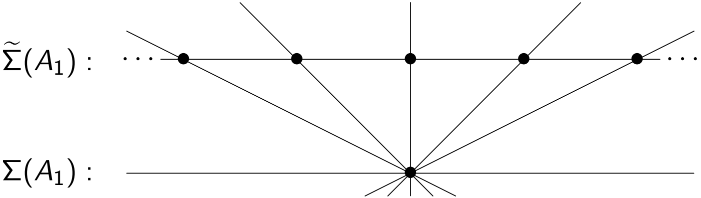
Affine descent sets characterize faces of the Steinberg torus:

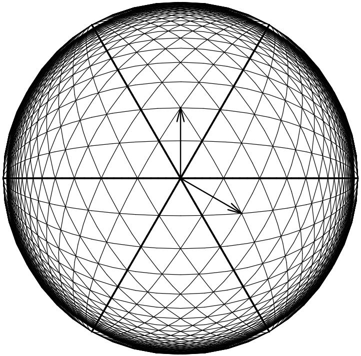
$$\{\textit{faces with color-set } J\} \leftrightarrow \{w \in W : \widetilde{D}(w) \subseteq J\},$$

i.e.,

$$\widetilde{\sigma}_J \leftrightarrow \widetilde{X}_J$$

The Tits cone





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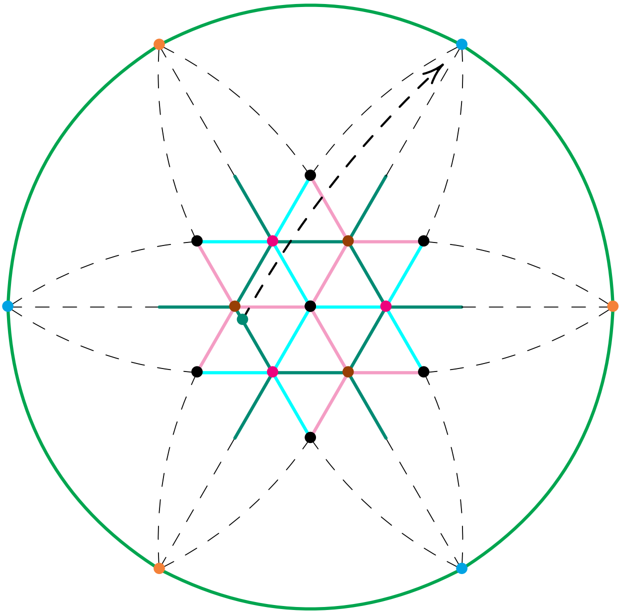
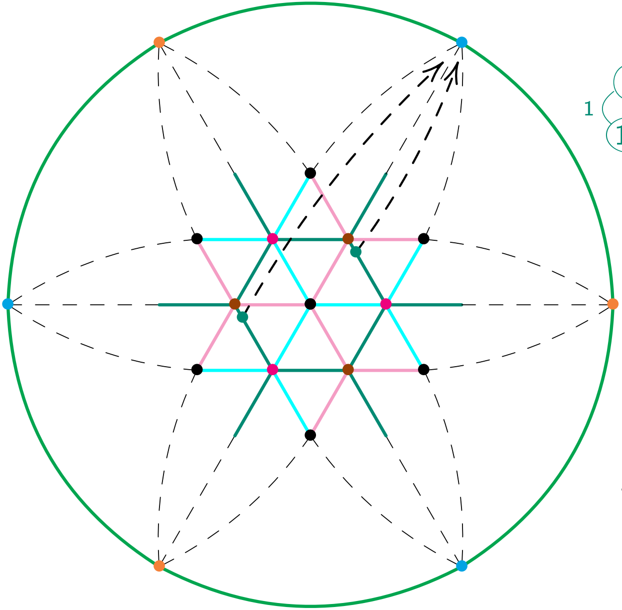
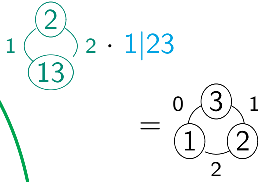
The affine descent module

The product $\widetilde{\Sigma} \times \Sigma \to \widetilde{\Sigma}$



The product $\Sigma_T \times \Sigma \to \Sigma_T$





The semigroup Σ acts on Σ_T

W-invariants

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acts on

$$(\Sigma_T)^W \cong \widetilde{\mathsf{Sol}}(W)$$

$$\widetilde{\sigma}_J \sigma_K = \sum_L c_{J,K}^L \widetilde{\sigma}_L \leftrightarrow X_K \widetilde{X}_J = \sum_L c_{J,K}^L \widetilde{X}_L$$

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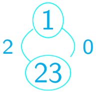
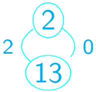


$+$


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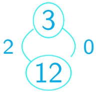

$\leftrightarrow 231 + 132 + 123$

Main result

Theorem (Aguiar-P.)

For any irreducible (finite) Weyl group

1. $\widetilde{\Sigma}(W)$ *is a right module over* $\Sigma(W)$
2. $\Sigma_T(W)$ *is a right module over* $\Sigma(W)$
3. $\widetilde{\mathsf{Sol}}(W)$ *is a left module over* $\mathsf{Sol}(W)$