A combinatorial method to find sharp lower bounds on flip distances

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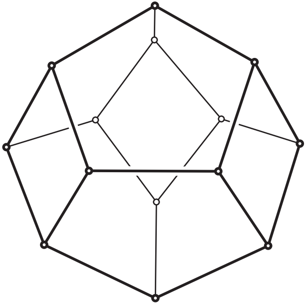
Theorem (arXiv:1207.6296)

The d-dimensional associahedron has diameter $2d - 4$ when d is greater than 9.



Summary

0. What are flip distances?

1. The problem
   - The diameters of flip-graphs,
   - The diameters of associahedra.

2. Main ideas of the proof
   - Two maximally distant triangulations,
   - Edge contractions in a polygon,
   - A recursive lower bound.

3. Related questions

Definition

A triangulation of a convex polygon π is a maximal set of edges that:

i. share their vertices with π,

ii. are pairwise non-crossing.

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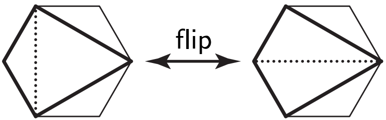
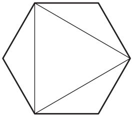
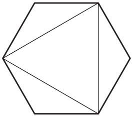


A **flip** consists in exchanging the diagonals of a quadrilateral within a triangulation.

0. What are flip distances?

The flip distance of two triangulations is the minimal number of flips needed to transform one into the other.





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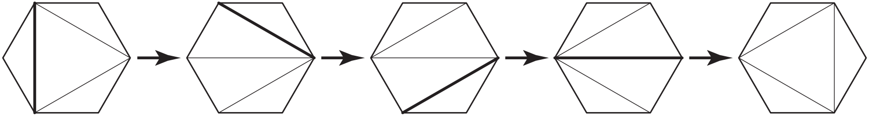

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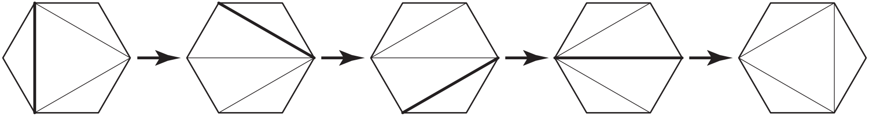
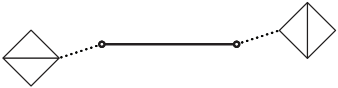


In other words, flip distances are distances measured within flip-graphs:

Definition

The flip-graph of a convex polygon π is the graph whose vertices are the triangulations of π, and whose edges correspond to flips.

0. What are flip distances?



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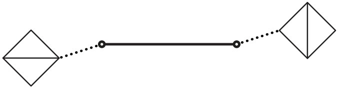

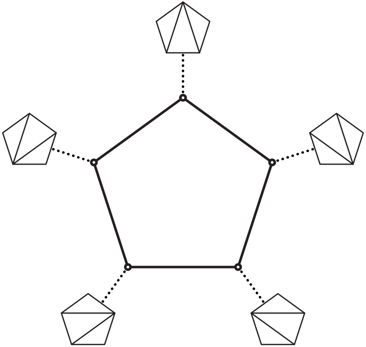

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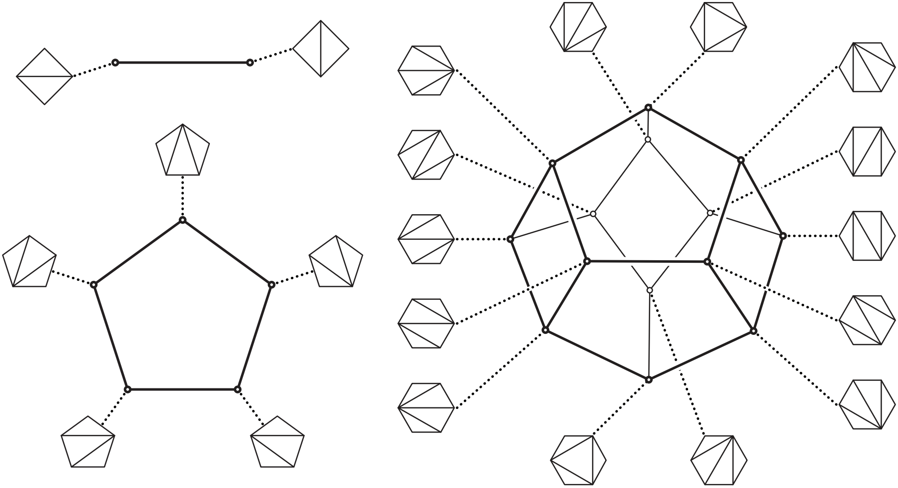


The flip-graph of a convex polygon with n vertices is the graph of the d-dimensional associahedron, where $n = d + 3$.

What is the diameter Δ_n of the flip-graph of a convex polygon with n vertices?

While working to the dynamic optimality conjecture, Daniel Sleator, Robert Tarjan, and William Thurston have shown that:

Theorem (1988)

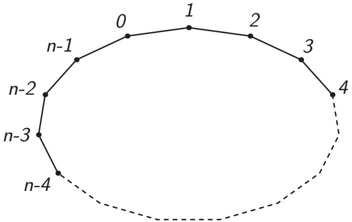
i. $\Delta_n \leq 2n - 10$ when n is greater than 12,

ii. $\Delta_n = 2n - 10$ when n is **large enough**.

Two problems remained open:

- Is there a combinatorial proof (of **ii.**)?
- Does **large enough** means **greater than** 12?

Consider a polygon with n vertices labeled from 0 to $n-1$ clockwise.



We search for two triangulations W_n^- and W_n^+ of this polygon so that:

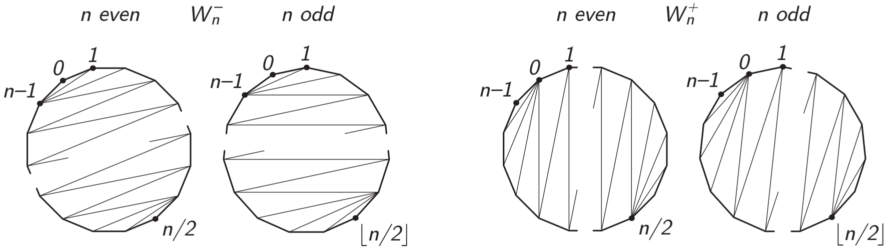
W_n^- and W_n^+ have flip distance $2n-10$ when $n > 12$,

There are two main difficulties:

- Finding triangulations W_n^- and W_n^+,
- Proving that their flip distance is indeed $2n-10$ when $n > 12$.

2. Main ideas

Two maximally distant triangulations

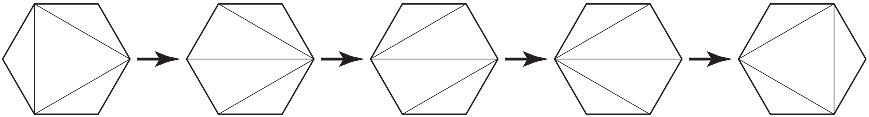


Call:
- A_n the pair $\{W_n^-, W_n^+\}$,
- $\delta(A_n)$ the flip distance of W_n^- and W_n^+.

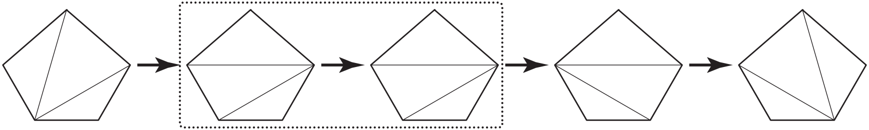
Claim

$$\delta(A_n) \geq \min(\delta(A_{n-1}) + 2, \delta(A_{n-2}) + 4, \delta(A_{n-5}) + 10, \delta(A_{n-6}) + 12).$$

Consider a path ψ of length k between two triangulations:



Contracting the edge at the top results in a path of length $k - j$:



where j is equal to the number of flips that modify the triangle incident to the contracted edge along path ψ.

Let U' and V' be the triangulations obtained by contracting a boundary edge ε of a convex polygon in two triangulations U and V of this polygon.

Theorem

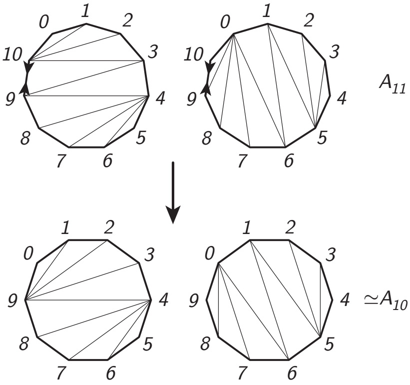
If ψ is a path of length k between U and V, then there exists a path of length $k-j$ between U' and V', where j is the number of flips along path ψ that modify the triangle incident to ε.

If in addition, ψ is a minimal path, then:

i. $k = \delta(\{U, V\})$,
ii. $\delta(\{U', V'\}) \leq k - j$.

One obtains the following inequality on flip distances as a consequence:

$$\delta(\{U, V\}) \geq \delta(\{U', V'\}) + j.$$


0
1
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9
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7
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0
1
2
10
3
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4
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7
6
A_{11}
1
2
0
3
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5
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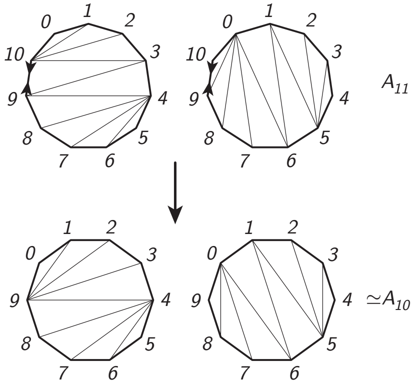
6
$\simeq A_{10}$

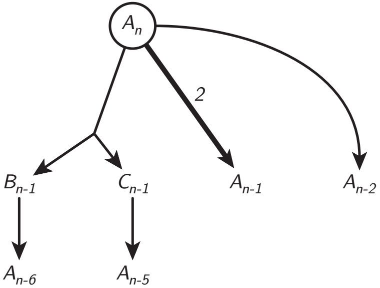


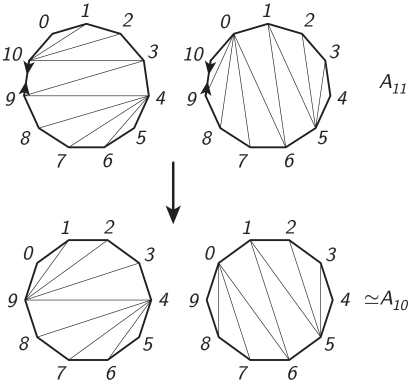
If there exists a minimal path from W_n^- to W_n^+ that modifies (at least) twice the triangle containing $\{n-2, n-1\}$, then:

$$\delta(A_n) \geq \delta(A_{n-1}) + 2.$$

2. Main ideas

A recursive lower bound





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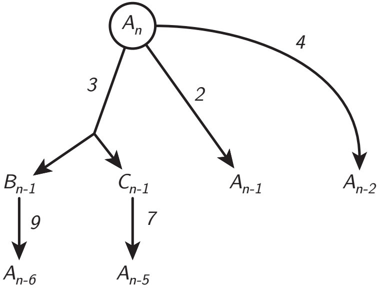
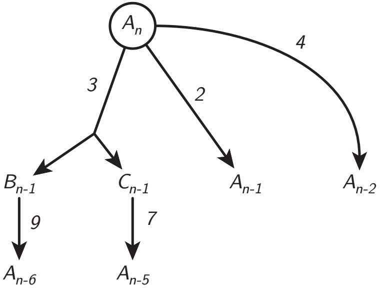


An arc of weight w from a pair P to a pair Q corresponds to the inequality $\delta(P) \geq \delta(Q) + w$ obtained under some condition (omitted here).

The conditions associated to the arcs with origin A_n exhaust all possibilities.



As $\delta(A_n) \geq 2n - 10$ when $7 \leq n \leq 12$,

Corollary

When n is greater than 12, the distance of W_n^- and W_n^+ is $2n - 10$.

Theorem

When n is greater than 12, the following inequality holds:

$$\delta(A_n) \geq \min(\delta(A_{n-1}) + 2, \delta(A_{n-2}) + 4, \delta(A_{n-5}) + 10, \delta(A_{n-6}) + 12).$$

What about the (maximal) flip distance of:

- multi-triangulations?
- centrally-symmetric triangulations?
- triangulations of an arbitrary surface?
- triangulations of arbitrary point configurations?