Beyond q:
Special functions on elliptic curves

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The q-disease

One surprisingly fruitful observation is that many objects arising in enumerative and algebraic combinatorics have "q-analogues"; e.g., replacing

$$n! \mapsto \prod_{1 \le i \le n} (1-q^i)/(1-q)$$

in binomial coefficients gives q-binomial coefficients, polynomials in q (counting subspaces of $\mathbb{F}_q^n$).

This works for Schur functions (irreducible characters of the unitary group); result (with additional parameter t) is Macdonald polynomials. [General principle: Lie groups $\to$ quantum groups]

q-Pochhammer symbol

Analogue of (reciprocal of) Gamma function:

$$(x;q) := \prod_{0\le j}(1-q^j x)$$

($|q|<1$ for convergence; often $(x;q)_\infty$ in the literature)

Convention: $(a,b,\dots,z;q) := (a;q)(b;q)\cdots(z;q)$.

"q-hypergeometric" $\equiv$ expressible as sums of ratios of q-Pochhammer symbols, e.g.

$$\sum_k \frac{z^k(q^k a_1, q^k a_2, \dots, q^k a_n; q)}{(q^k b_1, q^k b_2, \dots, q^k b_m; q)}$$

(See Gasper and Rahman, "Basic hypergeometric series")

The elliptic disease

General idea of q-analogues: replace rational functions in k by rational functions in q^k. Can think of this in terms of algebraic groups: replace rational functions on additive group by rational functions on multiplicative group.

Question (Baxter, Frenkel-Turaev, others): Why not work on an arbitrary (1-dimensional) algebraic group?

Only remaining possibility is an elliptic curve: replace rational functions of q^k by rational functions of $[k]q$ for q a point on the curve.

Aim of current talk: explain how this works for (certain) Macdonald polynomials.

p-elliptic functions

Nicest way to represent elliptic curve for present purposes is as quotient $\mathbb{C}^*/\langle p\rangle$, $|p| < 1$. A rational function on $\mathbb{C}^*/\langle p\rangle$ is just a p-periodic meromorphic function on $\mathbb{C} \setminus \{0\}$. (Relation to usual notion of doubly periodic meromorphic function is via composition with $\exp(2\pi\sqrt{-1}x)$)

Can construct these using theta functions:

$$\theta_p(x) := (x, p/x; p),$$
$$\theta_p(p/x) = \theta(x) \qquad \theta_p(1/x) = -x\theta(1/x)$$
$$\theta_p(px) = -x^{-1}\theta(x)$$

Elliptic functions come from products cancelling error in periodicity. Note $\theta_0(x) = 1 - x$.

Degenerations (an aside)

Ordinary and q-special functions can often degenerate by taking parameters to 0. Elliptic special functions only exist at the "top" level; any degeneration must also degenerate the curve. On the other hand, degenerating the curve can give very different-looking q-identities.

E.g., many of the q-special function identities in Gasper and Rahman's book are degenerations of the fact that a certain "elliptic" integral has a $W(E_7)$ symmetry. Includes various 4-term identities of series, integral representations, etc. (See work of van de Bult and the speaker classifying such identities.)

An elliptic MacMahon identity
(another aside)

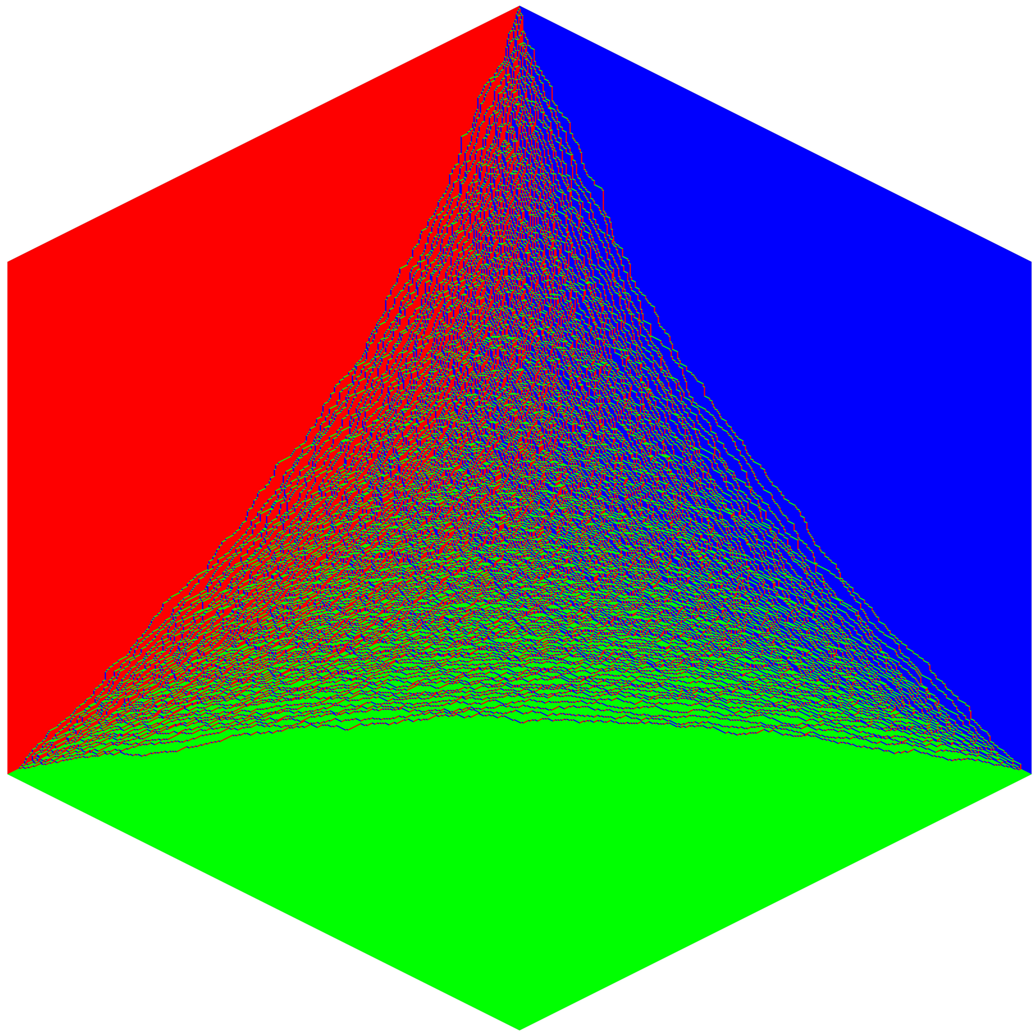
An example elliptic identity (Borodin/Gorin/Rains):

$$\sum_{\Pi} \prod_{(i,j,k)\in\Pi} \frac{q^3\theta_p(q^{j+k-2i}u_1/q, q^{i+k-2j}u_2/q, q^{i+j-2k}u_3/q)}{\theta_p(q^{j+k-2i}u_1 q, q^{i+k-2j}u_2 q, q^{i+j-2k}u_3 q)}$$
$$= \prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{q\theta_p(q^{i+j+k-1}, q^{j+k-i-1}u_1, q^{i+k-j-1}u_2, q^{i+j-k-1}u_3)}{\theta_p(q^{i+j+k-2}, q^{j+k-i}u_1, q^{i+k-j}u_2, q^{i+j-k}u_3)}$$

Here Π ranges over plane partitions (stacks of unit cubes packed in a corner) in an $a \times b \times c$ box, and $u_1u_2u_3 = 1$.

$u_1, u_2 \sim p^{1/3}$, $p \to 0$ gives usual MacMahon identity.

For suitable parameters, this is a probability distribution.



(Credit: D. Betea, generalizing exact sampling algorithm of Borodin/Gorin)

Macdonald polynomials

Given q, t with $|q|, |t| < 1$, Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$ is defined by (note $\mathbb{Z} \ni \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$ is a partition):

1. P_λ is invariant under permutations of variables.
2. P_λ has leading monomial $\prod_i x_i^{\lambda_i}$.
3. P_λ is an orthogonal polynomial:

$$\int P_\lambda(\ldots, x_i, \ldots) P_\mu(\ldots, x_i^{-1}, \ldots) \Delta dT \propto \delta_{\lambda\mu},$$

with

$$\Delta = \prod_{1 \leq i < j \leq n} \frac{(x_i/x_j, x_j/x_i; q)}{(tx_i/x_j, tx_j/x_i; q)},$$

$$dT = \prod_i \frac{dx_i}{2\pi\sqrt{-1}x_i}.$$

Note that if $q = t$, ΔdT is the eigenvalue measure for the unitary group, so P_λ is a Schur function.

First big surprise: They exist. "Leading monomial" is not terribly well-defined for multivariate polynomials, and the most natural notion (dominance ordering, from Lie theory) relates to a partial order. Gram-Schmidt on a partial order only gives orthogonality for vectors with comparable indices, but Macdonald polynomials are orthogonal anyway.

Macdonald "conjectures"

1. "Norm": The nonzero inner products factor nicely, as ratios of products of terms $(1 - q^a t^b)$, $a, b \in \mathbb{Z}$.

2. "Evaluation": The value of P_λ at $1, t, \ldots, t^{n-1}$ also factors nicely.

3. "Symmetry": For any other partition μ with $\leq n$ parts,

$$\frac{P_\lambda(\ldots, q^{\mu_i} t^{n-i}, \ldots)}{P_\lambda(\ldots, t^{n-i}, \ldots)} = \frac{P_\mu(\ldots, q^{\lambda_i} t^{n-i}, \ldots)}{P_\mu(\ldots, t^{n-i}, \ldots)}.$$

Koornwinder polynomials

Macdonald extended the definition to an arbitrary root system, proved existence and formulated analogues of Norm, Evaluation, and Symmetry conjectures (proved by Cherednik). For B/C root systems (hyperoctahedral/signed permutation symmetry), Koornwinder found a further generalization, and Macdonald extended the conjectures. Relevant density is

$$\prod_{1\le i\le n} \frac{(x_i^{\pm 2};q)}{(ax_i^{\pm 1}, bx_i^{\pm 1}, cx_i^{\pm 1}, dx_i^{\pm 1};q)} \prod_{1\le i<j\le n} \frac{(x_i^{\pm 1}x_j^{\pm 1};q)}{(tx_i^{\pm 1}x_j^{\pm 1};q)}.$$

(When $n = 1$, the Koornwinder polynomials become Askey-Wilson polynomials.) Macdonald's conjectures hold here, but (surprisingly for a classical root system) this was the hardest case (Sahi, using Cherednik's approach, a construction of Noumi and a reduction due to van Diejen).

An alternate proof for Koornwinder polynomials

Basic idea: Construct polynomials satisfying Evaluation and Symmetry (an overdetermined system of equations), and show that they're Koornwinder polynomials using difference equations they satisfy.

Koornwinder's proof of existence (like Macdonald's in other cases) used a self-adjoint difference operator. For general root systems, Macdonald had two constructions of such operators; the simpler used "minuscule" weights, but these don't exist for general BC. However, certain special cases *do* have minuscule weights.

If $c = q^{1/2}a$, $d = q^{1/2}b$, the Koornwinder polynomials are a special case of Macdonald's original construction (the BC/C case), and have a minuscule weight. Corresponding operator $D^{(n)}(a, b; q, t)$ takes hyperoctahedrally symmetric f to

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \le i \le n} \frac{(1 - a x_i^{\sigma_i})(1 - b x_i^{\sigma_i})}{1 - x_i^{2\sigma_i}} \prod_{1 \le i < j \le n} \frac{1 - t x_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}} \times f(\dots, q^{\sigma_i/2} x_i, \dots).$$

(N.b., Macdonald gave an explicit description of the operators for minuscule weights in all cases *except* this one.)

It turns out that this acts nicely on *general* Koornwinder polynomials:

$$D^{(n)}(a,b;q,t)K_\lambda(;q^{1/2}a,q^{1/2}b,c,d;q,t) \propto K_\lambda(;a,b,q^{1/2}c,q^{1/2}d;q,t).$$

In particular, K_λ is an eigenfunction of

$$D^{(n)}(c,d;q,t)D^{(n)}(q^{-1/2}a,q^{-1/2}b;q,t).$$

Essential property: For partitions μ,

$$\begin{aligned}(D^{(n)}&(a,b;q,t)f)(\dots,q^{\mu_i}t^{n-i}a,\dots)\\ &= \sum_\nu c_{\mu\nu}(a,b;q,t)f(\dots,q^{\nu_i}t^{n-i}q^{1/2}a,\dots)\end{aligned}$$

with ν ranging over partitions with $\mu_i - 1 \le \nu_i \le \mu_i$.

Interpolation polynomials

Symmetry conjecture involves evaluations of the form $K_\lambda(q^{\mu_i}t^{n-i}a)$. Okounkov constructed a family of symmetric polynomials with nice behaviour under such specializations:

Theorem [Okounkov]. There is a (unique) family of n-variable Laurent polynomials $P^{*(n)}_\lambda(;q,t,s)$ such that

1. $P^{*(n)}_\lambda(;q,t,s)$ is invariant under the hyperoctahedral group.
2. $P^{*(n)}_\lambda(;q,t,s)$ has leading monomial $\prod_i x_i^{\lambda_i}$.
3. For any other partition μ, if $\mu_k < \lambda_k$ for some k, then

$$P^{*(n)}_\lambda(\dots,q^{\mu_i}t^{n-i}s,\dots;q,t,s) = 0.$$

(Another overdetermined system of equations)

Key observations:

1. The evaluation and symmetry formulas uniquely determine the coefficients in the expansion of Koornwinder polynomials in interpolation polynomials. (Okounkov's "binomial formula") Coefficient of P^*_μ in K_λ is a simple factor times P^*_μ (with different parameter) evaluated at λ.

2. The interpolation polynomials satisfy a difference equation

$$D^{(n)}(s, u/s; q, t) P^{*(n)}_\lambda(; q, t, q^{1/2}s) \propto P^{*(n)}_\lambda(; q, t, s).$$

This implies that if we define polynomials using evaluation and symmetry, they automatically satisfy the difference equation w.r.to $D^{(n)}(a, b; q, t)$.

Slight technicality: we had to break the a, b, c, d symmetry to make this work, so don't get the $D^{(n)}(c, d; q, t)$ equation. But the interpolation polynomials satisfy enough identities to let one recover this symmetry, thus proving both Evaluation and Symmetry conjectures.

In particular, we can expand $P_{\lambda}^{*(n)}(; q, t, s')$ in terms of $P_{\lambda}^{*(n)}(; q, t, s)$, and those coefficients [plethystic specializations of Macdonald polynomials] themselves satisfy identities involving values of interpolation polynomials.

Can also use these identities to directly prove a special case of orthogonality (in which the contour integral degenerates to a sum of residues), with explicit nonzero inner products. This is a sufficiently general special case to prove the Norm conjecture.

Elliptic difficulties

There are some issues with extending this to the elliptic level:

0. What should the analogues of the Macdonald conjectures be?

1. We used "leading monomial" a lot; this makes even less sense at the elliptic level than it did at the q level. Elliptic functions don't even *have* monomials!

2. Elliptic functions also tend to have poles, and we'll need to specify something about where they are.

3. If we make the difference operator elliptic in the obvious way (replace $(1-x)$ by $\theta_p(x)$), the resulting coefficients aren't elliptic.

Elliptic difference operators

Extending the operators is pretty easy. The key point is that although $1 - x^2$ has degree 2, $\theta_p(x^2)$ has degree 4. That is, $\theta_p(x^2)$ has four zeros in any fundamental region. So we just need more numerator factors!

So we need more numerator factors in the difference operator. The simplest way to do this is:

$$\sum_{\sigma\in\{\pm1\}^n} \prod_{1\le i\le n} \frac{\theta_p(ax_i^{\sigma_i}, bx_i^{\sigma_i}, cx_i^{\sigma_i}, dx_i^{\sigma_i})}{\theta_p(x_i^{2\sigma_i})}$$
$$\prod_{1\le i<j\le n} \frac{\theta_p(tx_i^{\sigma_i}x_j^{\sigma_j})}{\theta_p(x_i^{\sigma_i}x_j^{\sigma_j})} f(\dots, q^{\sigma_i/2}x_i, \dots).$$

This is elliptic iff $t^{n-1}abcd = p$, so defines a family of operators $D^{(n)}(a, b, c; q, t)$.

Note that this operator tends to introduce poles, and move the existing poles. This phenomenon isn't too bad if the poles of f are on a suitable geometric progression: If

$$\Big(\prod_{\substack{1\le i\le n\\1\le j\le m}}\theta_p(q^{-j+1/2}bx_i^{\pm 1})\Big)f$$

is holomorphic, then so is

$$\Big(\prod_{\substack{1\le i\le n\\1\le j\le m}}\theta_p(q^{-j}bx_i^{\pm 1})\Big)D^{(n)}(a,b,c;q,t)f.$$

So we should replace "polynomial" by "has poles along such a geometric progression". (After all, polynomial just means "all poles are at infinity"...)

What about the notion of leading monomial? We can think of this as specifying asymptotics at infinity; equivalently, since we're allowing poles there, specifying the leading monomial is tantamount to specifying that after clearing the poles, the result vanishes to various orders.

Key idea: We're already using vanishing conditions to define interpolation polynomials, so let's just replace the leading monomial condition with another set of vanishing conditions!

[N.b., Coskun/Gustafson independently constructed these]

The resulting interpolation functions satisfy many identities; in particular, connection coefficients (change of basis coefficients under changing a parameter) are themselves values of interpolation functions. Moreover, we find that if we ask for a triangular linear combination of interpolation functions that satisfies an S_4 symmetry, there is essentially only one for which interpolation function identities prove the symmetry. We thus obtain our elliptic analogues of Koornwinder polynomials.

One issue is that the operator moves poles, so tends to break up orthogonality. We thus instead obtain *biorthogonal* functions (dual bases of two spaces with constrained poles). Similarly, the operators move poles in consistent directions, and thus never give rise to eigenvalue equations.

This is still good enough, though, that all of the Macdonald conjectures for *classical* root systems are limiting cases of results at the elliptic level, though. (The Koornwinder limit is easy, the Macdonald limit somewhat more subtle.)

Interestingly, though we introduced the interpolation functions just as an intermediate step in the construction, they're actually special cases of the biorthogonal functions! In particular, they satisfy their own version of evaluation and symmetry (the latter in sharp contrast to the interpolation polynomials; the limit breaks the symmetry).

Thus it would seem that any extension of this theory (either to nonsymmetric functions à la Cherednik, or to other root systems) would necessarily involve analogues of interpolation functions.

Recent developments (Work in progress)

1. A number of identities relating different classical groups have elliptic analogues; e.g., the fact that

$$\int_{O\in O_n} s_\lambda(O)dO$$

vanishes unless every part of λ is even. This generalizes to one of a family of $\geq$ 20 multivariate quadratic transformations related to Littlewood and Kawanaka identities. (Cauchy $\to W(E_7)$) (Proof uses formal power series (p a formal variable) and symmetric functions.)

2. Connections with noncommutative geometry, integrable systems, and ((double) affine) Hecke algebras.