Descent sets for oscillating tableaux

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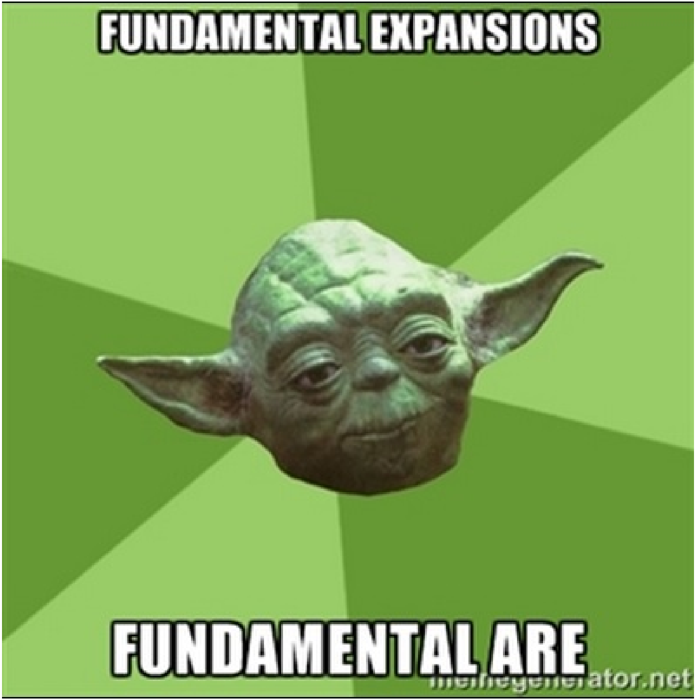
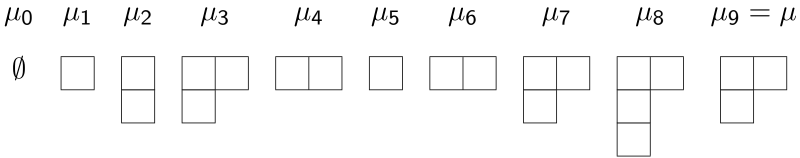

FUNDAMENTAL EXPANSIONS
FUNDAMENTAL ARE

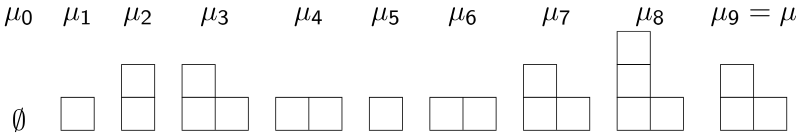
n-symplectic oscillating tableaux



an oscillating tableau is a sequence of partitions $(\mu_0, \mu_1, \ldots, \mu_r)$

- beginning with $\emptyset$
- Ferrers diagrams of consecutive partitions differ by precisely one cell

n-symplectic oscillating tableaux



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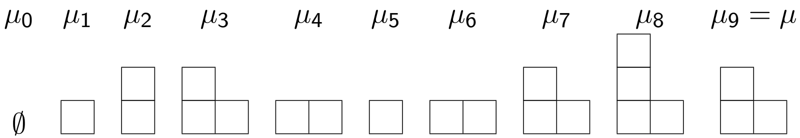

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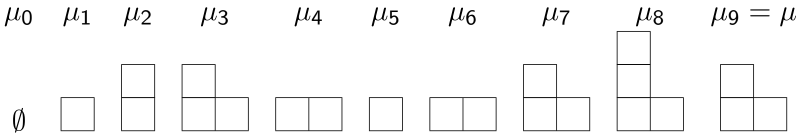
- r is the length $\quad r = 9$
- $\mu = \mu_r$ is the (final) shape $\quad \mu = (21)$
- n-symplectic if μ_i has at most n parts for all i $\quad n \geq 3$

n-symplectic oscillating tableaux



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why are n-symplectic oscillating tableaux interesting?

combinatorialist's answer

n-symplectic oscillating tableaux of length r and empty shape

and

$(n+1)$-noncrossing perfect matchings of $\{1, 2, \ldots, r\}$

are in bijection [Sundaram, Chen-Deng-Du-Stanley-Yan]!

but that's not for today...

Schur-Weyl duality

let V be the defining representation of the

$$\text{general linear group } \mathrm{GL}(n)$$

and consider its r-th tensor power $V^{\otimes r}$:

- $\mathrm{GL}(n)$ acts diagonally
- $\mathfrak{S}_r$ acts by permuting tensor positions

then

$$V^{\otimes r} \cong \bigoplus_{\substack{\mu \vdash r \\ \ell(\mu) \leq n}} V(\mu) \otimes S(\mu)$$

as $\mathrm{GL}(n) \times \mathfrak{S}_r$ modules.

($V(\mu)$ and $S(\mu)$ are the irreducible representations of $\mathrm{GL}(n)$ and $\mathfrak{S}_r$ corresponding to the partition μ)

Robinson-Schensted correspondence

the combinatorial counterpart of

$$V^{\otimes r} \cong \bigoplus_{\substack{\mu \vdash r \\ \ell(\mu) \leq n}} V(\mu) \otimes S(\mu)$$

is the Robinson-Schensted correspondence

$$\{1, \ldots, n\}^r \leftrightarrow \bigcup_{\substack{\mu \vdash r \\ \ell(\mu) \leq n}} \mathsf{SSYT}(\mu, n) \times \mathsf{SYT}(\mu)$$

- $V(\mu)$ has a basis indexed by $\mathsf{SSYT}(\mu, n)$,
  semistandard Young tableaux of shape μ, entries in $\{1, \ldots, n\}$
- $S(\mu)$ has a basis indexed by $\mathsf{SYT}(\mu)$,
  standard Young tableaux of shape μ

'symplectic' Schur-Weyl duality

let V be the defining representation of the

symplectic group $\mathrm{Sp}(2n)$

and consider its r-th tensor power $V^{\otimes r}$:

- $\mathrm{Sp}(2n)$ acts diagonally
- $\mathfrak{S}_r$ acts by permuting tensor positions

then

$$V^{\otimes r} \cong \bigoplus_{\ell(\mu)\leq n} V^{\mathrm{Sp}}(\mu) \otimes U(n,r,\mu)$$

as $\mathrm{Sp}(2n) \times \mathfrak{S}_r$ modules.

($V^{\mathrm{Sp}}(\mu)$ is the irreducible representations of $\mathrm{Sp}(2n)$
corresponding to the partition μ,
$U(n,r,\mu)$ is the isotypic component of type μ, an $\mathfrak{S}_r$ module)

Berele's correspondence

a combinatorial counterpart of

$$V^{\otimes r} \cong \bigoplus_{\ell(\mu)\leq n} V^{\mathrm{Sp}}(\mu) \otimes U(n,r,\mu)$$

is Berele's correspondence

$$\{\pm 1, \ldots, \pm n\}^r \leftrightarrow \bigcup_{\ell(\mu)\leq n} \mathsf{K}(\mu, n) \times \mathsf{Osc}(n,r,\mu)$$

- $V^{\mathrm{Sp}}(\mu)$ has a basis indexed by $\mathsf{K}(\mu, n)$, *King's n-symplectic semistandard tableaux* of shape μ, entries in $\{\pm 1, \ldots, \pm n\}$
- $U(n,r,\mu)$ has a basis indexed by $\mathsf{Osc}(n,r,\mu)$, *n-symplectic oscillating tableaux* of length r, shape μ

use n-symplectic oscillating tableaux to understand the isotypic components $U(n, r, \mu)$!

in particular, compute their Frobenius character

Frobenius character

the Frobenius map ch is a ring isomorphism between

- the ring of (virtual) characters of the symmetric group, and
- the ring of symmetric functions

set $\operatorname{ch} U = \operatorname{ch} \chi$ for a representation U with character χ

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example

let V be the defining representation of $\mathrm{GL}(n)$

by Schur-Weyl the isotypic component of type μ in $V^{\otimes r}$ is $S(\mu)$

its Frobenius character is

$$\mathsf{ch}\, S(\mu) = s_\mu$$

Sundaram's correspondence

to determine the Frobenius character of $U(n, r, \mu)$,
decompose it into $\mathfrak{S}_r$-irreducibles:

$$U(n, r, \mu) \cong \bigoplus_{\lambda \vdash r} a(\lambda, \mu) S(\lambda)$$

then

$$\mathrm{ch}\, U(n, r, \mu) = \sum_{\lambda \vdash r} a(\lambda, \mu)\, s_\lambda$$

Sundaram's correspondence

the combinatorial counterpart of

$$U(n, r, \mu) \cong \bigoplus_{\lambda \vdash r} a(\lambda, \mu) S(\lambda)$$

is Sundaram's correspondence

$$\mathrm{Osc}(n, r, \mu) \leftrightarrow \bigcup_{\substack{\lambda \vdash r \\ \beta \vdash r - |\mu| \\ \beta \text{ has even column lengths}}} \mathrm{LR}(n, \lambda/\mu, \beta) \times \mathrm{SYT}(\lambda)$$

- $a(\lambda, \mu)$ is the cardinality of $\mathrm{LR}(n, \lambda/\mu, \beta)$, the set of n-symplectic Littlewood-Richardson tableaux of shape λ/μ and weight β

the Frobenius character of $U(n, r, \mu)$

$$\operatorname{ch} U(n,r,\mu) = \sum_{\lambda \vdash r} \left(\sum_{\substack{\beta \vdash r - |\mu| \\ \beta \text{ has even column lengths}}} c^{\lambda}_{\mu,\beta}(n) \right) s_{\lambda}$$

where $c^{\lambda}_{\mu,\beta}(n) = \#\,\mathrm{LR}(n, \lambda/\mu, \beta)$

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we want something simpler!

quasisymmetric expansion

the fundamental quasisymmetric functions are

$$F_D = \sum_{\substack{i_1 \le \cdots \le i_r \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

a descent in a standard Young tableau is an entry k such that $k+1$ is in a lower row in English notation

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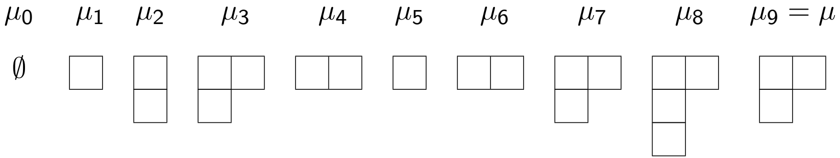
a descent in a standard Young tableau is an entry k such that $k+1$ is in a higher row

then, the Frobenius character of $S(\mu)$ can also be written as

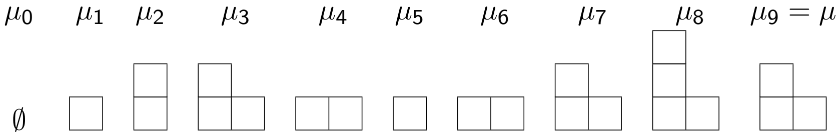
$$\operatorname{ch} S(\mu) = s_\mu = \sum_{Q \in \mathrm{SYT}(\mu)} F_{\mathrm{Des}(Q)}.$$

let's do the same for the symplectic group

descents for oscillating tableaux



descents for oscillating tableaux



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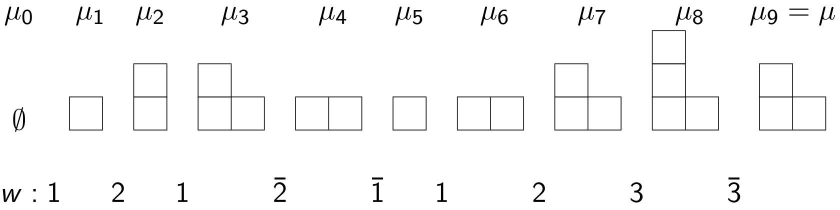
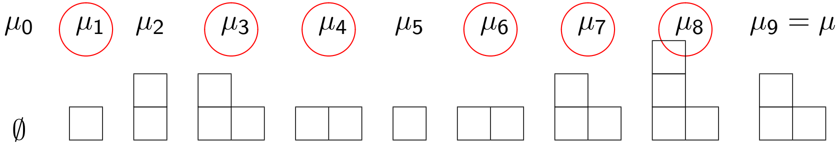


- convert the oscillating tableau to a highest weight word $w_1 w_2 \dots w_r$ with letters in $1 < 2 < \dots < n < \bar{n} < \dots < \bar{2} < \bar{1}$

descents for oscillating tableaux



$w: 1 \quad 2 \quad 1 \quad \bar{2} \quad \bar{1} \quad 1 \quad 2 \quad 3 \quad \bar{3}$

- convert the oscillating tableau to a highest weight word $w_1 w_2 \dots w_r$ with letters in $1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}$
- k is a descent if $w_k < w_{k+1}$

quasisymmetric expansion

Sundaram's correspondence

$$\mathrm{Osc}(n, r, \mu) \leftrightarrow \bigcup_{\substack{\lambda \vdash r \\ \beta \vdash r - |\mu| \\ \beta \text{ has even column lengths}}} \mathrm{LR}(n, \lambda/\mu, \beta) \times \mathrm{SYT}(\lambda)$$

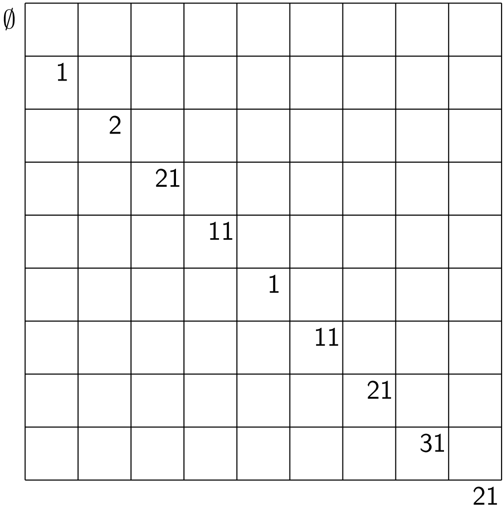
preserves descent sets:

$$O \leftrightarrow (L, Q) \Rightarrow \mathrm{Des}(O) = \mathrm{Des}(Q)$$

therefore

$$\mathrm{ch}\, U(n, r, \mu) = \sum_{O \in \mathrm{Osc}(n,r,\mu)} F_{\mathrm{Des}(O)}.$$

proof



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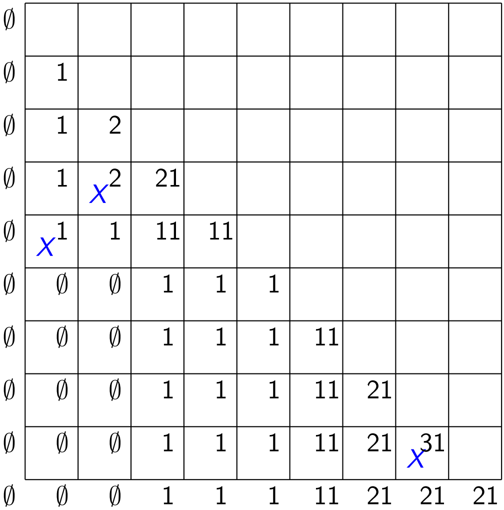
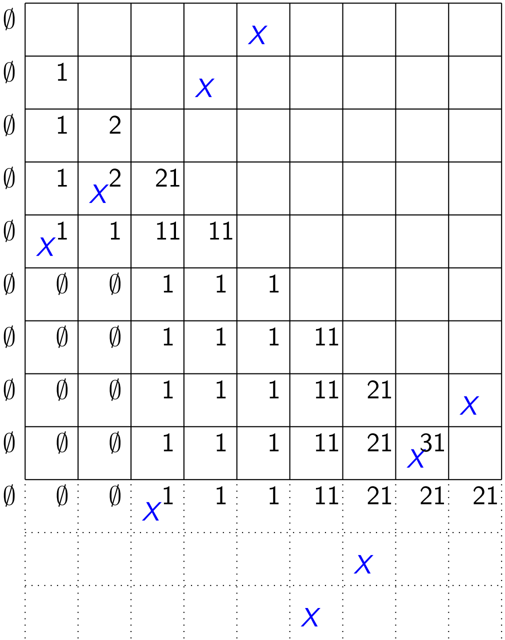


∅
∅ 1
∅ 1 2
∅ 1 X 2 21
∅ X 1 1 11 11
∅ ∅ ∅ 1 1 1
∅ ∅ ∅ 1 1 1 11
∅ ∅ ∅ 1 1 1 11 21
∅ ∅ ∅ 1 1 1 11 21 X 31
∅ ∅ ∅ 1 1 1 11 21 21 21

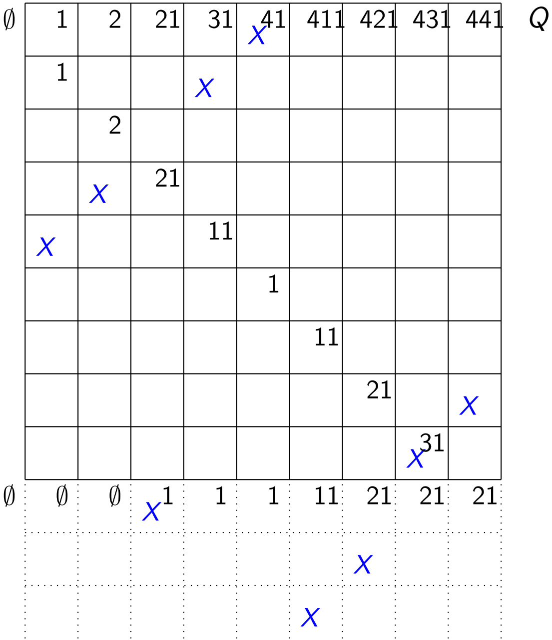
proof



proof

∅	1	2	21	31	X 41	411	421	431	441
∅	1	2	21	X 31	31	311	321	331	431
∅	1	2	21	21	21	211	221	321	421
∅	1	X 2	21	21	21	211	221	321	421
∅	X 1	1	11	11	11	111	211	311	411
∅	∅	∅	1	1	1	11	21	31	41
∅	∅	∅	1	1	1	11	21	31	41
∅	∅	∅	1	1	1	11	21	31	X 41
∅	∅	∅	1	1	1	11	21	X 31	31
∅	∅	∅	X 1	1	1	11	21	21	21
							X		
						X			

proof



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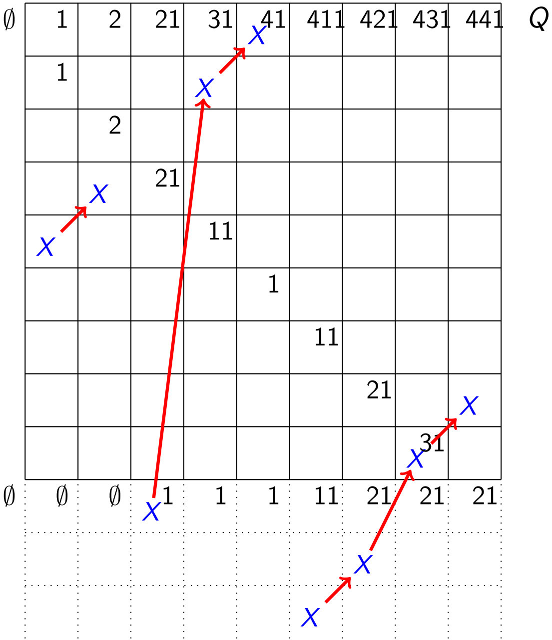

Summary

- let V the defining representation of $\mathrm{Sp}(2n)$
- let $\mathrm{Sp}(2n)$ act diagonally on $V^{\otimes r}$
- let $\mathfrak{S}_r$ act on $V^{\otimes r}$ by permuting tensor positions

then the Frobenius characteristic of the isotypic component of type μ in $V^{\otimes r}$ in terms of fundamental quasisymmetric functions is

$$\sum_{O \in \mathrm{Osc}(n,r,\mu)} F_{\mathrm{Des}(O)}$$

(this is easier to remember and to generalize than the expansion in terms of Schur functions due to Sundaram)

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outlook:

- defining representations of orthogonal groups and G_2
- cyclic sieving polynomials for promotion
- other representations