On Orbits of Order Ideals of Minuscule Posets

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What is a minuscule representation?

Definition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A representation V of $\mathfrak{g}$ is *minuscule* if the Weyl group W acts transitively on the weights of the Cartan subalgebra $\mathfrak{h}$.

Example

Let $\mathfrak{g} = \mathfrak{sl}_5$, and let $V = \bigwedge^2(\mathbb{C}^5)$. Then $\mathfrak{h}$ is the subalgebra of diagonal matrices, and $W = \mathfrak{S}_5$ acts transitively on $\{\varepsilon_i + \varepsilon_j + \varepsilon_k : 1 \leq i < j < k \leq 5\}$.

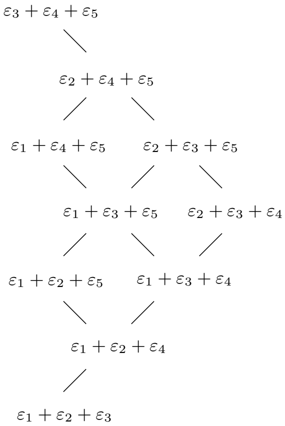
The weight poset

Definition

The *root order* on the weights is the transitive closure of the relations $\mu < \omega$ for all weights μ and ω such that $\omega - \mu$ is a simple root.

The weight poset, example

Here is the weight poset for the representation $\bigwedge^2(\mathbb{C}^5)$ of $\mathfrak{sl}_5$.



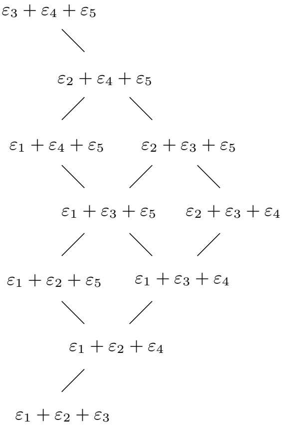
The minuscule poset

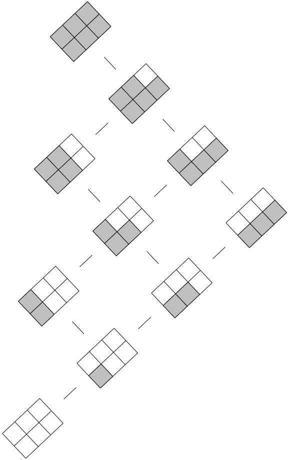
Let V be a minuscule representation of $\mathfrak{g}$.

- There exists a unique poset P for which the weight poset Q is the poset of order ideals of P.
- P is called the *minuscule poset* for V.

The minuscule poset, example

The minuscule poset for the representation $\bigwedge^2(\mathbb{C}^5)$ of $\mathfrak{sl}_5$ is the product of two chains.





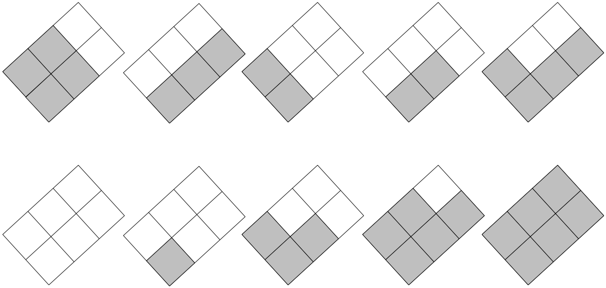
The Fon-Der-Flaass action

Let P be a poset, and let $J(P)$ be the poset of order ideals of P.

Definition

The *Fon-Der-Flaass action* Ψ maps each order ideal $I \in J(P)$ to the order ideal $\Psi(I)$ whose maximal elements coincide precisely with the minimal elements of $P \setminus I$.

Here are the orbits of the order ideals of the poset $P := [2] \times [3]$ under Ψ.



Query

Is there a uniform way to predict the orbit structure of the order ideals of minuscule posets under the Fon-Der-Flaass action?

Answer

Yes.

- P is a minuscule poset.
- $X = P \times [m]$.
- $X(q)$ is the rank-generating function of the poset of order ideals $J(X)$.

Theorem (Rush–Shi)

If $m \in \{1, 2\}$, then the triple $(X, X(q), \langle \Psi \rangle)$ exhibits the cyclic sieving phenomenon of Reiner, Stanton, and White.

What is the cyclic sieving phenomenon?

Definition

Let $\langle c \rangle$ be a cyclic action on a finite set X, and let n be the order of c. For $X(q) \in \mathbb{Z}[q]$, the triple $(X, X(q), \langle c \rangle)$ exhibits the *CSP* if, for all $d \in \mathbb{Z}$, the following equality holds:

$$|\{x \in X : c^d(x) = x\}| = X(e^{\frac{2\pi i d}{n}}).$$

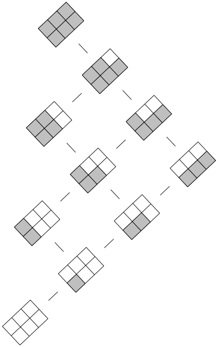
Let $\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \{i, j, k : 1 \leq i < j < k \leq 5\}$.

$\langle (12345) \rangle \subset \mathfrak{S}_5$ acts on X.

And $\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q, \langle (12345) \rangle \right)$ exhibits the CSP.

CSP, example

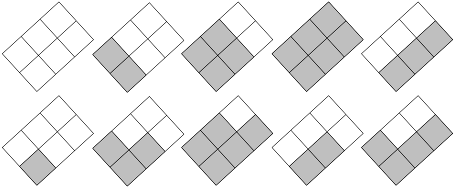
This is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.



$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$ is the rank-generating function.

CSP, example

These are the orbits of the order ideals under (12345).



This is in agreement with the predictions of the CSP.

CSP, minuscule setting

Suppose...

- W is a Weyl group;
- W_J is a maximal parabolic subgroup;
- W^J is the set of minimal-length representatives for the cosets of W/W_J.

Then...

- W^J is a poset under the Bruhat order;
- The action of W on W/W_J induces an action of W on W^J.

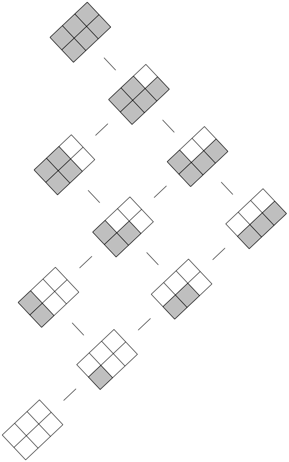
CSP, minuscule setting

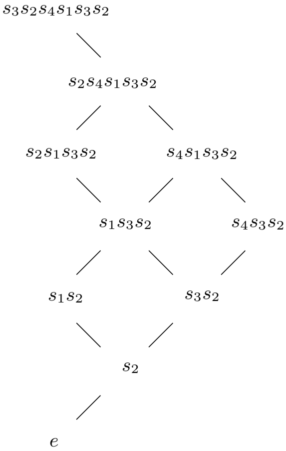
Theorem (Reiner–Stanton–White)

For $W^J(q) := \sum_{w \in W^J} q^{\ell(w)}$, *the triple* $(W^J, W^J(q), \langle c \rangle)$ *exhibits the CSP for all Coxeter elements* c *of* W.

CSP, minuscule setting, example

$$W = \mathfrak{S}_5;\ W_J = \mathfrak{S}_2 \times \mathfrak{S}_3, \text{ and } c = (12345).\ \ W^J(q) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q.$$





Bijection between settings

Suppose...

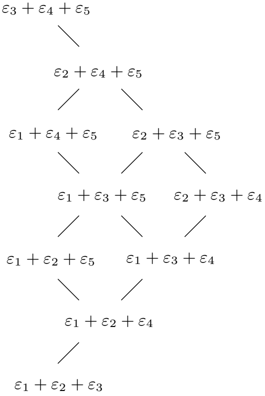
- V is a minuscule representation of a complex simple Lie algebra $\mathfrak{g}$ with Weyl group W;
- λ is the highest weight of V;
- W_J is the maximal parabolic subgroup stabilizing λ.

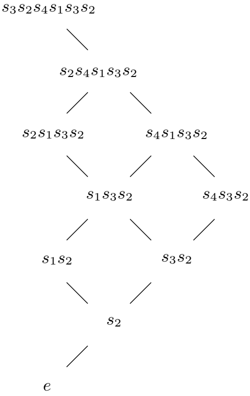
Then the map

$$\begin{array}{rcl} W^J & \to & Q \\ w & \mapsto & w_0 w \lambda \end{array}$$

is an isomorphism of posets.

Bijection between settings, example



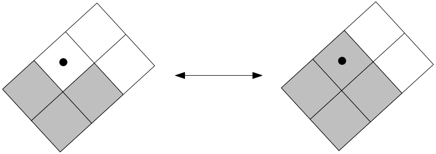


Here $\mathfrak{g}=\mathfrak{sl}_5$, $V=\bigwedge^2(\mathbb{C}^5)$, $W=\mathfrak{S}_5$, $\lambda=\varepsilon_3+\varepsilon_4+\varepsilon_5$, and $W_J=\mathfrak{S}_2\times\mathfrak{S}_3$.

Toggling order ideals

Let P be a poset. The *toggle group* acts on $J(P)$.

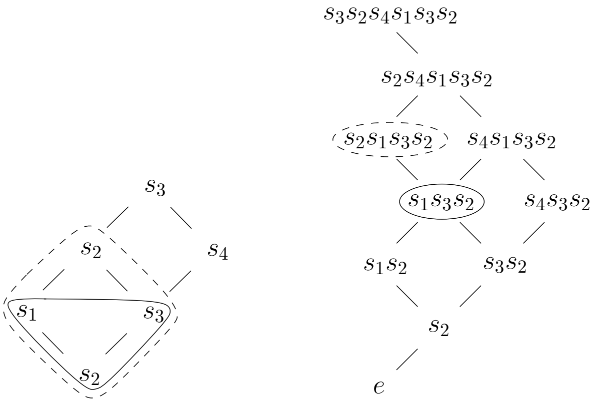
$$t_p(I) = \begin{cases} I\Delta\{p\} & \text{if } I\Delta\{p\} \in J(P) \\ I & \text{otherwise} \end{cases}$$



Toggling at the indicated element

Proof of the main result

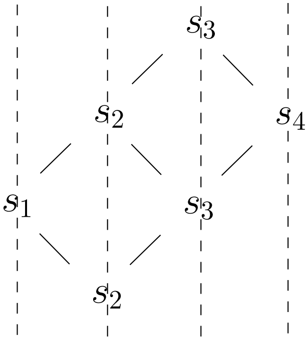
The isomorphism $J(P) \cong W^J$ is equivariant.



Multiplying $w \in W^J$ by s_i corresponds to toggling the corresponding order ideal $I \in J(P)$ at all elements labeled by s_i

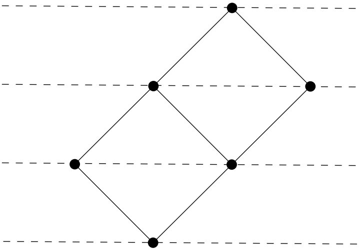
Proof of the main result (cont'd.)

Decomposing the Coxeter element action into toggles...



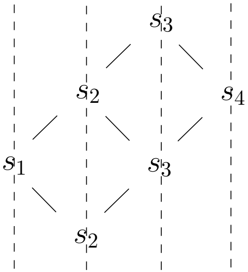
Proof of the main result (cont'd.)

... and decomposing the Fon-Der-Flaass action into toggles...

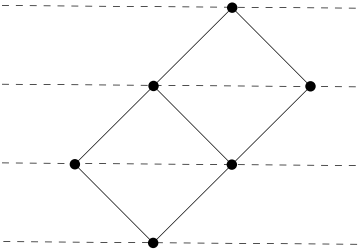


Proof of the main result (cont'd.)

... we see that the two actions are conjugate!



Toggle s_1, s_2, s_3, s_4.



Toggle ranks 1, 2, 3, 4.

Conclusion

Recall that the CSP predicts the orbit structure of the order ideals of $J(P)$ under the Coxeter element action.

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Hence it also predicts the orbit structure of the order ideals under the Fon-Der-Flaass action.

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Hence it also predicts the orbit structure of the order ideals under the Fon-Der-Flaass action.

Q.E.D.

Thanks

Merci!