Recent progress on the diameter of polyhedra and simplicial complexes

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Outline

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1. Introduction

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2. The maximum diameter of polyhedra and polytopes

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2. The maximum diameter of polyhedra and polytopes
3. The maximum diameter of simplicial complexes, and other abstractions.

Intro: the Hirsch Conjecture

Polyhedra and polytopes

Definition

A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

Definition

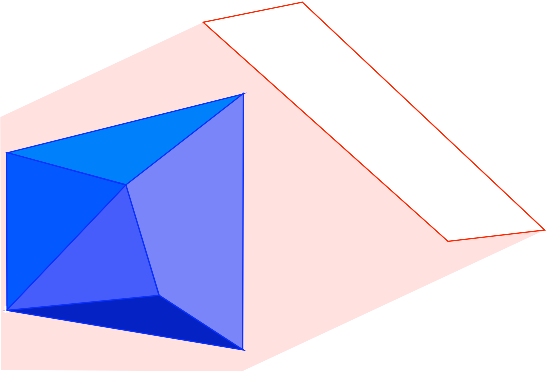
A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.

Polytope = bounded polyhedron.

The dimension of P is the dimension of its affine hull.

Faces of *P*

Let *P* be a polytope (or polyhedron) and let *H* be a hyperplane
not cutting,



Faces of P

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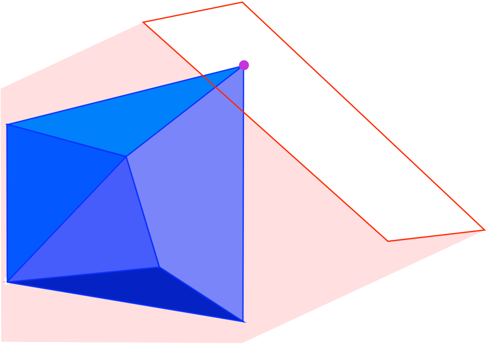
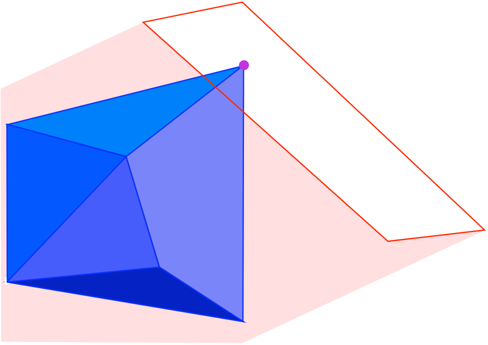


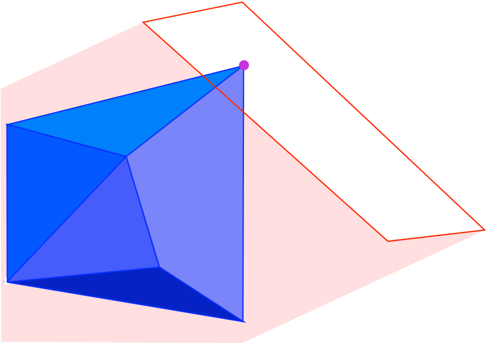
Faces of P

We say that $H \cap P$ is a face of P.



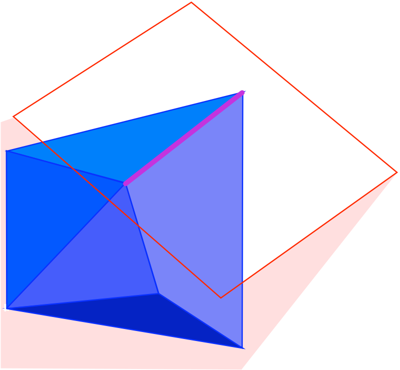
Faces of P

Faces of dimension 0 are called vertices.



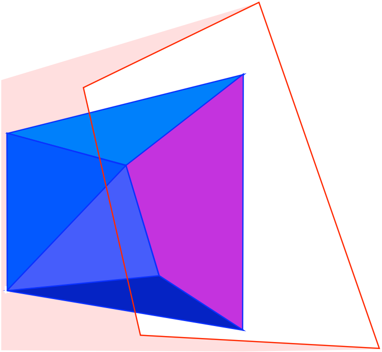
Faces of P

Faces of dimension 1 are called edges.



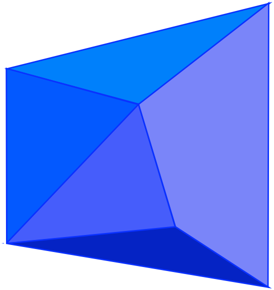
Faces of P

Faces of dimension $d - 1$ are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a graph (finite, undirected)



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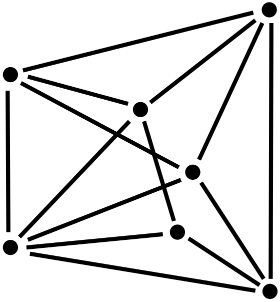

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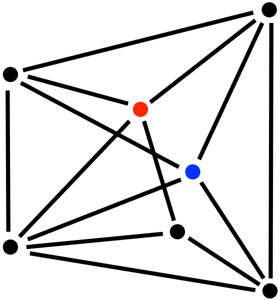

The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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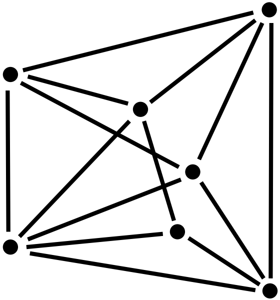

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\operatorname{diam}(P) = \max\{d(u,v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture (W. M. Hirsch, 1957)

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polytope	facets	dimension	$n - d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k + 2$	3	$k - 1$	$\lfloor k/2 \rfloor + 1$
n-cube	$2n$	n	n	n

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3. In 1967, Klee and Walkup disproved the unbounded case. In 2010 I disproved the bounded case.
4. The constructions do not produce polytopes whose diameter is more than a small constant times the Hirsch bound.
5. In the general case **we do not even know of a polynomial bound** for diam(P) in terms of n and d.

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- In particular, a polynomial pivot rule for the simplex method would prove that Linear Programming can be performed in *strongly polynomial* time (one of Smale's "problems for the next century").

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$$H(n, d) \leq n^k, \qquad \forall n, d.$$

General bounds and known cases

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Corollary (d-step theorem)

In order to bound $H(n, d)$ it suffices to bound $H(2d, d)$

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Theorem (Larman, 1970; Barnette, 1974, linear in fixed d)

$$H(n, d) \leq n2^{d-3}, \qquad \forall n, d.$$

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Let us consider the following particular case:

Definition

A pure simplicial complex is called flag if it equals the clique complex of a graph (that is, every clique defines a simplex).

Theorem (Adiprasito-Benedetti, 2013+)

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The proof works for all pure and normal flag simplicial complexes. Pure normal s. c. include all simplicial manifolds, with or w.o. boundary.

Counter-examples to Hirsch

(and what can we expect from them)

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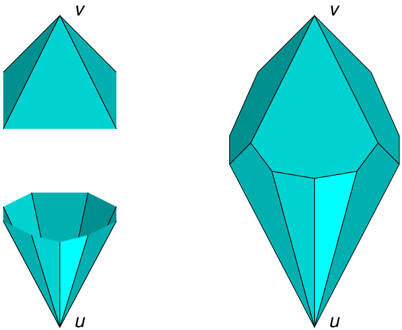
Theorem

- $H_b(86,43) \geq 44$ *(S. 2012)*
- $H_b(40,20) \geq 21$ *(Matschke-S.-Weibel, $\geq$ 2012)*

Spindles

Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).



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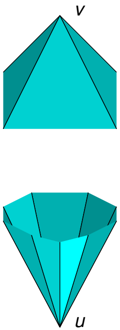

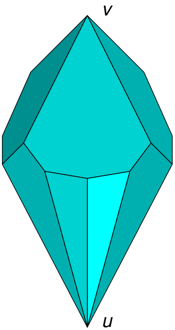

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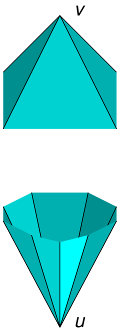

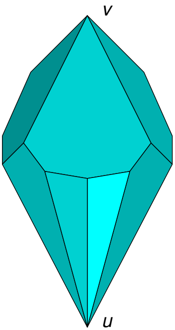

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Exercise

3-spindles have length ≤ 3.

Spindles

Theorem (S., 2012; "Strong d-step theorem for spindles")

If a d-spindle P has length $l > d$ then there is another spindle P' (of dimension $n - d$, with $2n - 2d$ facets, and length $l + n - 2d > n - d$) that violates the Hirsch conjecture.

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- "Highly non-Hirsch" 5-spindles exist, with $l \sim \sqrt{n/96}$ (S.-Matschke-Weibel, 2012+).

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Once we have a non-Hirsch polytope we can derive more via:

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E. g.: The excess of our non-Hirsch polytope with $n - d = 20$ and with diameter 21 is

$$\frac{21-20}{20} = 5\%.$$

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Corollary

For each $k \in \mathbb{N}$ there is an infinite family of non-Hirsch polytopes of fixed dimension $20k$ and with excess (tending to)

$$0.05\left(1-\frac{1}{k}\right).$$

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Lemma

Via the strong d-step Theorem, a spindle of a certain excess produces non-Hirsch polytopes of that same excess.

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Corollary

Using the Strong d-step Theorem for 5-spindles it is impossible to violate the Hirsch conjecture by more than 33%.

Non-decomposable polyhedra

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...where (strongly) decomposable complexes are defined with an extra recursive condition.

- All *polytopal* $(d-1)$-spheres and $(d-1)$-balls are $(d-1)$-decomposable, since this is equivalent to shellable.

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(A negative answer would imply the polynomial Hirsch conjecture).

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Bounds in terms of coefficients

The role of coefficients

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- If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ is defined by a totally unimodular matrix A then $\mathrm{diam}(P) \leq O(d^{16} n^3 (\log(dn))^3)$ (Dyer-Frieze 1994).

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This last result has been recently generalized to great extent.

Polytopes with bounded subdeterminants

Theorem (Bonifas-Di Summa-Eisenbrand-Hähnle-Niemeier, 2011+)

Let $P = \{x \in \mathbb{R}^d \colon Ax \le b\}$ *be a polytope defined by an integer matrix* $A \in \mathbb{Z}^{n\times d}$ *and suppose all subdeterminants of* A *are bounded in absolute value by a certain* $M \in \mathbb{N}$*. The, the diameter of* P *is bounded by* $O\left(M^2 d^{3.5} \log dM\right)$*.*

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- Plugging $M = 1$ (totally unimodular matrix) this result specializes to a drastic improvement of the Dyer-Frieze bound.

Spherical volumes

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- Fix two cones c_v and c_u and study how the volume spanned by respective “breadth first search” trees from both ends grows.
- When both volumes can be guaranteed to be at least half of the unit sphere, we have found a path from u to v.

□

Spherical volumes

The crucial step is the following “volume expansion” result:

Lemma

Let U_i denote the spherical volume covered by all cones at distance at most i from an initial cone c_u. Then, while $\mathrm{vol}(U_i)$ is less than half of the volume of the d-sphere we have:

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General simplicial complexes

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$H_c(n, d)$ is the (dual) diameter; two simplices are considered adjacent if they differ by a single vertex.

A simple, yet interesting, observation

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Lemma

- *$H_c(n, d)$ is attained at a corridor (in particular, at a pseudo-manifold) for every n, d.*
- *$H_c(n, d)$ equals the length of the maximum induced path in $J(n, d)$.*

The maximum diameter of pure simplicial complexes

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Theorem (S., 2013+)

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Corollary (S., 2013+)

$$\Omega(n^{2d/3}) \leq H_c(n,d) \leq O(n^d).$$

Normal complexes

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Manifolds (w. or wo. boundary) are normal, but pseudo-manifolds are not, in general.

The importance of being normal

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Theorem

$$H_n(n,d) \leq n^{\log d + 1}, \qquad H_n(n,d) \leq 2^{d-2} n.$$

An abstraction of normality

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Definition (Eisenbrand-Hähnle-Razborov-Rothvoss, 2010)

A connected layer family (CLF) of rank d on n symbols is a pure simplicial complex K of dimension $d-1$ with n vertices, together with a map

$$\lambda : \text{facets}(K) \to \mathbb{Z}$$

with the following property: for every simplex (of whatever dimension) $\tau \in K$ the values taken by λ in the star of τ form an interval.

The length of a CLF is the difference between the maximum and the minimum values taken by λ.

Example: A CLF of rank 2 and length $\sim 3n/2$

λ	0	1	2	3	4	5	6	7	8	9
		13	14		35	36		57	58	
Δ	12			34			56			78
		24	23		46	45		68	67	

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Let $H_{clf}(n, d) :=$ max length of a CLF of rank d on n symbols.
The example shows that:

$$H_{clf}(n, 2) \geq \left\lfloor \frac{3n}{2} \right\rfloor$$

C.l.f.'s versus normal complexes

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If a pure simplicial complex K is normal, then K is a clf with respect to the map $\lambda(v) = d(u, v)$. $\square$

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"Idea of proof": Adjacent simplices have $|\lambda(X) - \lambda(Y)| \leq n-1$.

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4. $H_{clf}(n, n/4) \geq \Omega(n^2 / \log n)$.

Connected Layer Multi-families

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Complete and injective clmf’s

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Theorem (polymath3, 2010)

The lengths of clmf's still satisfy the Kalai-Kleitman ($n^{\log d+1}$) and the Larman-Barnette ($2^{d-1}n$) bounds.

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If true, it would imply:

Conjecture

The diameter of a d-polytope with n-facets cannot exceed

$$d(n-d)+1.$$

THANK YOU