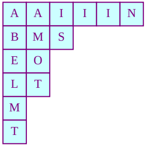
Combinatorics of non-ambiguous trees

Matteo Silimbani

(LaBRI - Bordeaux)

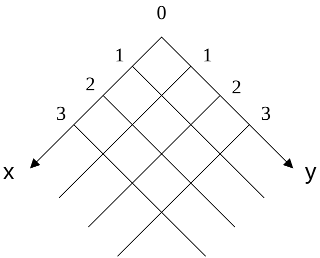


FPSAC 2013 - Université Sorbonne Nouvelle (Paris)

(joint work with J.-C. Aval, A. Boussicault and M. Bouvel)

Some conventions

Objects will be drawn in a $\mathbb{N} \times \mathbb{N}$ grid.



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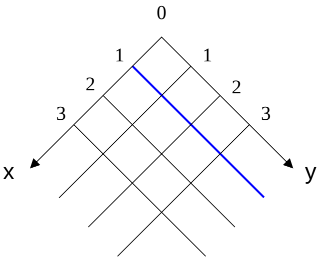

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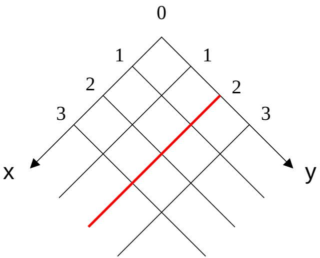
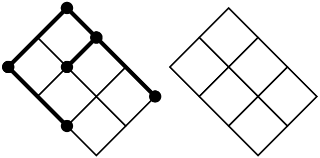


y-oriented lines = rows
x-oriented lines = columns

Intuitive definition of non ambiguous trees

A non-ambiguous tree is a binary tree embedded in the grid in such a way that the embedding of its vertices in the grid determines the tree completely (i.e. determines its edges)



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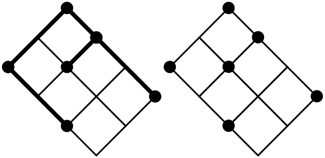

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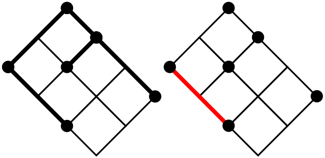

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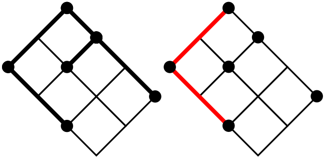

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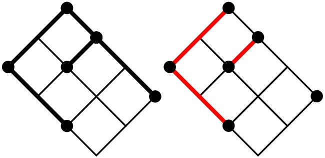

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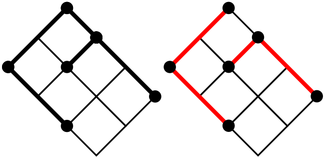

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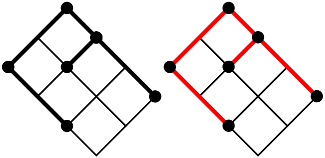

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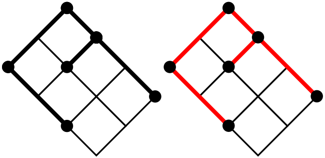

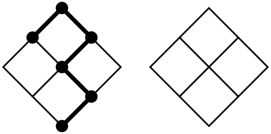

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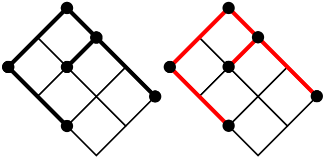

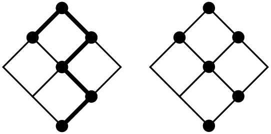

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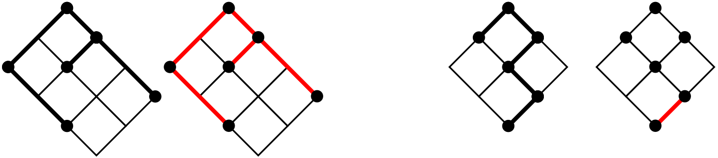

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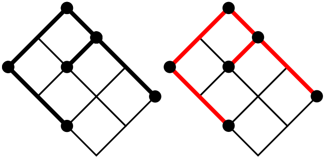

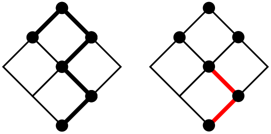

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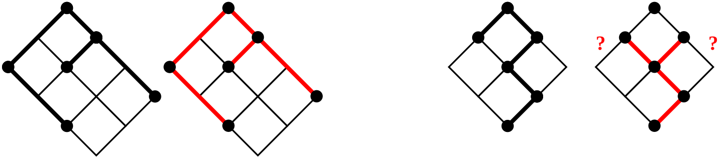
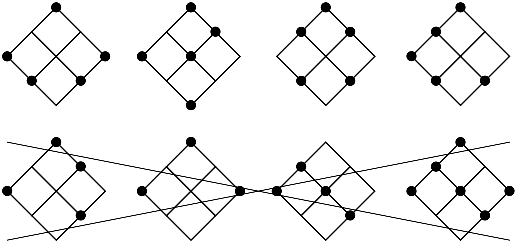


Definition of a non-ambiguous tree

A non-ambiguous tree of size n is a set A of n points $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that:

- $(0, 0) \in A$ (the root)
- every point (except the root) has a "parent"
- the pattern  is forbidden
- compactness: there is no empty row/column



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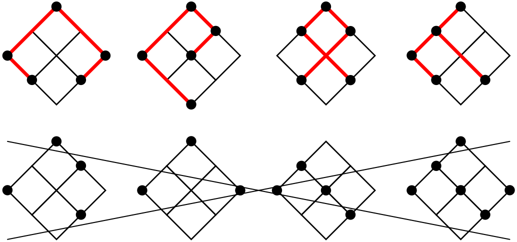


The context

Permutation tableaux and alternative tableaux are objects used to study:

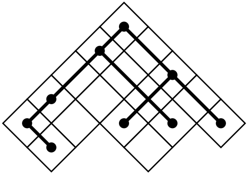
- the PASEP model in physics;
- $2-31$ pattern inside a permutation;
- excedences and cycles of a permutation;
- Laguerre polynomials;
- ...

[Burstein, Corteel, Dasse-Hartaut, Hitczenko, Josuat-Vergès, Nadeau, Postnikov, Steingrímsson, Viennot, Williams, Kim, Novelli, Thibon 2005-2012]

The context

Tree-like tableaux have been introduced to simplify and to explain some of the previous results.
[Aval, Boussicault, Nadeau, 2011]

A tree-like tableau can be defined as a Ferrers diagram containing a non-ambiguous tree



Outline of the talk

1. Enumeration of non-ambiguous trees inside a rectangular box
2. Enumeration of non-ambiguous trees with a given underlying binary tree
3. Complete non-ambiguous trees and the Bessel function
4. A bijection between parallelogram polyominoes and binary trees

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Non-ambiguous trees inside a box

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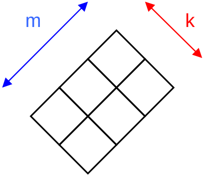

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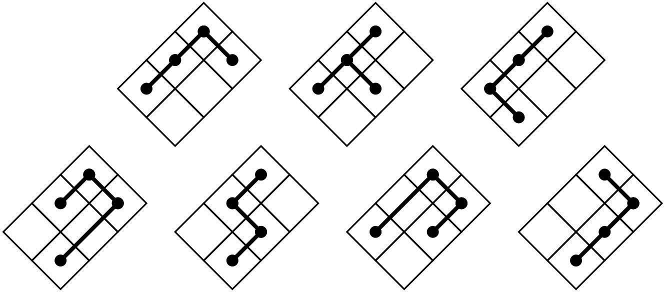

$A(3, 2) = 7$

Non-ambiguous trees inside a box

Theorem [Aval, Boussicault, Bouvel, and S. 2012]

$$\sum_{k=1}^{n} s(n,k)\, A(m,k) = n^{m-1}\, n!\,,$$

where $s(n,k)$ are the Stirling numbers of first kind.

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Proposition

$A(m,k) = p(m+k-1,m) = \#$ permutations of size $m+k-1$ having m excedences at positions $1,2,\ldots,m$

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Theorem [Ehrenborg, Steingrímsson 2000]

$$p(m+k,m) = \sum_{i=1}^{k+1} (-1)^{k+1-i} S(k+1,i)\, i!\, i^m\,,$$

where $S(n,i)$ are the Stirling numbers of second kind.

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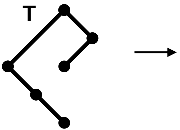
Non-ambiguous trees associated with a given binary tree

$NA(T) = \#$ non-ambiguous trees whose underlying binary tree is T.



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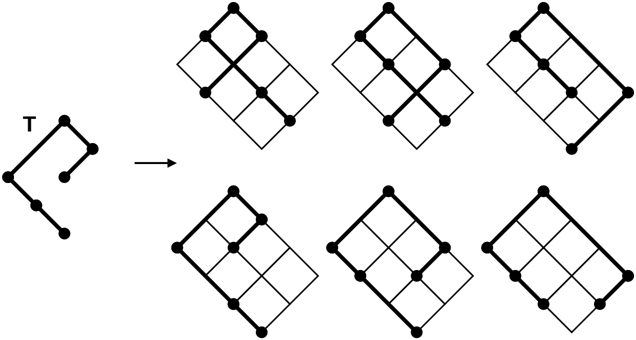

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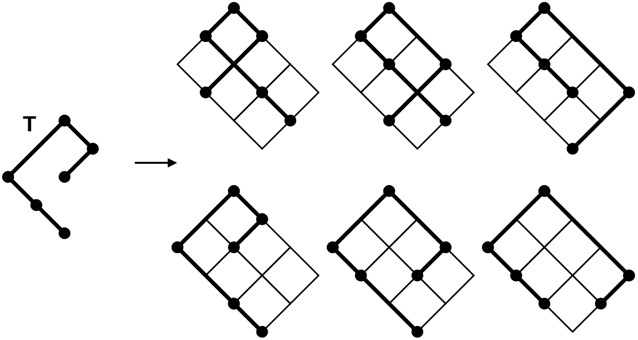

In this example, $NA(T) = 6$.

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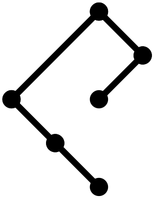
To distinguish the vertices of a binary tree T, we label them by integers to 1 to $|T|$:



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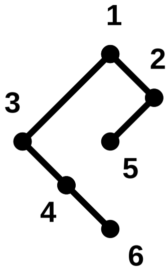

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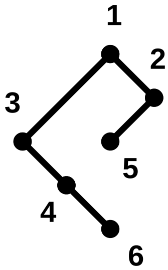

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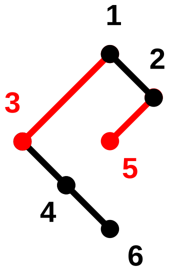

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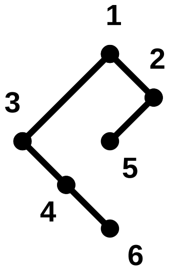

$V_L = \{$ left children $\}$ (here $V_L = \{3, 5\}$)
$V_R = \{$ right children $\}$

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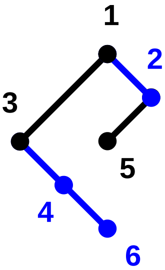
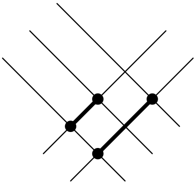
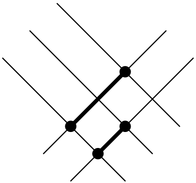


$V_L = \{$ left children $\}$ (here $V_L = \{3, 5\}$)
$V_R = \{$ right children $\}$ (here $V_R = \{2, 4, 6\}$)

The coordinates of left and right children

Two left children cannot belong to the same row





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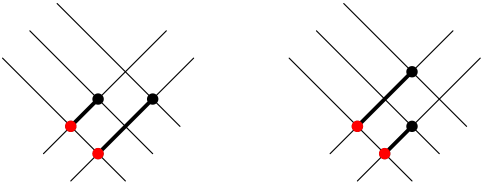

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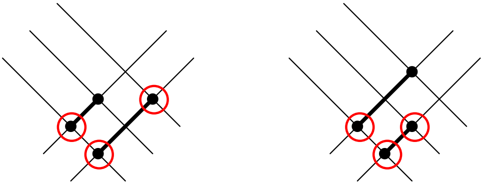

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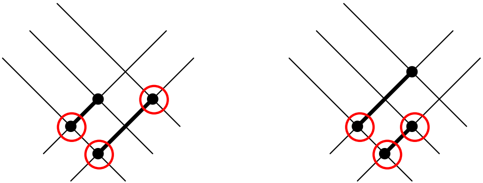

Hence, all the elements in V_L have different x-coordinates, and these coordinates form the interval $[1, |V_L|]$.

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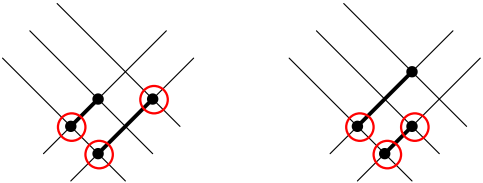
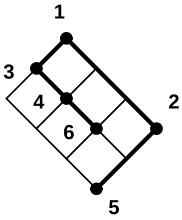


Hence, all the elements in V_L have different x-coordinates, and these coordinates form the interval $[1, |V_L|]$.

For the same reason, all the elements in V_R have different y-coordinates, and these coordinates form the interval $[1, |V_R|]$

The words α_L and α_R



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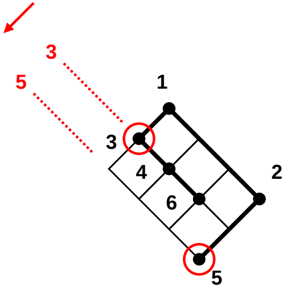

$\alpha_L = 3\,5$

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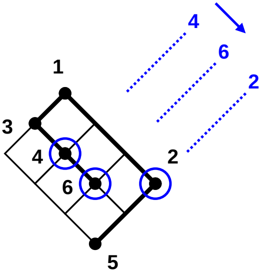
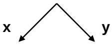


$$\alpha_L = 3\,5$$

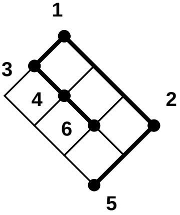
$$\alpha_R = 4\,6\,2$$

A triple representing a non-ambiguous tree

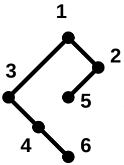


$$\Phi(A) = (T, \alpha_L, \alpha_R)$$

$A =$



$\Leftrightarrow$ ($T =$



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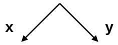

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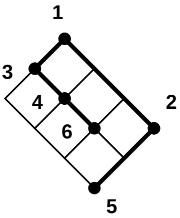
$A =$ 

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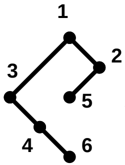
$\Bigg($ $T =$ 

$, \quad \alpha_L = 3\,5, \quad \alpha_R = 4\,6\,2 \Bigg)$

We can reconstruct A from $\Phi(A)$: $X(1) = Y(1) = 0$ and

$$\begin{cases} X(i) = \alpha_L^{-1}(i) \text{ and } Y(i) = Y(\text{parent}(i)) & i \text{ left child} \\ Y(j) = \alpha_R^{-1}(j) \text{ and } X(j) = X(\text{parent}(j)) & j \text{ right child} \end{cases}$$

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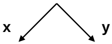
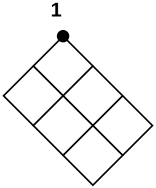
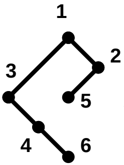


$$\Phi(A) = (T, \color{red}{\alpha_L}, \color{blue}{\alpha_R})$$

$A =$ 

$\Leftrightarrow$ $\left(T = \right.$ 

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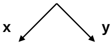
A triple representing a non-ambiguous tree



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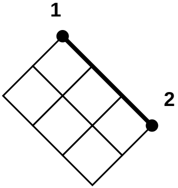
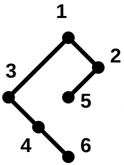
$A =$ 

$\Leftrightarrow$ $\Bigg($ $T =$ 

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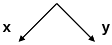

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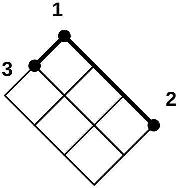
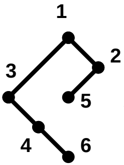
$A =$ 

$\Leftrightarrow$ $\Bigg($ $T =$ 

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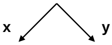

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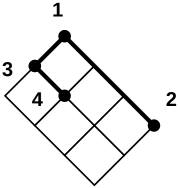
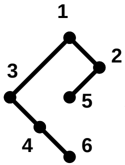
$A =$ 

$\Leftrightarrow$ $\Bigg($ $T =$ 

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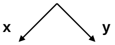

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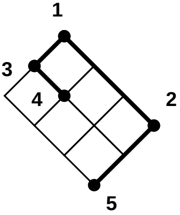

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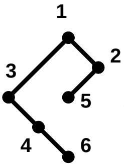

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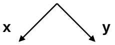

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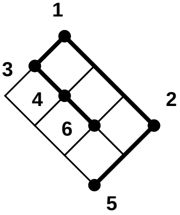
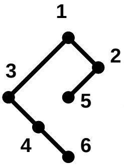
$A =$ 

$\Leftrightarrow \left(T = \right.$ 

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We can reconstruct A from $\Phi(A)$, hence Φ is injective.

Question: what is the image of Φ? Are there some constraints on α_L and α_R?

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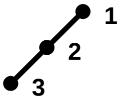

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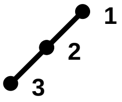


Here we must have $\alpha_L = 2\,3$

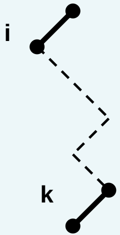
Actually, we have some constraints on α_L and α_R

- α_L must be a linear extension of a particular poset defined on V_L
- α_R must be a linear extension of a particular poset defined on V_R

The posets defined on V_L and V_R

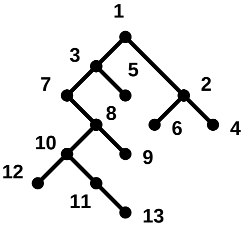
The order relation $<_{V_L}$

Given $i, k \in V_L$



$$i <_{V_L} k \quad \Longleftrightarrow \quad i \text{ is an ancestor of } k.$$

The posets defined on V_L and V_R



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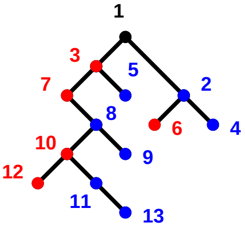

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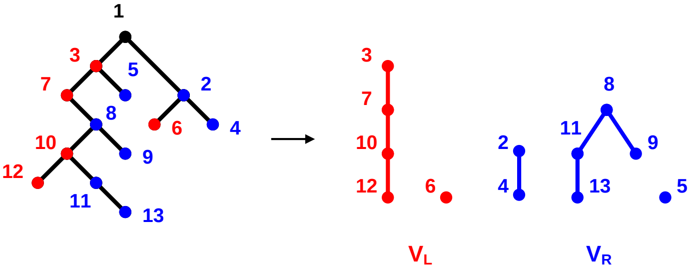

Drawing convention: In these Hasse diagrams, the minimal elements are the topmost ones.

The constraints on α_L and α_R

If i is an ancestor of k, and we must also have $X(i) < X(k)$. This means that i appears to the left of k in α_L.

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If i is an ancestor of k, and we must also have $X(i) < X(k)$. This means that i appears to the left of k in α_L.

Constraints

α_L must be a linear extension of the poset $(V_L, <_{V_L}) \rightarrow \alpha_L \in \mathcal{L}(V_L)$
α_R must be a linear extension of the poset $(V_R, <_{V_R}) \rightarrow \alpha_R \in \mathcal{L}(V_R)$

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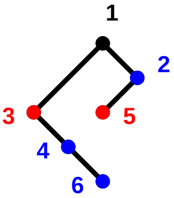
We can show that these coniditions are also sufficient.

The linear extensions of V_L and V_R



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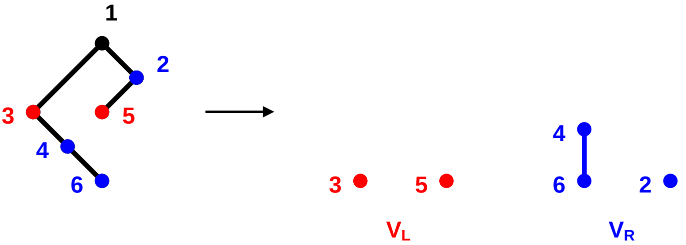

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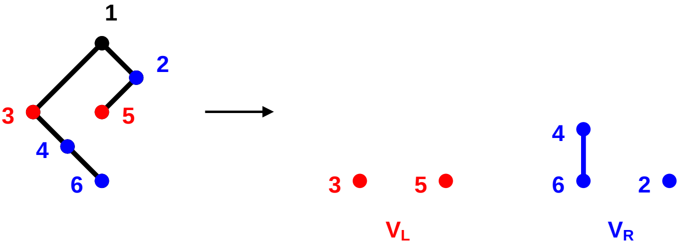
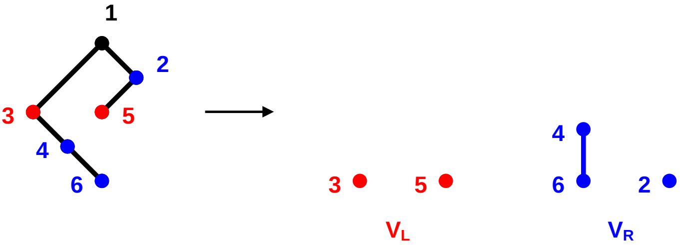


$\mathcal{L}(V_L) = \{35, 53\}$

The linear extensions of V_L and V_R



$\mathcal{L}(V_L) = \{35, 53\} \quad \mathcal{L}(V_R) = \{246, 426, 462\}$

The linear extensions of V_L and V_R

$$\mathcal{L}(V_L) \times \mathcal{L}(V_R) = \{(35, 246), (35, 426), (35, 462), (53, 246), (53, 426), (53, 462)\}$$

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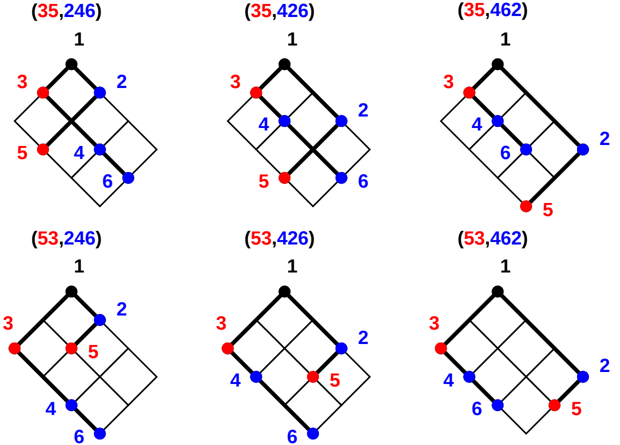

Enumerative consequences

Theorem [ABBS 2012]

$$NA(T) = |\mathcal{L}(V_L)| \times |\mathcal{L}(V_R)|$$

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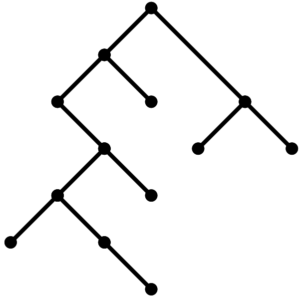
The Hasse diagrams of $(V_L, <_{V_L})$ and $(V_R, <_{V_R})$ are forests.

We can use Knuth's hook formula for forests to compute the value of $NA(T)$.

A hook formula for $NA(T)$



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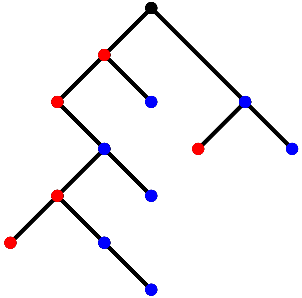

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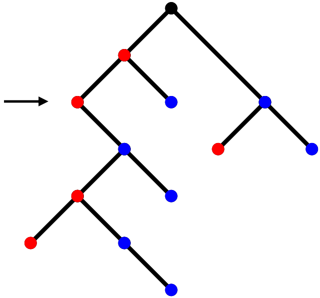

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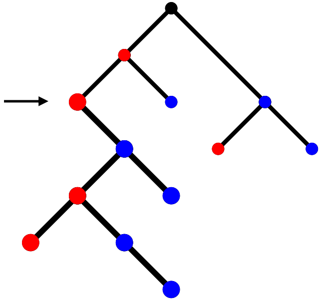

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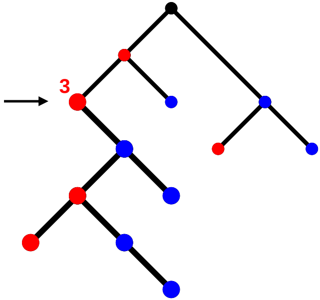

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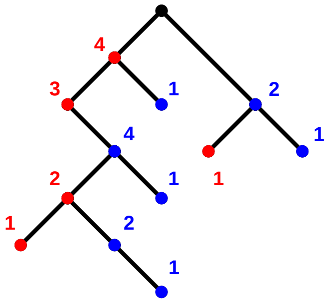

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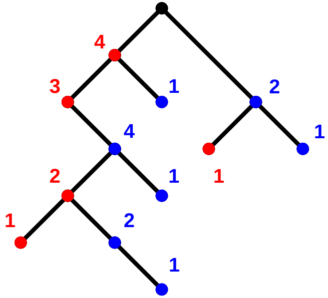


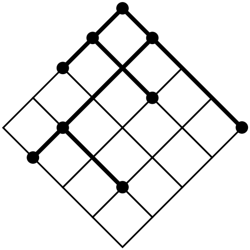
$$NA(T) = \frac{|V_L|!}{\prod_{e \in V_L} \lambda(e)} \cdot \frac{|V_R|!}{\prod_{e \in V_R} \lambda(e)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} \cdot \frac{7!}{4 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1} = 1575$$

Outline of the talk

1. Enumeration of non-ambiguous trees inside a rectangular box
2. Enumeration of non-ambiguous trees with a given underlying binary tree
3. Complete non-ambiguous trees and the Bessel function
4. A bijection between parallelogram polyominoes and binary trees

Complete non-ambiguous trees

A complete non-ambiguous tree is a non-ambiguous tree whose underlying binary tree is complete (i.e. every vertex has 0 or 2 children)



size =

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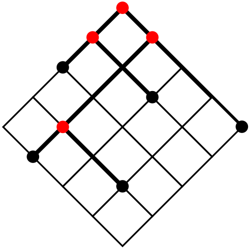

size $=$ $\#$ internal vertices

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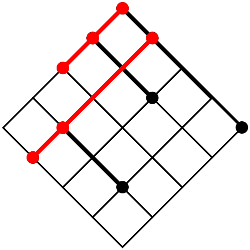

size = # internal vertices = # left edges

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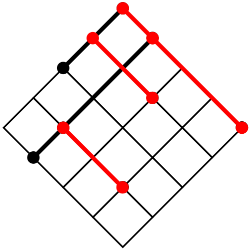

size = # internal vertices = # left edges = # right edges

The number of complete non-ambiguous trees

$b_n = \#$ complete non-ambiguous trees of size n

The sequence $(b_k)_{k \geq 0}$ appears on the OEIS:

$$\texttt{A002190} = (b_n)_{n \geq 0} = (1, 1, 4, 33, 456, 9460, \dots)$$

with no combinatorial interpretation.

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Theorem [ABBS 2012]

$$-\ln (J_0(x)) \quad = \quad \sum_{k \geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2}$$

where J_0 is the Bessel function of order 0

$$J_0(x) = \sum_{i \geq 0} \frac{(-1)^i}{(i!)^2} \left(\frac{x}{2}\right)^{2i}$$

The number of complete non-ambiguous trees

Proof The Bessel function of order 0 is the solution of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0,$$

such that $y(0) = 1$ and $y'(0) = 0$.

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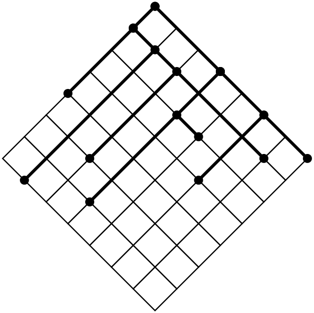
such that $y(0) = 1$ and $y'(0) = 0$. The function

$$y(x) = \exp\left(-\sum_{k\geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2}\right).$$

solves this differential equation if and only if the sequence b_n satisfies the following recurrence

$$b_{n+1} = \sum_{u+v=n} \binom{n+1}{u}\binom{n+1}{v} b_u b_v$$

Recursive enumeration of complete non-ambiguous trees



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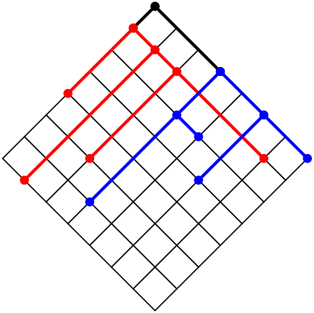

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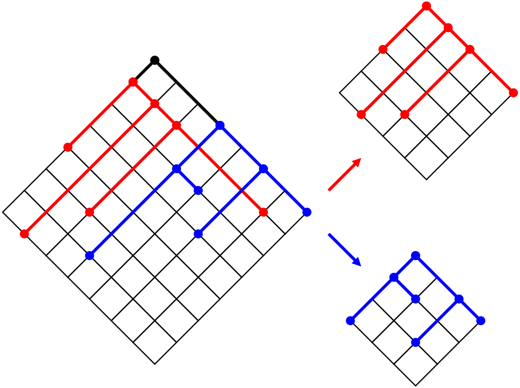

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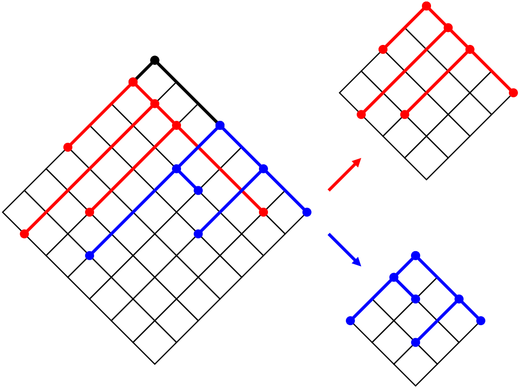
Recursive enumeration of complete non-ambiguous trees



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Another identity involving complete non-ambiguous trees

Theorem [Carlitz 1963, combinatorial proof ABBS 2012]

For every $n \geq 1$

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n-1}{k} b_k = 1$$

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Gridded trees

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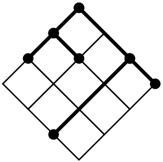

A complete non-ambiguous tree of size 3

Gridded trees

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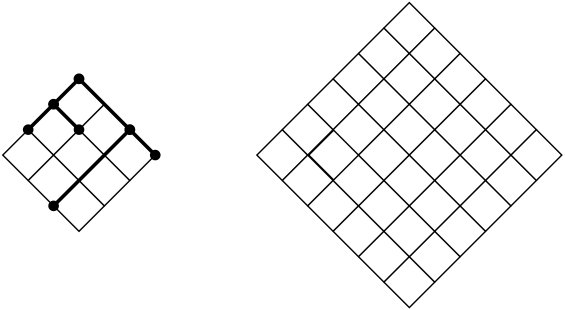

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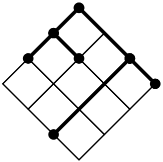

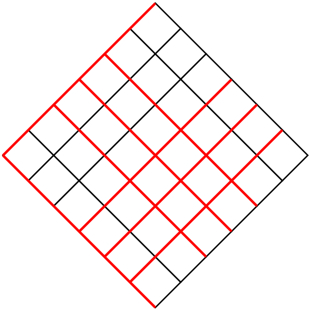

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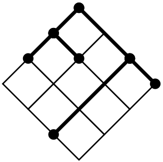

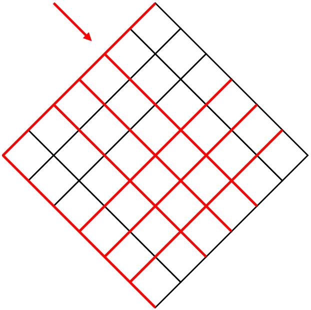

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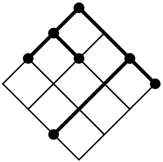

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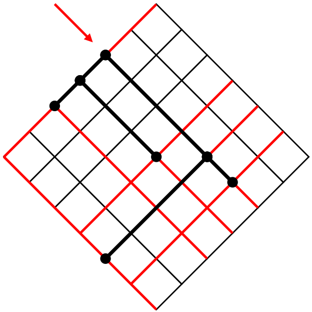

A gridded tree of size $(3, 7)$

Gridded trees

The number of gridded trees of size (k, n) is

$$g_{k,n} = \binom{n}{k+1}\binom{n-1}{k} b_k$$

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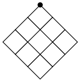
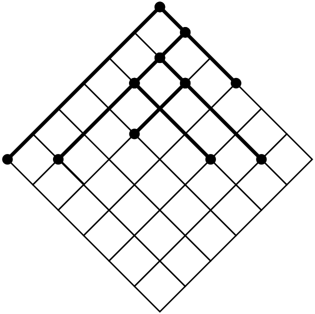


An involution on gridded trees



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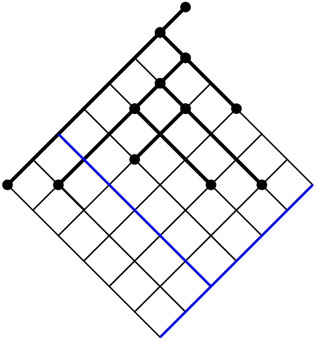

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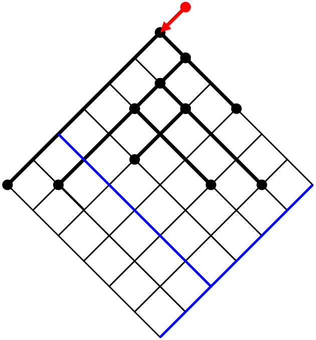

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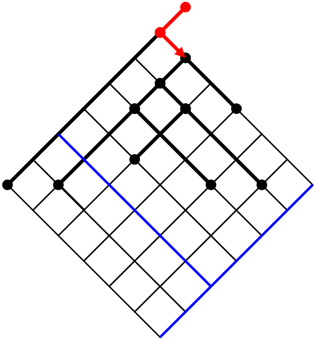

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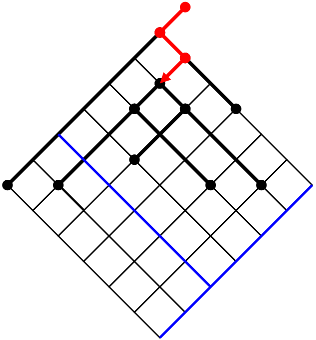

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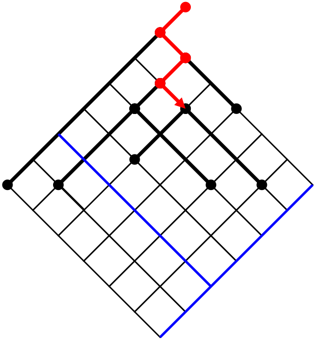

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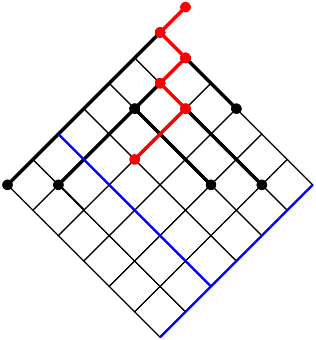

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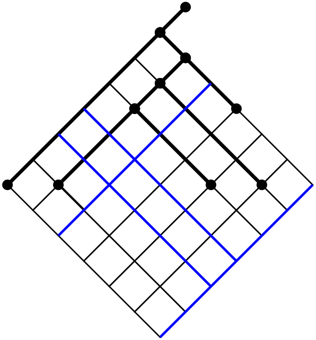

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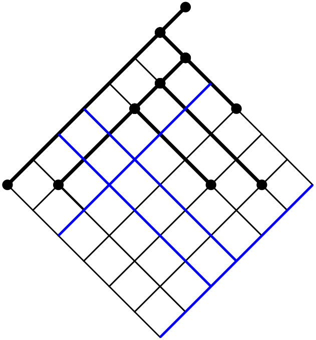

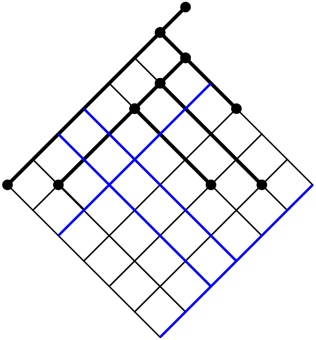

Case 2: the zig-zag path crosses an empty row/column

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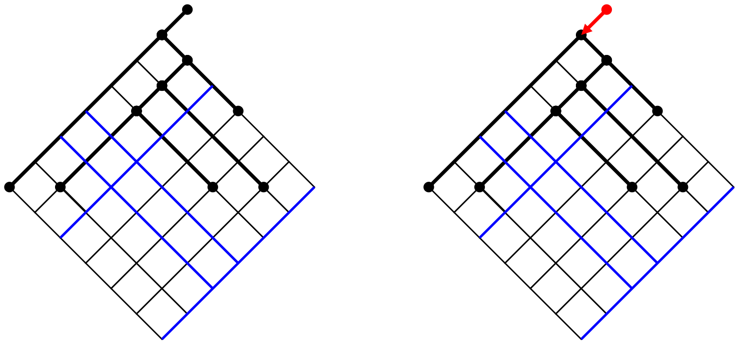

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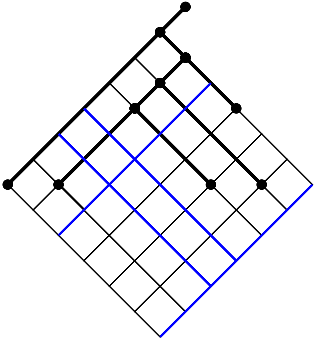

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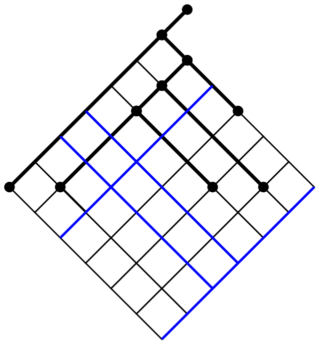

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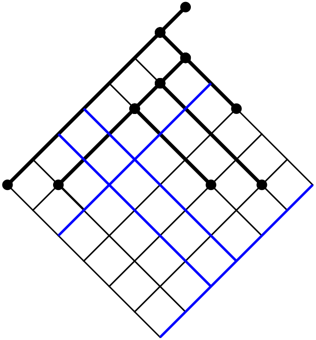

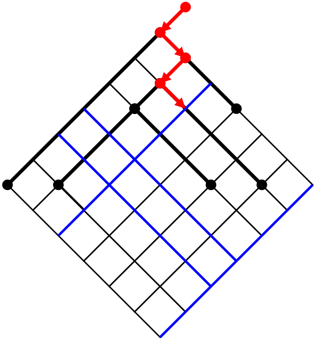

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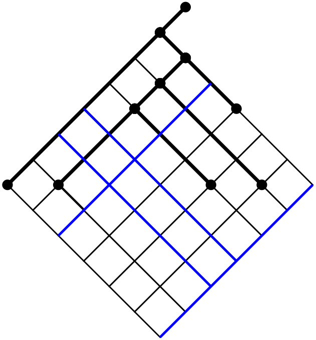

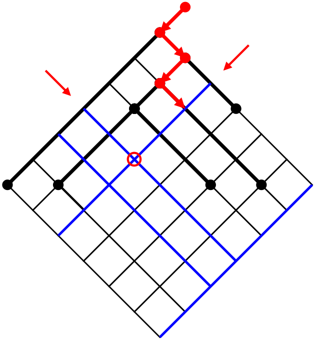

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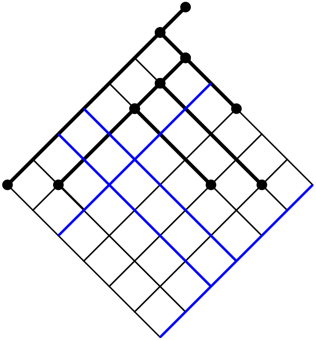

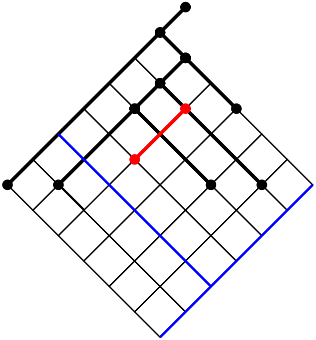

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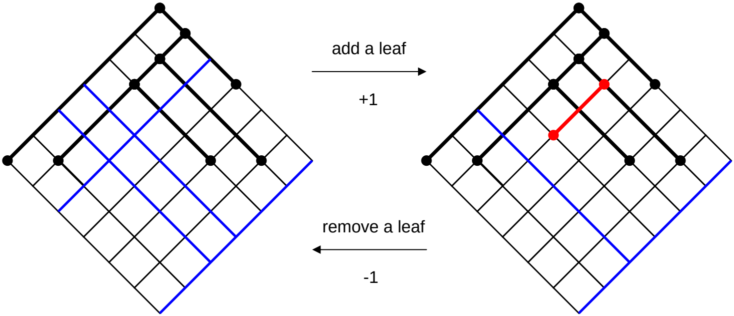


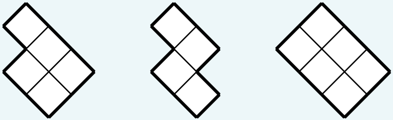
Outline of the talk

1. Enumeration of non-ambiguous trees inside a rectangular box
2. Enumeration of non-ambiguous trees with a given underlying binary tree
3. Complete non-ambiguous trees and the Bessel function
4. A bijection between parallelogram polyominoes and binary trees

Parallelogram polyominoes

A parallelogram polyomino of size n is a pair of lattice paths of lengths $n+1$ with south-west and south-east steps starting at the same point, ending at the same point and never meeting each other.

Some examples for $n = 4$



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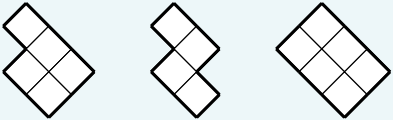
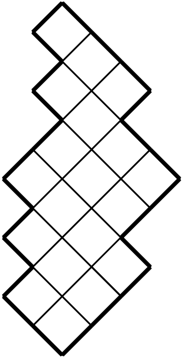


Theorem [Delest, Viennot, . . .]

Parallelogram polyominoes of size n are in bijection with binary trees with n vertices.

A new bijection S between PPs and binary trees



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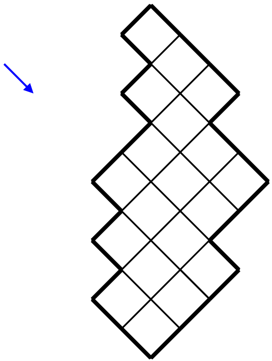

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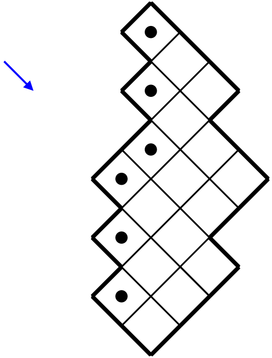

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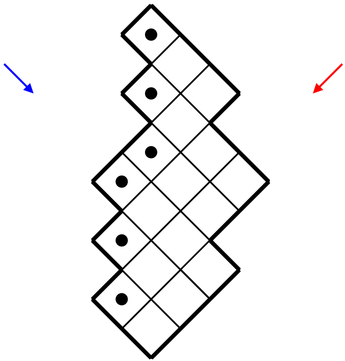

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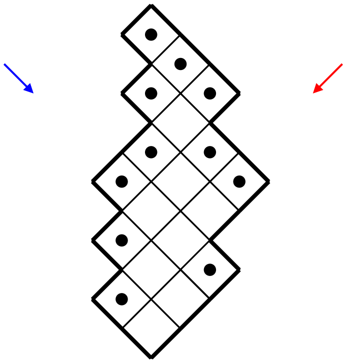

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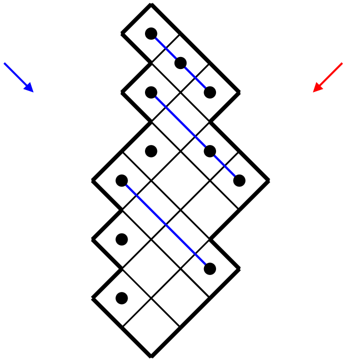

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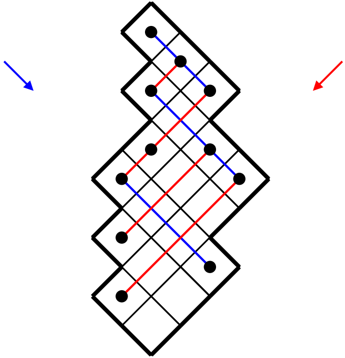

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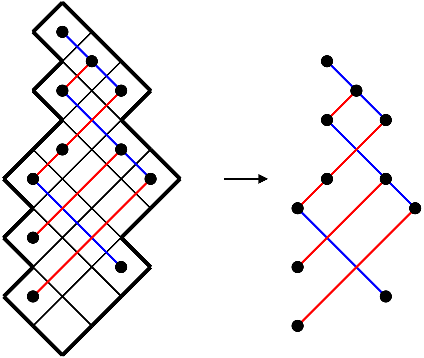

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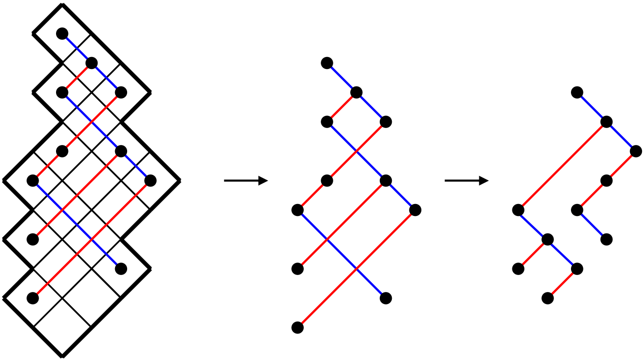

Some perspectives

- Find the OGF/EGF of non-ambiguous trees and of complete non-ambiguous trees
- Find an analogue of complete non-ambiguous trees enumerated by the coefficients in the expansion of $-\ln(J_k(x))$ for other values of k
- Define an analogue of non-ambiguous trees in higher dimensions
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Thank you for your attention!