Network parametrizations for Grassmannians

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Outline

- Some decompositions of the Grassmannian
  - → Schubert cells
  - → Positroid strata
  - → Deodhar components
  - → Matroid strata
- ⅃-diagrams and total positivity
- Go-diagrams and networks parametrizing Deodhar components
- A characterization of Deodhar components in terms of vanishing/non-vanishing of certain Plücker coordinates

Plücker coordinates in $Gr_{k,n}(\mathbb{K})$

We can represent points in $Gr_{k,n}$ by full rank $k \times n$ matrices, e.g.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & -5 & 7 \\ 0 & 0 & 1 & 0 & 2 & -9 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix} \in Gr_{3,6}.$$

The Plücker coordinates $\left(P_J : J \in \binom{[n]}{k}\right)$ of a point are its $k \times k$ minors, indexed by their column sets. I.e. P_J is the $k \times k$ minor with columns J.

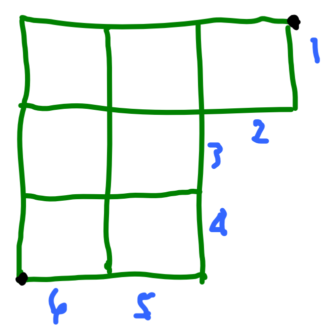
Schubert cell decomposition of $Gr_{k,n}$

If our matrix representing a point in $Gr_{k,n}$ is in row echelon form, we can "see" a partition λ.

Eg:

$$\begin{array}{c} \begin{bmatrix} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix} \\ \begin{array}{cccccc} 1 & \;2 & \;3 & \;4 & \;5 & \;6 \end{array} \end{array}$$

$I(\lambda) = \{1, 3, 4\}$.

$\lambda =$ 

Schubert cell:

$$\Omega_\lambda = \left\{ A \in Gr_{k,n} : P_{I(\lambda)} \text{ is the lex min'l non-vanishing Plücker coord} \right\}$$

Positroid stratification of $Gr_{k,n}$

Take the common refinement of the n "cyclically shifted Schubert cell decompositions" to get the positroid stratification.

Recall: $\Omega_\lambda = \left\{ A \in Gr_{k,n} : \begin{array}{l} P_{I(\lambda)} \text{ lex min'l } \neq 0 \\ \text{WRT } 1 < 2 < \cdots < n \end{array} \right\}$

Shifted: For $1 \leq a \leq n$,

$$\Omega_\lambda^a = \left\{ A \in Gr_{k,n} : \begin{array}{l} P_{I(\lambda)} \text{ lex min'l } \neq 0 \\ \text{WRT } a <_a a+1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a-1 \end{array} \right\}.$$

The common refinement of these n decompositions is the positroid stratification.

Each stratum $S_{\mathcal{I}}$ is indexed by an n-tuple $\mathcal{I} = \{I_1, \ldots, I_n\}$, where each P_{I_a} is the lex min'l Plücker coord WRT $<_a$.

Positroid strata

For most choices of $\mathcal{I} = (I_1, \ldots, I_n)$, the corresponding intersection of shifted Schubert cells is empty.

When this intersection is nonempty, $\mathcal{I}$ is called a Grassmann necklace and we can use it to characterize our positroid stratum:

$$S_{\mathcal{I}} = \left\{ \begin{array}{ll} A \in Gr_{k,n} : & P_{I_1}(A) \neq 0 \\ & P_J(A) = 0 \text{ if } J <_{lex 1} I_1 \\ & P_{I_2}(A) \neq 0 \\ & P_J(A) = 0 \text{ if } J <_{lex 2} I_2 \\ & \vdots \\ & P_{I_n}(A) \neq 0 \\ & P_J(A) = 0 \text{ if } J <_{lex\, n} I_n \end{array} \right\}$$

⅃-diagrams

A ⅃-diagram is a filling of a partition in a $k \times (n-k)$ box with 0 and $+$ which avoids $\rightsquigarrow$

$$\begin{array}{ccc} & & \boxed{+} \\ & & \vdots \\ \boxed{+} & \cdots & \boxed{0} \end{array}.$$

Eg:

$$\begin{array}{|cccc|} + & 0 & + & + \\ 0 & 0 & 0 & \\ + & 0 & + & \end{array}$$

$k = 3$
$n = 8$
$\lambda = (4, 3, 3)$, $I(\lambda) = \{2, 4, 5\}$.

$\exists$ Bijection $\left\{ \begin{array}{c} \text{Grassmann} \\ \text{necklaces} \end{array} \right\} \longleftrightarrow \{ \text{⅃-diagrams} \}$,

So ⅃-diagrams also index our positroid strata S_I.

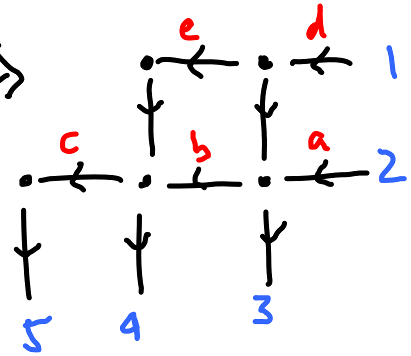
⅃-diagrams and total positivity

Def: The TNN Grassmanian $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ with all Plücker coords ≥ 0.

Thm (Postnikov): $S_{\mathcal{I}} \cap (Gr_{k,n})_{\geq 0}$ is a cell i.e. homeomorphic to $(\mathbb{R}_{>0})^d$ for some d.

Proof idea: ⅃-diag $\rightsquigarrow$ network $\rightsquigarrow$ parametrization.

$$\boxed{\begin{matrix} 0 & + & + \\ + & + & + \end{matrix}} \rightsquigarrow$$



$$\longrightarrow \begin{bmatrix} 1 & 0 & -d & -(bd+de) & -(bcd+cde) \\ 0 & 1 & a & ab & abc \end{bmatrix}$$

$\pm$ i,j entry counts paths $i \rightsquigarrow j$.

Then, $\varphi : (\mathbb{R}_{>0})^5 \longrightarrow Gr_{2,5}$ is a homeomorphism onto the "TNN positroid cell" $S_{\mathcal{I}} \cap (Gr_{2,5})_{\geq 0}$.

Motivating Question:

Q: Can we extend this construction and use networks to parametrize all of $Gr_{k,n}$?

A1: Naive approach - letting edge weights range over all of $\mathbb{R}$ (not just $\mathbb{R}_{>0}$) - does NOT work.

Get many more points in $Gr_{k,n}$, but not all.

A2: Using the Deodhar decomposition DOES work.

"Go-diagrams" indexing Deodhar components $\rightsquigarrow$ networks $\rightsquigarrow$ parametrization.

The Deodhar decomposition of $Gr_{k,n}$

A refinement of the positroid stratification w/

- Components indexed by Go-diagrams, a generalization of J-diagrams
- Networks corresponding to Go-diagrams which parametrize the components (some nonplanar)
- Each component homeomorphic to $(\mathbb{K}^*)^a \times \mathbb{K}^b$ f.s. a,b.
- A characterization by vanishing / non-vanishing of certain Plücker coordinates

The matroid stratification of $Gr_{k,n}$

Refine the Deodhar decomposition. Each stratum characterized by requiring certain P_J to be nonzero and <u>all others</u> to be zero.
(Strata can be topologically terrible.)

Some Deodhar history

- Deodhar gave a decomposition of G/B, used to study Kazhdan-Lusztig polys.
- Marsh-Rietsch parametrized Deodhar components in G/B using factorization by tridiagonal matrices $x_i(m)$, $y_i(p)$, s_i.
- Deodhar components in $Gr_{k,n}$ are just projections of Deodhar components in G/B.
- Deodhar decomposition of $Gr_{k,n}$ is used in studying soliton solutions of KP-equation.
  (Kodama-Williams).

Deodhar decomposition for $Gr_{k,n}$

Assume we are in Ω_λ. Obtain a reduced word in S_n as follows.

11	10	9	8
7	6	5	
4	3	2	
1			

reading order

s_4	s_3	s_2	s_1
s_5	s_4	s_3	
s_6	s_5	s_4	
s_7			

simple transp. $s_i = (i, i+1)$

$$w = s_7 \, s_4 \, s_5 \, s_6 \, s_3 \, s_4 \, s_5 \, s_1 \, s_2 \, s_3 \, s_4$$

M-R param, cont

Look for <u>distinguished</u> <u>subwords</u> of w.

Rule: Working L→R, if choosing s_i would decrease the length of the subword so far, you <u>must</u> take it.

To make a <u>GD-diagram</u>,

record $+$ if you don't choose the letter
$\circ$ if you do and length so far increases
$\bullet$ " " " " " " decreases

Eg: $W = \underline{s_7}\ \underline{s_4}\ s_5\ \underline{s_6}\ s_3\ \underline{s_4}\ s_5\ s_1\ s_2\ \underline{s_3}\ s_4$

forced!

11	10	9	8
7	6	5	
4	3	2	
1			

subword

s_4	s_3	s_2	s_1
s_5	s_4	s_3	
s_6	s_5	s_4	
s_7			

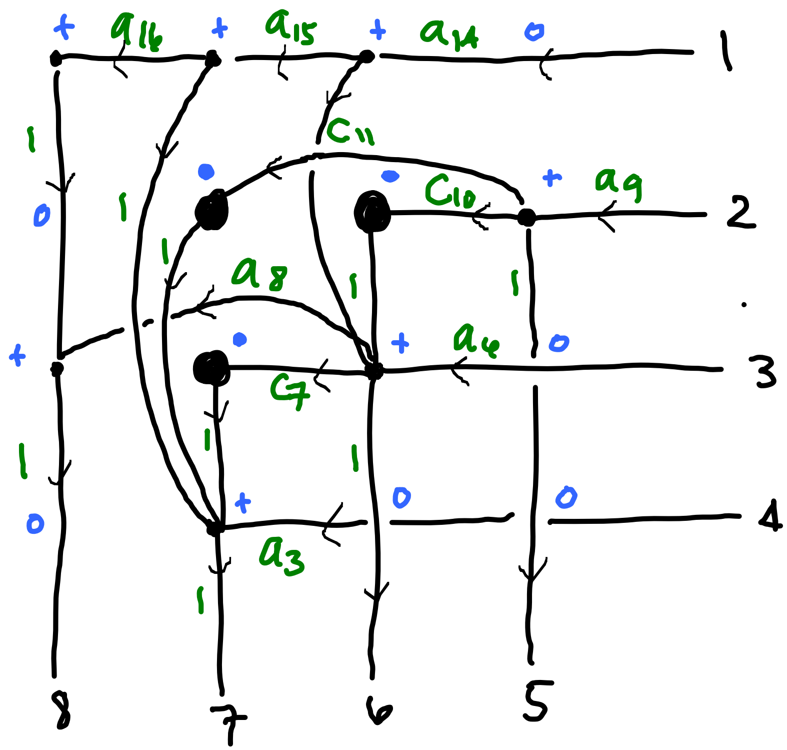
$\rightsquigarrow$

$+$	$\circ$	$+$	$+$
$+$	$\bullet$	$+$	
$\circ$	$+$	$\circ$	
$\circ$			

Go-diagrams and networks

Eg:

$$D = \begin{array}{|c|c|c|c|} \hline + & + & + & 0 \\ \hline 0 & \bullet & \bullet & + \\ \hline + & \bullet & + & 0 \\ \hline 0 & + & 0 & 0 \\ \hline \end{array} \rightsquigarrow \quad N_D =$$



$a_i \in \mathbb{K}^*$

$c_i \in \mathbb{K}$

$Gr_{4,8}$

4×8 matrix $A = (A_{ij})$, where $\pm A_{ij}$ enumerates paths $i \rightsquigarrow j$.

Prop: ⅃-diagrams are precisely the Go-diagrams with no $\boxed{\bullet}$.

<u>Thm 1</u> (T-W) If D is a Go-diagram with r $\boxed{+}$'s and s $\boxed{\bullet}$'s, then this map

$$\varphi : (\mathbb{K}^*)^r \times \mathbb{K}^s \longrightarrow Gr_{k,n}(\mathbb{K})$$

(edge weights) $\longmapsto$ matrix counting paths

is a homeomorphism onto a subset of $Gr_{k,n}$, called a <u>Deodhar component</u> R_D, and

$$Gr_{k,n} = \bigcup{}^{*} R_D$$

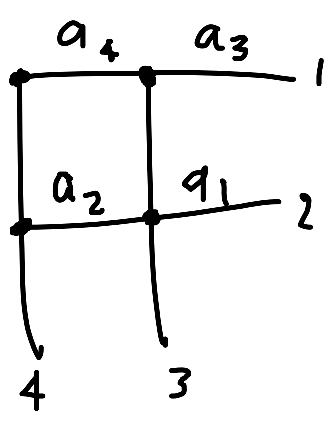
$D =$ Go-diagram in 

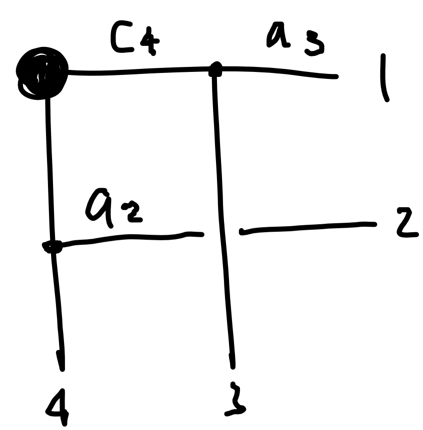
$Gr_{2,4}$ example

$$S_{\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}} = R_{\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}} \cup R_{\begin{array}{|c|c|}\hline \bullet & + \\ \hline + & 0 \\ \hline\end{array}}$$

$$R_{\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}} \cong (\mathbb{K}^*)^4 \qquad R_{\begin{array}{|c|c|}\hline \bullet & + \\ \hline + & 0 \\ \hline\end{array}} \cong (\mathbb{K}^*)^2 \times \mathbb{K}$$

Networks





Corollaries to Thm 1.

- Every point in $Gr_{k,n}(\mathbb{K})$ can be represented by a unique network constructed from a Go-diagram.
- So can every matrix.
- Given a Go-network, all Plücker coords can be read off as polynomials in the edge weights by looking at appropriate families of vertex-disjoint paths (as in work of Lindström, Gessel-Viennot).

Theorem 2 (Talaska-Williams)

The Deodhar component S_D, where D is of shape λ, is characterized by:

$$A \in S_D \text{ iff } \begin{cases} P_{I(\lambda)}(A) \neq 0 & \\ P_J(A) = 0 & \text{if } J <_{lex} I(\lambda) \\ P_{I_b}(A) \neq 0 & \text{if } b = \boxed{+} \\ P_{I_b}(A) = 0 & \text{if } b = \boxed{0}, \end{cases}$$

where each box b in D has a special minor I_b associated to it (obtained by applying a certain permutation to $I(\lambda)$.)

A couple open questions:

1) Recall ⅃-diagrams are characterized by the forbidden pattern

$$\begin{array}{ccc} & & \boxed{+} \\ & & \vdots \\ \boxed{+} & \cdots & \boxed{0} \end{array}.$$

Is there such a rule for Go-diagrams?

2) Enumeration of Go-diagrams.