Rainbow supercharacters and a poset analogue to q-binomial coefficients

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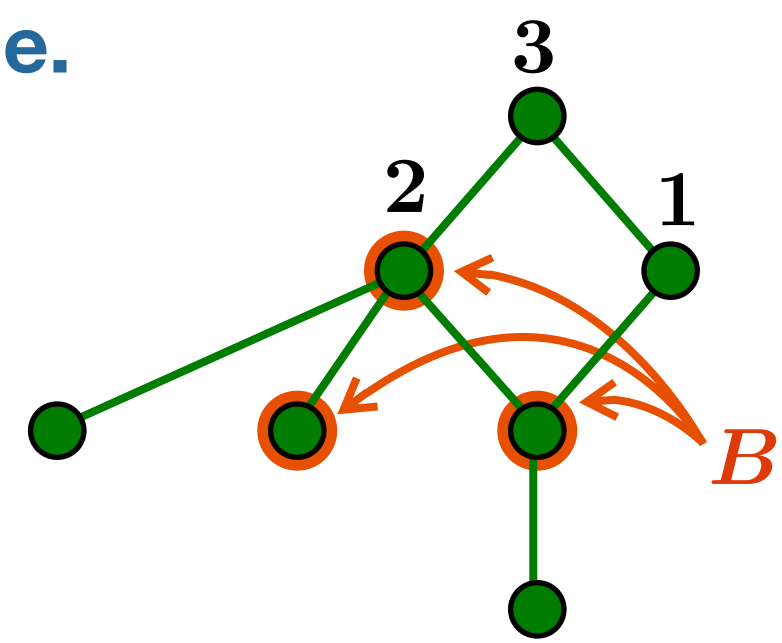
Binomial coefficients?

Let $\mathcal{P}$ be finite poset. The weight $\text{wt}_{\mathcal{P}}(B)$ of a subset $B \subseteq \mathcal{P}$ is given by

$$\text{wt}_{\mathcal{P}}(B) = \#\{(b,p) \in B \times \mathcal{P} \mid b \prec_{\mathcal{P}} p\}.$$

Example.

$\mathcal{P} =$ 

$$\text{wt}_{\mathcal{P}}(B) = 2 + 3 + 1$$

For a fixed (prime power) q and $k \in \mathbb{Z}_{\geq 0}$, we get a q-binomial coefficient

$$\begin{bmatrix} \mathcal{P} \\ k \end{bmatrix}_q = \sum_{\substack{B \subseteq \mathcal{P} \\ |B| = k}} q^{\text{wt}_{\mathcal{P}}(B)}.$$

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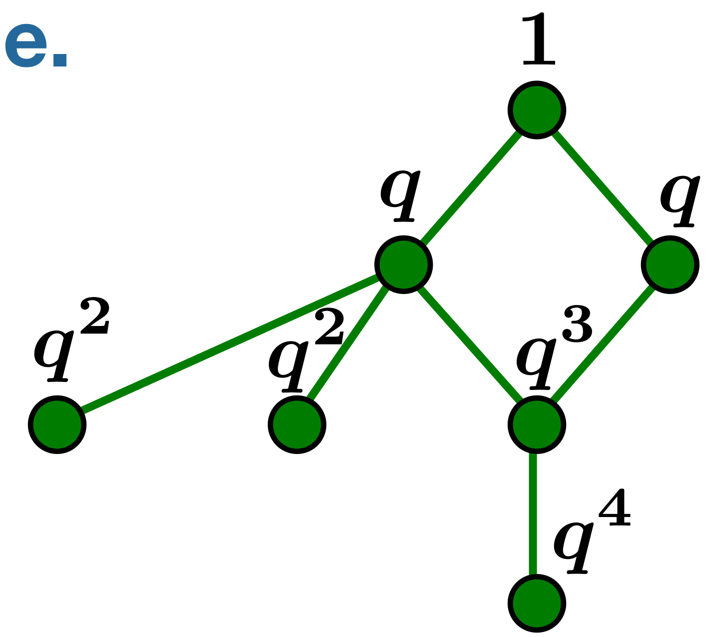
$\mathcal{P} =$ 

$$\begin{bmatrix} \mathcal{P} \\ 2 \end{bmatrix}_q = 2q + 3q^2 + 5q^3 + 4q^4 + 4q^5 + 2q^6 + q^7$$

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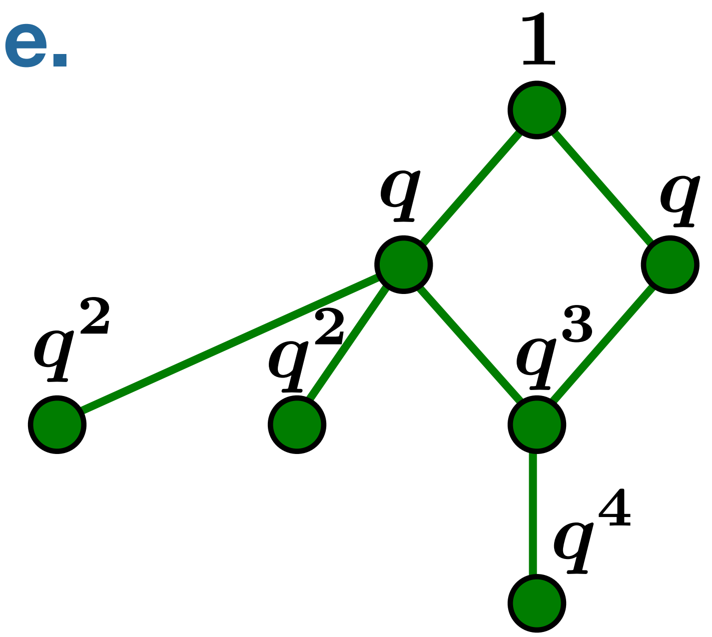
$\mathcal{P} =$ 

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Binomial coefficients?

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- If $\mathcal{P}$ has no relations, then $\begin{bmatrix}\mathcal{P}\\k\end{bmatrix}_q = \binom{|\mathcal{P}|}{k}$.
- If $\mathcal{P}$ is a total order, then $\begin{bmatrix}\mathcal{P}\\k\end{bmatrix}_q = q^{\binom{k}{2}} \begin{bmatrix}|\mathcal{P}|\\k\end{bmatrix}_q$.
- If $\mathcal{P}$ is an appropriate spanning tree of the Bruhat graph then

$$\begin{bmatrix}\mathcal{P}\\1\end{bmatrix}_q = \sum_{w \in S_n} q^{\ell(w)} = [n]!$$

Usual q-binomial

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- While in general $\begin{bmatrix} \mathcal{P} \\ k \end{bmatrix}_q \neq \begin{bmatrix} \mathcal{P} \\ |\mathcal{P}|-k \end{bmatrix}_q$, there is a pleasing relation among the coefficients.

Binomial coefficients?

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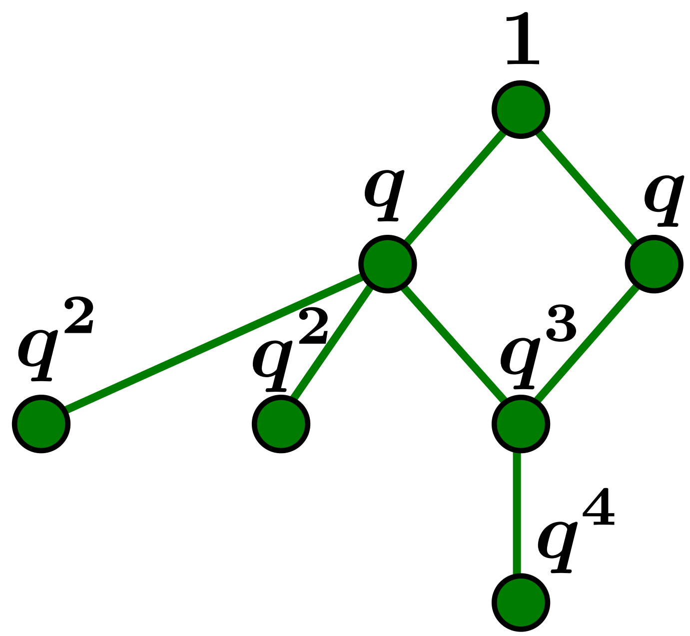

$$\begin{bmatrix} \mathcal{P} \\ 2 \end{bmatrix}_q = 2q + 3q^2 + 5q^3 + 4q^4 + 4q^5 + 2q^6 + q^7$$

$$\begin{bmatrix} \mathcal{P} \\ 5 \end{bmatrix}_q = 2q^{12} + 3q^{11} + 5q^{10} + 4q^9 + 4q^8 + 2q^7 + q^6$$

- **Note that for each choice of element $x \in \mathcal{P}$, the removal of the element gives us a recursion relation amongst these coefficients.**
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The why (via representation theory)

The group of unipotent upper-triangular matrices

$$U_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \;\middle|\; * \in \mathbb{F}_q \right\}$$

WILD

has a notoriously tricky representation theory.

- **For starters, it is provably impossible to enumerate its conjugacy classes.**

However, Aguiar–Bergeron–T showed that if we glue together the class functions of all these groups, one obtains a Hopf structure using the representation theoretic functors of inflation and restriction.

- **Here we obtain either a Hopf monoid or a Hopf algebra, depending on your preference.**

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Some notable substructures

(Grinning Ninnies)

$$\mathsf{cf} = \bigoplus_n \mathsf{cf}(U_n) = \{\text{class functions}\}$$

$$\cup\!\mid$$

set partitions

$$\mathsf{scf} = \bigoplus_n \mathsf{scf}(U_n)$$

Symmetric functions in noncommuting variables

WSym Π NCSym

$$\cup\!\mid$$

integer compositions

$$\mathsf{lc} = \bigoplus_n \mathsf{lc}(U_n) =$$

Noncommutative symmetric functions

Sym NSym

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set partitions: Symmetric functions in noncommuting variables — WSym Π NCSym

integer compositions: Noncommutative symmetric functions — Sym NSym

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However, Aguiar–Bergeron–T showed that if we glue together the class functions of all these groups, one obtains a Hopf structure using the representation theoretic functors of inflation and restriction.

?

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Central question for this talk

Sc

Symmetric functions in noncommuting variables

WSym $\quad\Pi\quad$ NCSym

{class functions}

set partitions

$$\text{scf} = \bigoplus_n \text{scf}(U_n) = \{\text{superclass functions}\}$$

$$\cup\!\mid \qquad\qquad = \mathbb{C}\text{-span}\{\text{supercharacters}\}$$

integer compositions

$$\text{lc} = \bigoplus_n \text{lc}(U_n) =$$

Noncommutative symmetric functions

Sym $\quad$ NSym

Supercharacters and set partitions

As our favorite basis of scf(U_n), supercharacters are

- Characters (traces of representations) that partition the irreducibles of U_n,
- Indexed by set partitions of $\{1, \ldots, n\}$.

A set partition λ of $\{1, 2, \ldots, n\}$ is a subset

$$\lambda \subseteq \{i \frown j \mid 1 \leq i < j \leq n\},$$

such that for $i \frown k, j \frown l \in \lambda$, $i = j$ if and only if $k = l$.

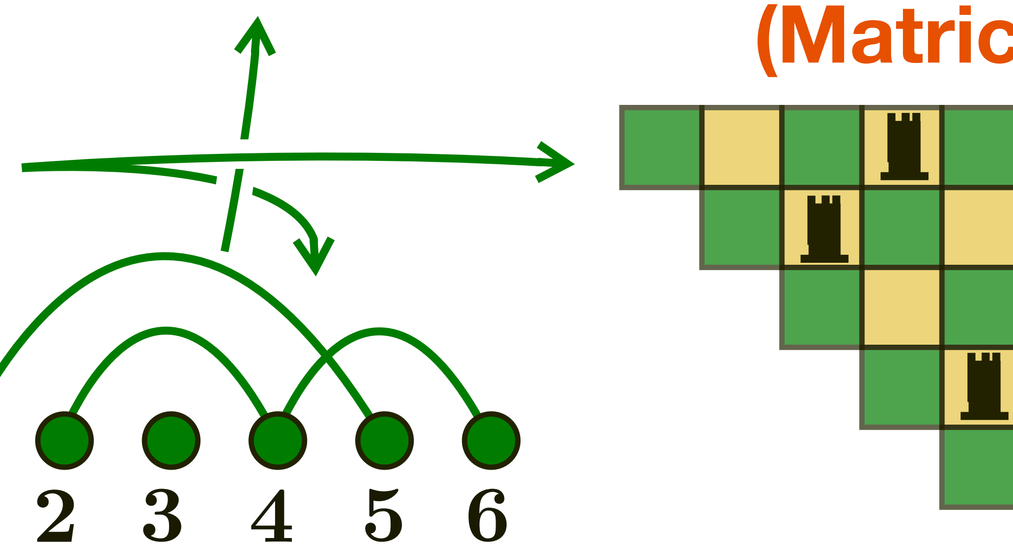
Example.

(Usual) $\{\{1, 5\}, \{2, 4, 6\}, \{3\}\}$

(Matrices?)

$\lambda = \{1 \frown 5, 2 \frown 4, 4 \frown 6\}$

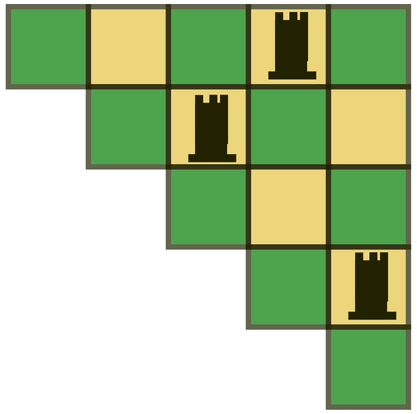
$(n = 6)$



(Preferred)

Supercharacters and set partitions

(Matrices?)



Supercharacters and set partitions

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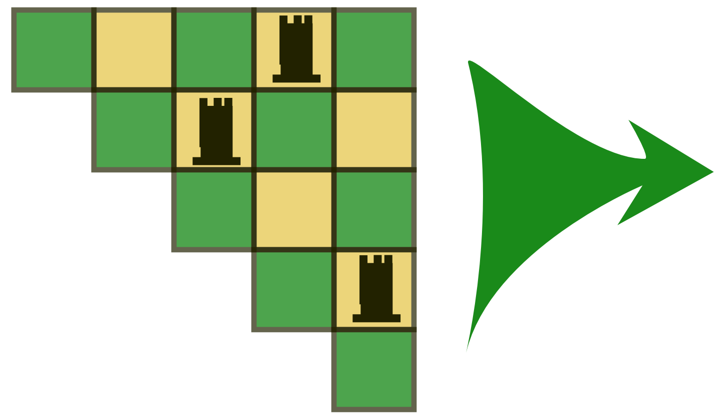

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We say a matrix $u \in U_n$ has partition type λ if

$u -$ **Id** $\xrightarrow[\text{add rows up}]{\text{add columns right}}$ **rook placement corresponding to λ**

+scaling

The Hopf structure scf is exactly the space of functions that depend only on the partition type of a matrix.

nearly orthonormal?

The supercharacters $\{\chi^\lambda\}$ form ~~an orthogonal~~ basis for this space.

Problem. **For $U_k \subseteq U_n$, decompose Res$_{U_k}^{U_n}(\chi^\lambda)$.**

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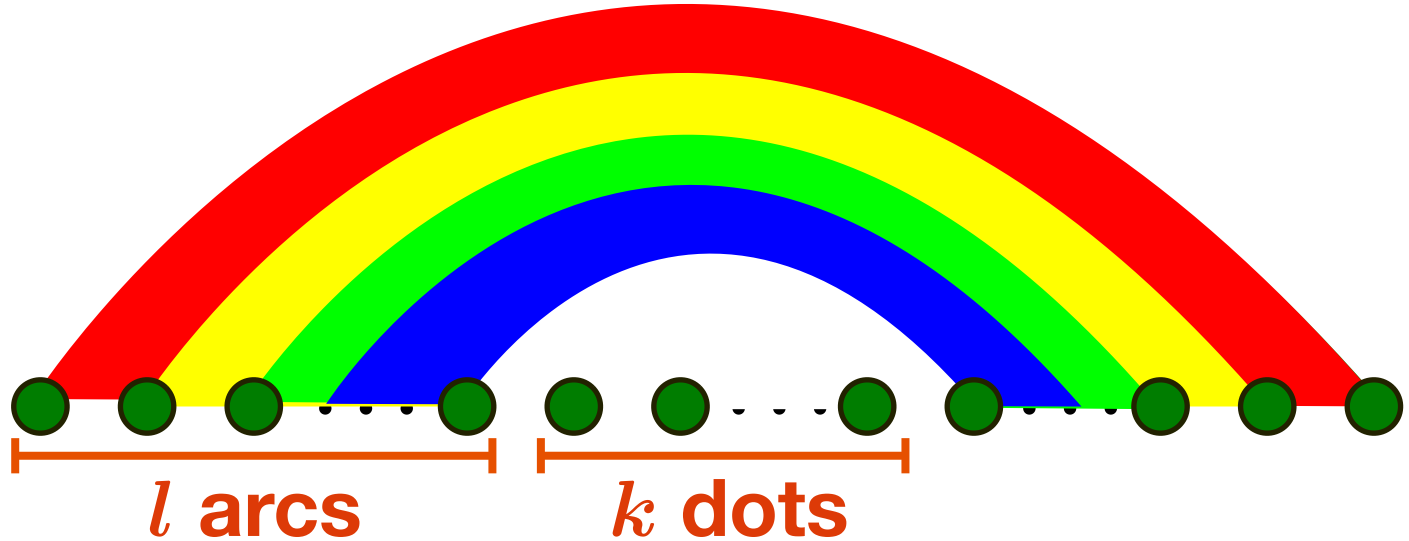
Easier Problem. For $U_k \subseteq U_n$, decompose $\mathrm{Res}_{U_k}^{U_n}(\chi^{\lambda})$, where

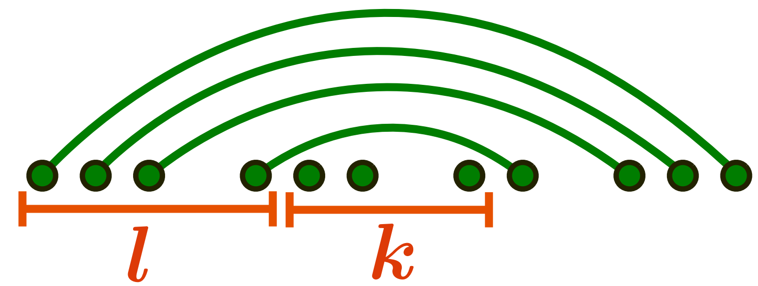
- $n = 2l + k$
- $U_k \cong \left\{ \left[\begin{array}{c|c|c} \mathrm{Id}_l & 0 & 0 \\ \hline 0 & U_k & 0 \\ \hline 0 & 0 & \mathrm{Id}_l \end{array} \right] \right\} \subseteq U_n$
- $\lambda =$



Notable property. Every supercharacter of U_k appears with a nonzero coefficient for l sufficiently large.

Rainbow supercharacters

Theorem. (B–T) For $\lambda =$



$$\mathrm{Res}^{U_{2l+k}}_{U_k}(\chi^{\lambda}) \in \mathbb{C}\text{-span}\{\psi_0, \psi_1, \ldots, \psi_k\} \subseteq \mathrm{scf}(U_k),$$

ψ_j only depends on $|\mu|$

where if $u_\mu \in U_k$ has partition type μ,

$$\psi_j(u_\mu) = \begin{bmatrix} \mathcal{T}_\mu \\ j \end{bmatrix}_q, \quad \left(= \sum_{\substack{B \subseteq \mathcal{T}_\mu \\ |B| = j}} q^{\#\{(b,t) \in B \times \mathcal{T}_\mu \mid b \prec t\}} \right)$$

and $\mathcal{T}_\mu$ is a total order on $k - |\mu|$ elements.

Remark. ψ_0 is the trivial character of U_k and ψ_k is the regular character of U_k. In general, the modules of these characters are very natural.

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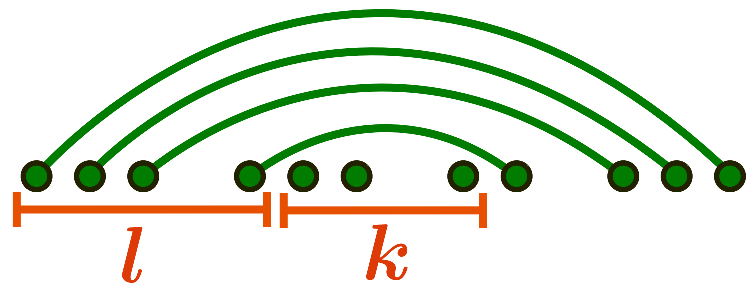

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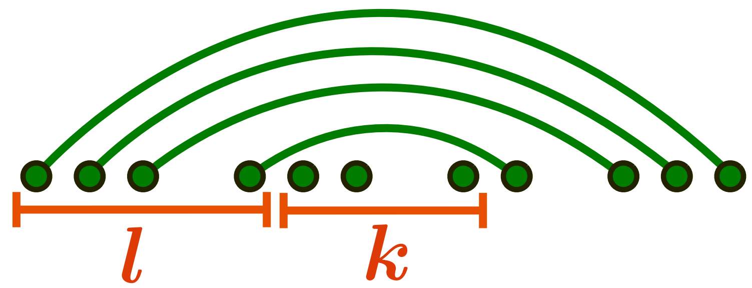
Theorem. (B–T) For $0 \leq j \leq k$,

$$\psi_j = \sum_{\substack{\nu \text{ a set partition} \\ \text{of } \{1, 2, \ldots, k\}}} q^{\mathrm{nst}^{\nu}_{\nu}} \begin{bmatrix} \mathcal{P}(\tilde{\nu}) \\ j - |\nu| \end{bmatrix}_q \chi^{\nu}.$$

Theorem. (B–T) For $\lambda =$ 

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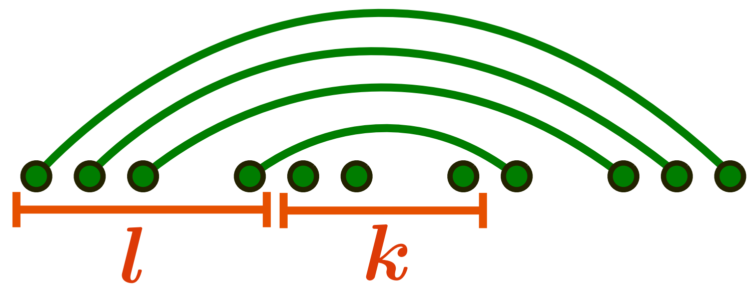
Theorem. (B–T) For $\lambda =$ 

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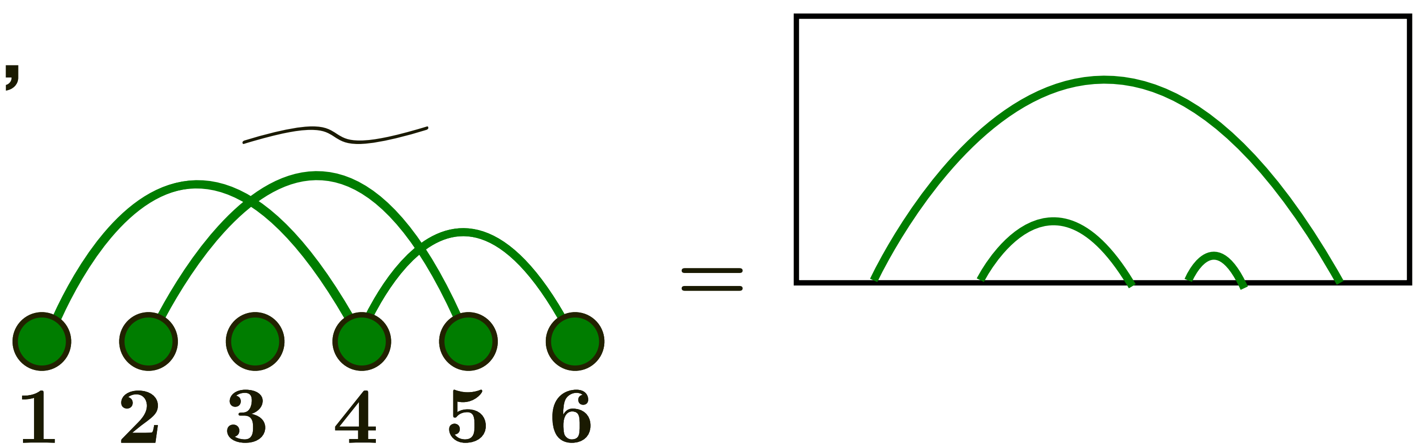

That is, the set partition with the same left and right endpoints, but no crossings

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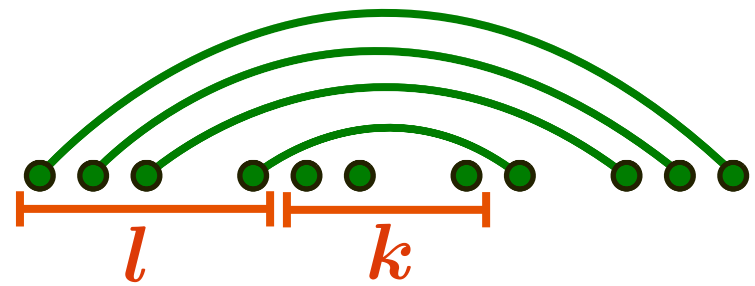

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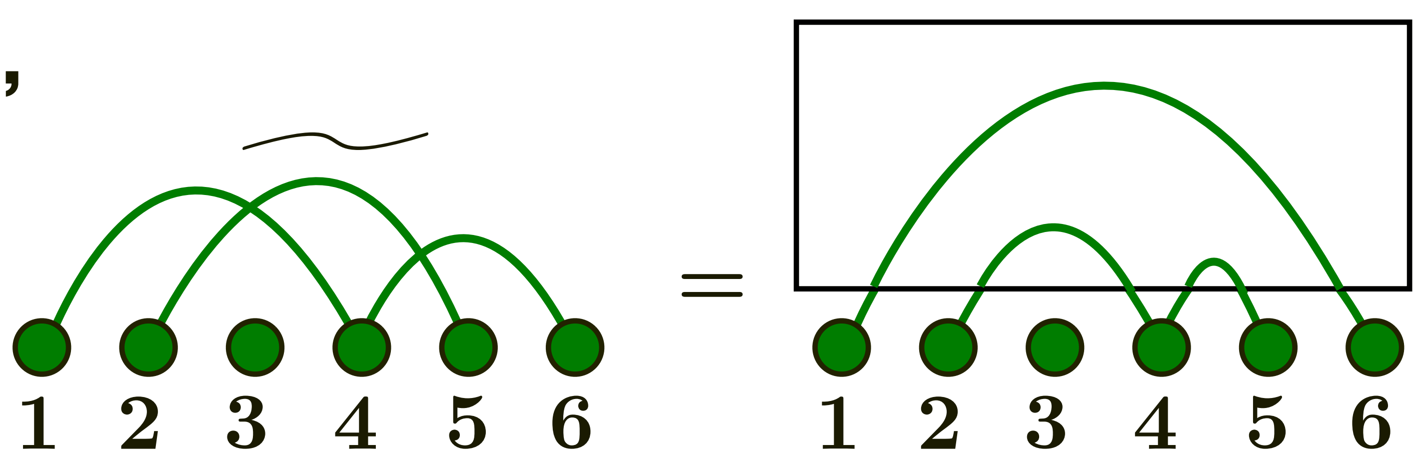

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Rainbow supercharacters

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nested arcs

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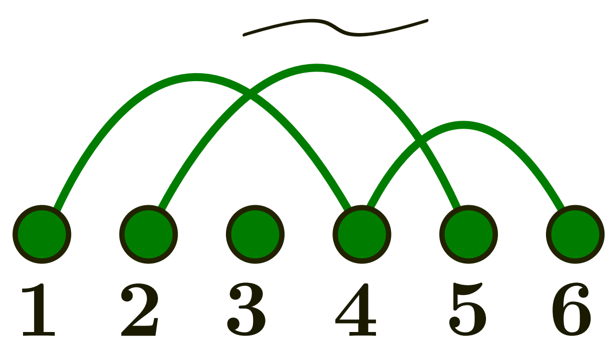

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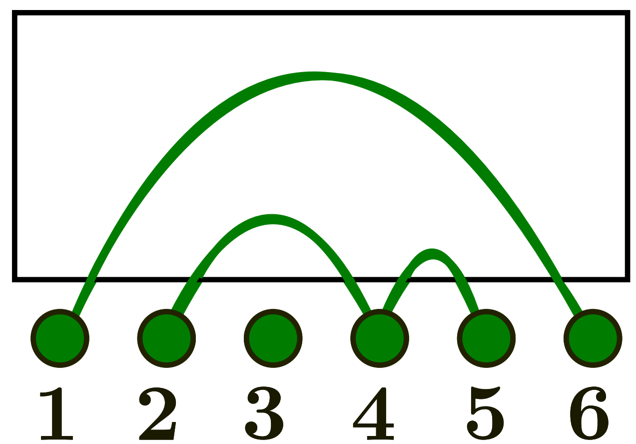

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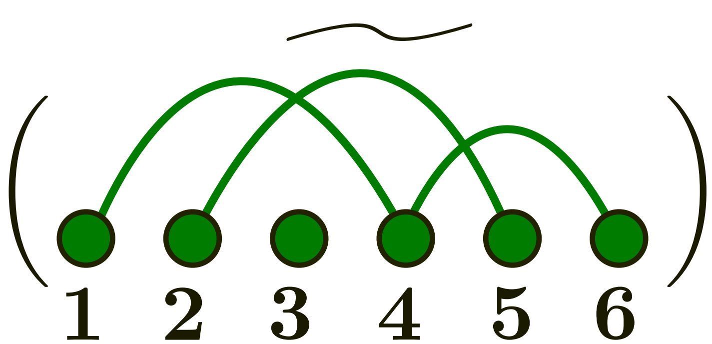
$\mathcal{P}\Big($$\Big) =$

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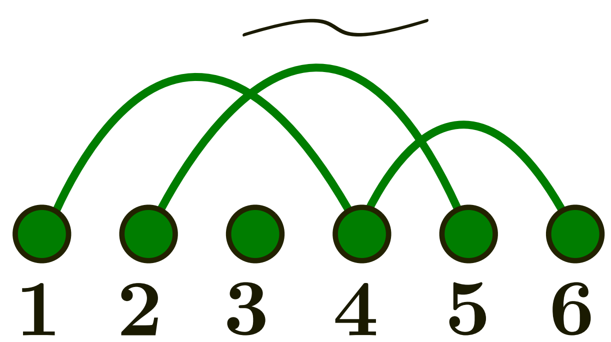

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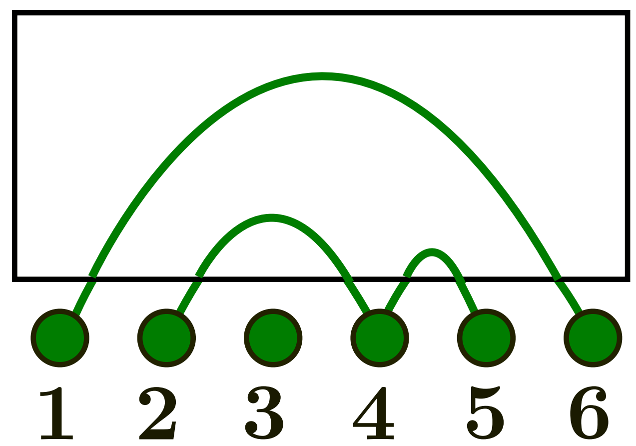

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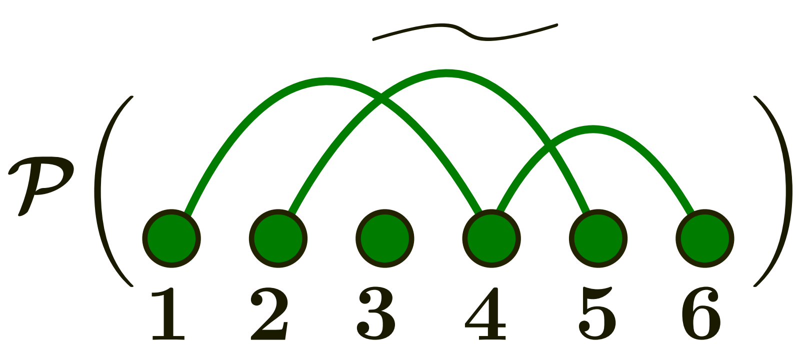

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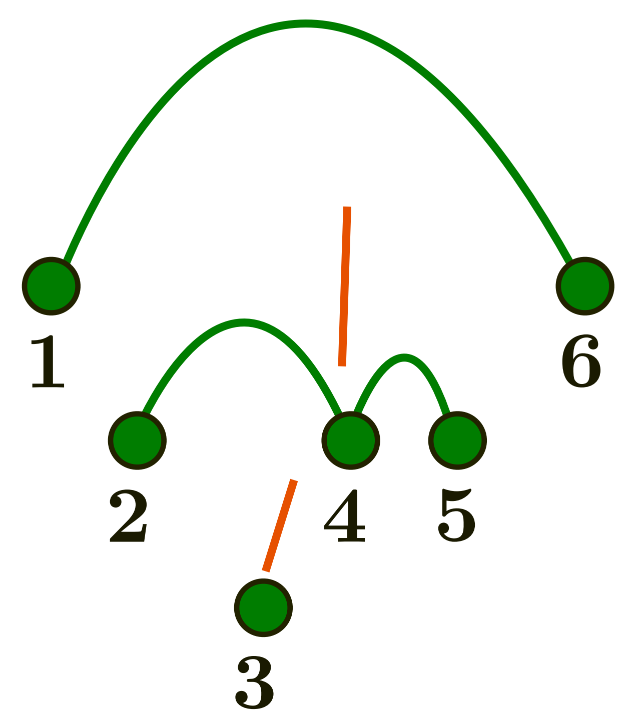

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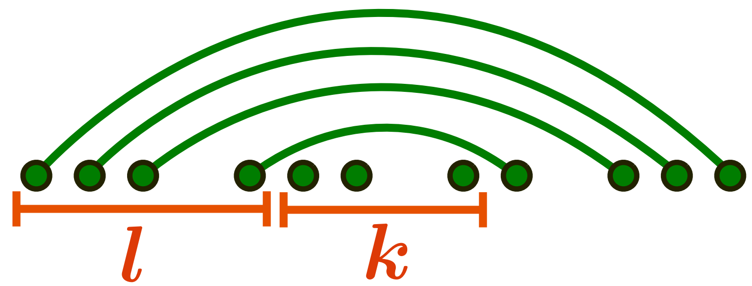
Theorem. (B–T) For $\lambda =$ 

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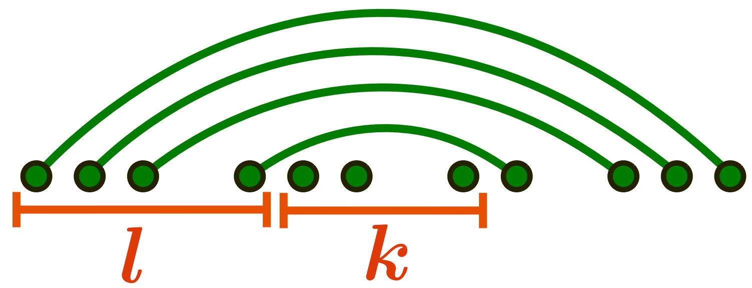
Corollary. For $0 \le j \le k$,

$$\sum_{\substack{\nu \text{ a set partition} \\ \text{of } \{1,2,\ldots,k\}}} \frac{q^{\mathrm{nst}_\nu^\nu + \dim(\nu)}}{q^{|\nu|}} (q-1)^{|\nu|} \begin{bmatrix} \mathcal{P}(\tilde{\nu}) \\ j - |\nu| \end{bmatrix}_q = q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q.$$

Proof. Apply the previous theorem to the element 1. □

Back to the original problem

Theorem. (B–T) For $\lambda =$ 

the coefficient of χ^{ν} in $\mathrm{Res}_{U_k}^{U_{2l+k}}(\chi^{\lambda})$ is

$$\sum_{j=|\nu|}^{\min(l,k)} q^{\mathrm{nst}_{\nu}^{\nu}+l(l-1)}(q-1)^{2l-j}\frac{[l]!}{[j]!}\begin{bmatrix}\mathcal{P}(\tilde{\nu})\\ j-|\nu|\end{bmatrix}_q.$$

Remarks.

- This is a reasonable formula, but we don't know yet what it counts.
- This approach seems to generalize (with some complications) to a more general restriction problem.
- Has anyone seen these kind of binomial coefficients?