

# A BIJECTION FOR COVERED MAPS ON ORIENTABLE SURFACES

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ABSTRACT. *Unicellular maps* are a natural generalisation of plane trees to higher genus surfaces. In this article we study *covered maps*, which are maps together with a distinguished unicellular spanning submap. We prove that the covered maps of genus  $g$  with  $n$  edges are in bijection with pairs made of a plane tree with  $n$  edges and a bipartite unicellular map of genus  $g$  with  $n + 1$  edges. This generalises to any genus the bijection given in [2] between planar *tree-rooted maps* (maps with a distinguished spanning tree) and pairs made of a tree with  $n$  edges and a tree with  $n + 1$  edges. In the special case of genus 1, a duality argument allows us to obtain a bijective proof of a formula of Lehman and Walsh [4] about the number of tree-rooted maps of genus 1.

## 1. INTRODUCTION

We consider maps on orientable surfaces of arbitrary genus. A map is *unicellular* if it has a single face, that is, if the complement of the map is simply connected. A unicellular map on the torus is represented in Figure 4(b). A *covered map* is a map together with a distinguished spanning unicellular submap. A map of genus  $g$  can have spanning submaps of any genus in  $\{0, \dots, g\}$ . An example of covered map is given in Figure 1. The main goal of this article is to exhibit a bijection  $\Psi$  between covered maps of genus  $g$  and size  $n$  and pairs made of a plane tree of size  $n$  and a unicellular map of genus  $g$  and size  $n + 1$ .

Covered maps are a natural generalisation of *tree-rooted maps*, that is, maps together with a distinguished spanning tree. In the planar case these two notions coincide and our bijection  $\Psi$  specialise into the bijection found in [2] in order to give a bijective explanation of a result of Mullin [6]: the number of planar tree-rooted maps of size  $n$  is  $T_n^0 = C_n C_{n+1}$ , where  $C_n = \frac{(2n)!}{n!(n+1)!}$  is the  $n^{\text{th}}$  Catalan number i.e. the number of plane trees with  $n$  edges. In the case of the torus, a duality argument shows that exactly half of the covered maps of size  $n$  are tree-rooted maps. Therefore, our bijection  $\Psi$  give a bijective explanation to the formula of Lehman and Walsh [4]: the number of tree-rooted maps of genus 1 is  $T_n^1 = \frac{1}{2} C_n B_{n+1}^1$ , where  $B_n^1 = \frac{(2n-1)!}{6n!(n-3)!}$  is the number of bipartite unicellular maps with  $n$  edges.

We first recall some definitions. A *map* is a connected graph *embedded* in an orientable surface considered up to homeomorphism. By *embedded*, one means drawn on the surface in such a way the edges do not intersect and the *faces* (connected components of the complement of the graph) are simply connected. An example is given in Figure 1 (forget the thick lines for the time being). The *genus* of a map is the genus of the surface in which it is embedded and its *size* is the number of edges. A *planar map* is a map of genus 0. A map is *unicellular* if it has a single face. For instance, *plane trees* are the unicellular planar maps. A map is *bipartite* if the underlying graph is. A unicellular bipartite map of genus 1 is represented in Figure 4(b).

The embedding of a map defines a cyclic order (the counterclockwise order) of the half-edges around each vertex. There is, in fact, a one-to-one correspondence between maps and connected graphs together with a cyclic order of the edges around each vertex [5]. Equivalently, a map can be defined as a triple  $M = (H, \sigma, \alpha)$ , where  $H$  is a finite set whose element are the *half-edges*,  $\alpha$  is an involution of  $H$  without fixed point, and  $\sigma$  is a permutation of  $H$  such that the group generated by  $\alpha$  and  $\sigma$  acts transitively on  $H$ . The cycles of the involution  $\alpha$  are the edges and the cycles of the permutation  $\sigma$  are the vertices together with the counterclockwise order of half-edges around them. For instance, the map in Figure 1 is  $M = (H, \sigma, \alpha)$ , where  $H = \{1, 1', 2, 2', \dots, 9, 9'\}$ ,  $\alpha = (1, 1')(2, 2') \cdots (9, 9')$  and  $\sigma = (1, 2, 6)(1', 2', 3, 5')(3', 4')(5, 9')(4, 8', 9)(6', 7', 8, 7)$ . Observe that the faces of  $M$  are in bijection with the cycles of the permutation  $\phi = \sigma\alpha$ . For the map of Figure 1,  $\phi = (1, 2', 6, 7', 6')(1', 2, 3, 4', 8', 7, 8, 9, 5)(3', 5', 9', 4)$ . A map is *rooted* if one of the half-edges is distinguished as the *root*; we denote by  $M = (H, r, \sigma, \alpha)$  the map  $(H, \sigma, \alpha)$  having root  $r$ . In the following

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maps are rooted and are considered up to *isomorphism* (relabelling of the half-edges).

Given a subset  $S$  of  $H$ , the *restriction of  $\pi$  to  $S$* , denoted by  $\pi|_S$  is the permutation of  $S$  whose cycles are obtained from the cycles of  $\pi$  by erasing the elements not in  $S$ . For instance, if  $\pi = (a, b, c)(d, e)(f, g, h, i)$  and  $S = \{b, c, f, g, i\}$ , then  $\pi|_S = (b, c)(f, g, i)$ . A *submap* of a map  $M = (H, \sigma, \alpha)$  is a map of the form  $N = (S, \alpha|_S, \sigma|_S)$ , where  $S \subseteq H$ . It is *spanning* if every cycle of  $\sigma$  contains an element of  $S$ . A submap of a map of genus  $g$  has genus less or equal to  $g$ . For instance, the map  $M$  in Figure 1 has genus 1 while the spanning submap  $T = (S, \alpha|_S, \sigma|_S)$  induced by the set  $S = \{1, 1', 3, 3', 6, 6', 8, 8', 9, 9'\}$  (thick lines) has genus 0. A pair  $(M, T)$  made of a map  $M$  and a unicellular spanning submap  $T$  is a *covered map*. A covered map  $(M, T)$  is represented in Figure 1. Given a covered map  $(M, T)$ , a half-edge is called *internal* if it belongs to the submap  $T$  and *external* otherwise. An *orientation* of a map  $M = (H, \sigma, \alpha)$  is a partition  $H = I \uplus O$  such that the involution  $\alpha$  maps the set  $I$  on the set  $O$ ; the half-edges in  $I$  and  $O$  are respectively called *ingoing* and *outgoing*. The orientation  $I = \{1', 2', \dots, 9'\}$  and  $O = \{1, 2, \dots, 9\}$  of the map  $M$  is represented in Figure 2(a).

## 2. BIJECTION

We now define the mapping  $\Psi$  which associates to a covered map  $(M, T)$  a pair made of a (rooted plane) tree  $\Psi_1(M, T)$  and a bipartite unicellular map  $\Psi_2(M, T)$ . The mapping  $\Psi$  has two steps. At the first step, one defines an orientation  $(I, O) = \delta_M(T)$  of the map  $M$  which is closely related to the order in which half-edges of  $M$  appear *around the submap  $T$* . At the second step, the map is broken into two parts: a plane tree  $\Psi_1(M, T)$  containing every edge of  $M$  and a bipartite unicellular map  $\Psi_2(M, T)$  which roughly speaking *describes how to fold the tree  $\Psi_1(M, T)$  in order to obtain the map  $M$*  (and the orientation  $(I, O)$ ).

**Step 1: orientation.** Consider a map  $M = (H, r, \alpha, \sigma)$ . We denote by  $\phi = \sigma\alpha$  the permutation corresponding to the faces of  $M$ . For any unicellular spanning submap  $T$  of  $M$ , we call the *motion function around  $T$*  the mapping  $\theta$  on  $H$  defined by  $\theta(h) = \sigma(h)$  if  $h$  is external and  $\theta(h) = \phi(h)$  otherwise. It can be shown that the motion function  $\theta$  is a cyclic permutation of  $H$  if and only if  $T$  is a unicellular map. In this case, the motion function  $\theta$  induces a total order on the set of half-edges  $H$  by setting  $r <_T \theta(r) <_T \theta^2(r) \dots <_T \theta^{|H|-1}(r)$ . For instance, the order induced by the spanning submap  $T$  in Figure 1 is  $1 < 2' < 3 < 4' < 3' < 5' < 1' < 2 < 6 < 7' < 8 < 9 < 5 < 9' < 4 < 8' < 7 < 6'$ . We are now ready to define the orientation  $\delta_M(T)$  which is represented in Figure 2.

**Definition 2.1.** Let  $M$  be a map. The mapping  $\delta_M$  associates to a unicellular submap  $T$  of  $M$  the orientation  $\delta_M(T) = (I, O)$ , where the set of ingoing half-edges  $I$  contains the internal half-edges such that  $\alpha(h) <_T h$  and the external half-edges such that  $h <_T \alpha(h)$  (and  $O = H - I$ ).

**Step 2: unfolding.** Let us first describe the *unfolding step* informally. At this step, each vertex of the map  $M$  is broken according to the rule described in Figure 3(a). The rule is the following: given a vertex, that is, a cycle  $v = (h_1, \dots, h_k)$  of  $\sigma$  we consider the indices  $1 \leq i_1 < i_2 < \dots < i_l = k$  of the ingoing half-edges incident to  $v$ . At the unfolding step, the vertex  $v$  is decomposed into  $l$  vertices  $v_1 = (h_1, \dots, h_{i_1}), v_2 = (h_{i_1+1}, \dots, h_{i_2}), \dots, v_l = (h_{i_{l-1}+1}, \dots, h_{i_l})$ . Note that the decomposition of  $v$  can be written as:  $v = v_1 v_2 \dots v_l \pi_\bullet$ , where  $\pi_\bullet(h) = h$  if  $h \in O$  and  $\pi_\bullet(h_{i_j}) = h_{i_{j+1}}$  for  $j = 1, \dots, l$ . Figure 3(a) shows the topological representation of the decomposition of a vertex incident to 3 ingoing half-edges. After unfolding, one gets the vertices  $v_1, v_2, v_3$  (they will be vertices of the plane tree  $\Psi_1(M, T)$ ) and a big black vertex corresponding to the permutation  $\pi_\bullet$  (it will be a vertex of the unicellular bipartite map  $\Psi_2(M, T)$ ).

We now describe the unfolding step in more details. Let  $(I, O) = \delta_M(T)$  be the orientation of  $M$  associated to the unicellular map  $T$ . Let  $i$  and  $o$  be two new elements not in  $H$ . We define  $\sigma'$  (resp.  $\phi'$ ) as the permutation of  $I' = I \cup \{i\}$  (resp.  $O' = O \cup \{o\}$ ) obtained from  $\sigma$  by inserting the new half-edge  $i$  (resp.  $o$ ) just before the root  $r$  in the cycle of  $\sigma$  (resp.  $\phi$ ) containing  $r$ . We also consider the restrictions  $\pi_\bullet = \sigma'_{I'}$  and  $\pi_o = \phi'_{O'}$ . In our favourite example, we get  $\pi_\bullet = (i)(1', 2', 5')(3', 4')(6', 7')(8')(9')$  and  $\pi_o = (o, 1, 6)(2, 3, 7, 8, 9)(4)$ . We now define  $\pi = \pi_\bullet \pi_o^{-1}$  and  $\tau' = \sigma' \pi_\bullet^{-1}$  (here we make a slight abuse of notation by considering that  $\pi_\bullet = \sigma'_{I'}$  acts as the identity on  $O'$  and that  $\pi_o = \phi'_{O'}$  acts as the identity on  $I'$ ). We are now ready to define the mapping  $\Psi$ .

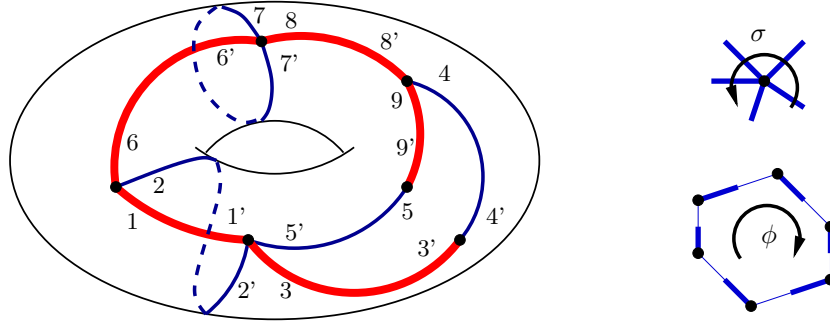


FIGURE 1. A map  $M$  (rooted on the half-edge 1) and a unicellular spanning submap  $T$  (thick lines).

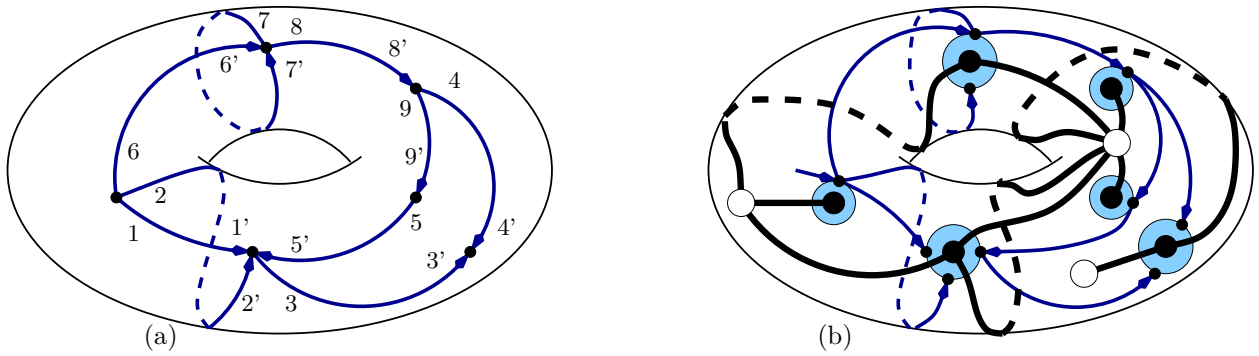


FIGURE 2. (a) Orientation  $(O, I) = \delta_M(T)$ . (b) Unfolding

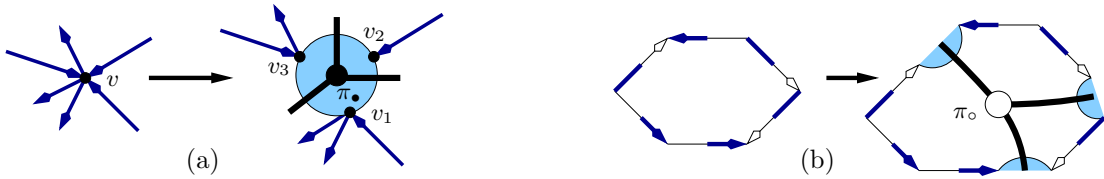


FIGURE 3. Topological representation of the *unfolding* around a vertex (a) and around a face (b).

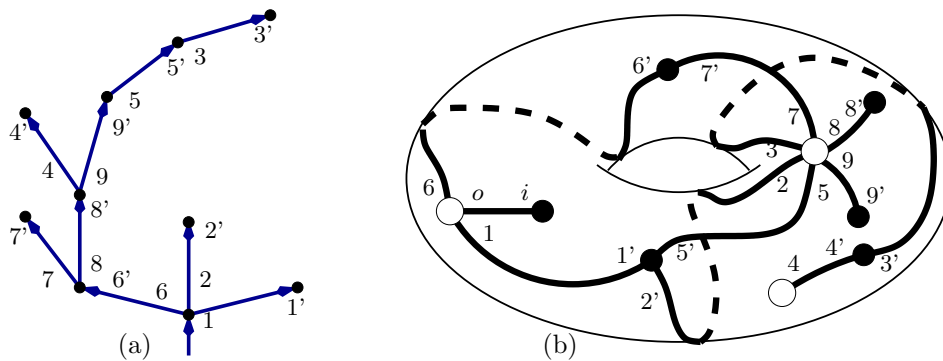


FIGURE 4. (a) The tree  $\Psi_1(M, T)$ . (b) The unicellular map  $\Psi_2(M, T)$ .

**Definition 2.2.** Let  $M = (H, r, \sigma, \alpha)$  be a map and let  $T$  be a unicellular spanning submap. The mapping  $\Psi$  associates to the covered map  $(M, T)$  the pair  $(\Psi_1(M, T), \Psi_2(M, T))$  defined by:  $\Psi_1(M, T) = (H, t, \tau, \alpha)$  and  $\Psi_2(M, T) = (H', i, \pi, \alpha)$  where  $\tau = \tau'_{|H}$  and  $t = \tau'(i)$ .

The image of the covered map in Figure 1 by  $\Psi_1$  and  $\Psi_2$  are represented respectively in Figure 4 (a) and (b). Our main result is the following:

**Theorem 2.3.** *The mapping  $\Psi : (M, T) \mapsto (\Psi_1(M, T), \Psi_2(M, T))$  is a bijection between covered maps of size  $n$  and genus  $g$  and pairs made of a tree of size  $n$  and a bipartite unicellular map of size  $n+1$  and genus  $g$ .*

### 3. ENUMERATIVE CORROLARIES.

The immediate enumerative corrolary of Theorem 2.3 is the following.

**Corollary 3.1.** *The number of covered maps of size  $n$  and genus  $g$  is  $S_n^g = C_n B_{n+1}^g$ , where  $C_n = \frac{(2n)!}{n!(n+1)!}$  is the  $n^{\text{th}}$  Catalan number and  $B_n^g$  is the number of bipartite unicellular maps with  $n$  edges.*

In [3], an expression is given for the number  $B_n^g$  of bipartite unicellular maps. In particular, it is shown there that for a given genus  $g$  the asymptotic of  $B_n^g$  is

$$B_n^g \sim_{n \rightarrow \infty} \frac{1}{\sqrt{\pi} g! 48^g} \cdot n^{3g-3/2} 4^n.$$

Using this formula, we obtain the following asymptotic result.

**Proposition 3.2.** *Let  $g$  be a non-negative integer. The asymptotic number of covered maps of genus  $g$  and size  $n$  is*

$$(1) \quad S_n^g \sim \frac{4}{\pi g! 96^g} \cdot n^{3g-3} 16^n.$$

**Covered maps vs tree-rooted maps.** As mentioned in the introduction, the notion of covered map generalise the well studied notion of tree-rooted map. In the planar case (genus 0), the two notions coincide. In the toroidal case (genus 1), a duality argument shows that exactly half of the covered maps of size  $n$  are tree-rooted maps. This property, together with the expression of  $B_n^1$  given in [3] allows one to recover a result obtained by Lehman and Walsh:

**Proposition 3.3.** [4] *The number  $T_n^1$  of tree-rooted maps of size  $n$  on the torus is*

$$T_n^1 = \frac{1}{2} A_n^1 = \frac{1}{2} C_n B_n^1 = \frac{(2n)!(2n+1)!}{12(n-2)!n!((n+1)!)^2}.$$

For genus  $g$  greater than 1, no nice relation seems to hold between the number  $S_n^g$  of covered maps of size  $n$  and the number  $T_n^g$  of tree-rooted maps of size  $n$ . However, it is proved in [1] that the asymptotic number of tree-rooted maps of genus  $g$  is

$$(2) \quad T_n^g \sim \frac{4}{\pi g! 48^g} \cdot n^{3g-3} 16^n.$$

Comparing this result with (1) shows that  $S_n^g \sim 2^g T_n^g$ . In other words, the probability that a covered map of genus  $g$  is a tree-rooted map tends to  $1/2^g$ . As an algorithmic consequence of this fact, our bijection could be used to provide an optimal coding of tree-rooted maps of genus  $g$ , using only  $4 + o(1)$  bits per edge.

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