A BIJECTION FOR COVERED MAPS ON ORIENTABLE SURFACES

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Abstract. *Unicellular maps* are a natural generalisation of plane trees to higher genus surfaces. In this article we study *covered maps*, which are maps together with a distinguished unicellular spanning submap. We prove that the covered maps of genus g with n edges are in bijection with pairs made of a plane tree with n edges and a bipartite unicellular map of genus g with $n+1$ edges. This generalises to any genus the bijection given in [2] between planar *tree-rooted maps* (maps with a distinguished spanning tree) and pairs made of a tree with n edges and a tree with $n+1$ edges. In the special case of genus 1, a duality argument allows us to obtain a bijective proof of a formula of Lehman and Walsh [4] about the number of tree-rooted maps of genus 1.

1. Introduction

We consider maps on orientable surfaces of arbitrary genus. A map is *unicellular* if it has a single face, that is, if the complement of the map is simply connected. A unicellular map on the torus is represented in Figure 4(b). A *covered map* is a map together with a distinguished spanning unicellular submap. A map of genus g can have spanning submaps of any genus in $\{0\dots,g\}$. An example of covered map is given in Figure 1. The main goal of this article is to exhibit a bijection Ψ between covered maps of genus g and size n and pairs made of a plane tree of size n and a unicellular map of genus g and size $n+1$.

Covered maps are a natural generalisation of *tree-rooted maps*, that is, maps together with a distinguished spanning tree. In the planar case these two notions coincide and our bijection Ψ specialise into the bijection found in [2] in order to give a bijective explanation of a result of Mullin [6]: the number of planar tree-rooted maps of size n is $T_n^0 = C_nC_{n+1}$, where $C_n = \frac{(2n)!}{n!(n+1)!}$ is the n^{th} Catalan number i.e. the number of plane trees with n edges. In the case of the torus, a duality argument shows that exactly half of the covered maps of size n are tree-rooted maps. Therefore, our bijection Ψ give a bijective explanation to the formula of Lehman and Walsh [4]: the number of tree-rooted maps of genus 1 is $T_n^1 = \frac{1}{2}C_nB_{n+1}^1$, where $B_n^1 = \frac{(2n-1)!}{6n!(n-3)!}$ is the number of bipartite unicellular maps with n edges.

We first recall some definitions. A *map* is a connected graph *embedded* in an orientable surface considered up to homeomorphism. By *embedded*, one means drawn on the surface in such a way the edges do not intersect and the *faces* (connected components of the complement of the graph) are simply connected. An example is given in Figure 1 (forget the thick lines for the time being). The *genus* of a map is the genus of the surface in which it is embedded and its *size* is the number of edges. A *planar map* is a map of genus 0. A map is *unicellular* if it has a single face. For instance, *plane trees* are the unicellular planar maps. A map is *bipartite* if the underlying graph is. A unicellular bipartite map of genus 1 is represented in Figure 4(b).

The embedding of a map defines a cyclic order (the counterclockwise order) of the half-edges around each vertex. There is, in fact, a one-to-one correspondence between maps and connected graphs together with a cyclic order of the edges around each vertex [5]. Equivalently, a map can be defined as a triple $M = (H,\sigma,\alpha)$, where H is a finite set whose element are the *half-edges*, α is an involution of H without fixed point, and σ is a permutation of H such that the group generated by α and σ acts transitively on H. The cycles of the involution σ are the edges and the cycles of the permutation σ are the vertices together with the counterclockwise order of half-edges around them. For instance, the map in Figure 1 is $M = (H,\sigma,\alpha)$, where $H = \{1,1',2,2',\dots,9,9'\}$, $\alpha = (1,1')(2,2')\cdots(9,9')$ and $\sigma = (1,2,6)(1',2',3,5')(3',4')(5,9')(4,8',9)(6',7',8,7)$. Observe that the faces of M are in bijection with the cycles of the permutation $\phi = \sigma\alpha$. For the map of Figure 1, $\phi = (1,2',6,7',6')(1',2,3,4',8',7,8,9,5)(3',5',9',4)$. A map is *rooted* if one of the half-edges is distinguished as the *root*; we denote by $M = (H,r,\sigma,\alpha)$ the map (H,σ,α) having root r. In the following

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maps are rooted and are considered up to *isomorphism* (relabelling of the half-edges).

Given a subset S of H, the *restriction of* π *to* S, denoted by $\pi_{|S}$ is the permutation of S whose cycles are obtained from the cycles of π by erasing the elements not in S. For instance, if $\pi = (a,b,c)(d,e)(f,g,h,i)$ and $S = \{b,c,f,g,i\}$, then $\pi_{|S} = (b,c)(f,g,i)$. A *submap* of a map $M = (H,\sigma,\alpha)$ is a map of the form $N = (S, \alpha_{|S}, \sigma_{|S})$,where $S \subseteq H$. It is *spanning* if every cycle of σ contains an element of S. A submap of a map of genus g has genus less or equal to g. For instance, the map M in Figure 1 has genus 1 while the spanning submap $T = (S, \alpha_{|S}, \sigma_{|S})$ induced by the set $S = \{1,1',3,3',6,6',8,8',9,9'\}$ (thick lines) has genus 0. A pair (M,T) made of a map M and a unicellular spanning submap T is a *covered map*. A covered map (M,T) is represented in Figure 1. Given a covered map (M,T), a half-edge is called *internal* if it belongs to the submap T and *external* otherwise. An *orientation* of a map $M = (H,\sigma,\alpha)$ is a partition $H = I \uplus O$ such that the involution α maps the set I on the set O; the half-edges in I and O are respectively called *ingoing* and *outgoing*. The orientation $I = \{1',2',\dots,9'\}$ and $O = \{1,2,\dots,9\}$ of the map M is represented in Figure 2(a).

2. Bijection

We now define the mapping Ψ which associates to a covered map (M,T) a pair made of a (rooted plane) tree $\Psi_1(M,T)$ and a bipartite unicellular map $\Psi_2(M,T)$. The mapping Ψ has two steps. At the first step, one defines an orientation $(I,O) = \delta_M(T)$ of the map M which is closely related to the order in which half-edges of M appear *around the submap* T. At the second step, the map is broken into two parts: a plane tree $\Psi_1(M,T)$ containing every edge of M and a bipartite unicellular map $\Psi_2(M,T)$ which roughly speaking *describes how to fold the tree* $\Psi_1(M,T)$ *in order to obtain the map* M (and the orientation (I,O)).

Step 1: orientation. Consider a map $M = (H,r,\alpha,\sigma)$. We denote by $\phi = \sigma\alpha$ the permutation corresponding to the faces of M. For any unicellular spanning submap T of M, we call the *motion function around* T the mapping θ on H defined by $\theta(h) = \sigma(h)$ if h is external and $\theta(h) = \phi(h)$ otherwise. It can be shown that the motion function θ is a cyclic permutation of H if and only if T is a unicellular map. In this case, the motion function θ induces a total order on the set of half-edges H by setting $r <_T \theta(r) <_T \theta^2(r) \cdots <_T \theta^{|H|-1}(r)$. For instance, the order induced by the spanning submap T in Figure 1 is $1 < 2' < 3 < 4' < 3' < 5' < 1' < 2 < 6 < 7' < 8 < 9 < 5 < 9' < 4 < 8' < 7 < 6'$. We are now ready to define the orientation $\delta_M(T)$ which is represented in Figure 2.

Definition 2.1. Let M be a map. The mapping δ_M associates to a unicellular submap T of M the orientation $\delta_M(T) = (I,O)$, where the set of ingoing half-edges I contains the internal half-edges such that $\alpha(h) <_T h$ and the external half-edges such that $h <_T \alpha(h)$ (and $O = H - I$).

Step 2: unfolding. Let us first describe the *unfolding step* informally. At this step, each vertex of the map M is broken according to the rule described in Figure 3(a). The rule is the following: given a vertex, that is, a cycle $v = (h_1,\dots,h_k)$ of σ we consider the indices $1 \le i_1 < i_2 < \cdots < i_l = k$ of the ingoing half-edges incident to v. At the unfolding step, the vertex v is decomposed into l vertices $v_1 = (h_1, \dots h_{i_1}), v_2 = (h_{i_1+1},\dots,h_{i_2}),\dots,v_l = (h_{i_{l-1}+1},\dots,h_{i_l})$. Note that the decomposition of v can be written as: $v = v_1 v_2 \cdots v_l \pi_\bullet$, where $\pi_\bullet(h) = h$ if $h \in O$ and $\pi_\bullet(h_{i_j}) = h_{i_{j+1}}$ for $j = 1,\dots,l$. Figure 3(a) shows the topological representation of the decomposition of a vertex incident to 3 ingoing half-edges. After unfolding, one gets the vertices v_1, v_2, v_3 (they will be vertices of the plane tree $\Psi_1(M,T)$) and a big black vertex corresponding to the permutation $\pi_\bullet$ (it will be a vertex of the unicellular bipartite map $\Psi_2(M,T)$).

We now describe the unfolding step in more details. Let $(I,O) = \delta_M(T)$ be the orientation of M associated to the unicellular map T. Let i and o be two new elements not in H. We define σ' (resp. ϕ') as the permutation of $I' = I \cup \{i\}$ (resp. $O' = O \cup \{o\}$) obtained from σ by inserting the new half-edge i (resp. o) just before the root r in the cycle of σ (resp. ϕ) containing r. We also consider the restrictions $\pi_\bullet = \sigma'_{|I'}$ and $\pi_\circ = \phi'_{|O'}$. In our favourite example, we get $\pi_\bullet = (i)(1',2',5')(3',4')(6',7')(8')(9')$ and $\pi_\circ = (o,1,6)(2,3,7,8,9)(4)$. We now define $\pi = \pi_\bullet \pi_\circ^{-1}$ and $\tau' = \sigma' \pi_\bullet^{-1}$ (here we make a slight abuse of notation by considering that $\pi_\bullet = \sigma'_{|I'}$ acts as the identity on O' and that $\pi_\circ = \phi'_{|O'}$ acts as the identity on I'). We are now ready to define the mapping Ψ.

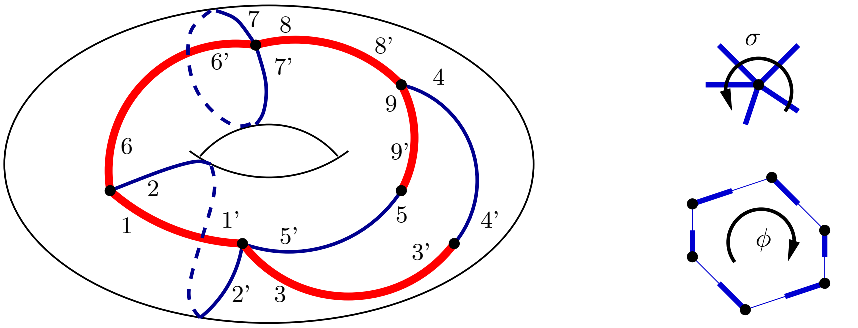

FIGURE 1. A map M (rooted on the half-edge 1) and a unicellular spanning submap T (thick lines).

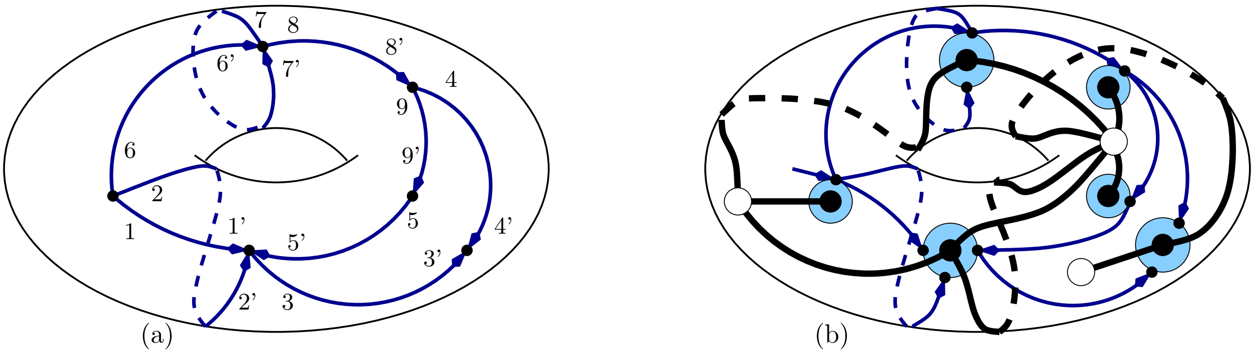

FIGURE 2. (a) Orientation $(O, I) = \delta_M(T)$. (b) Unfolding

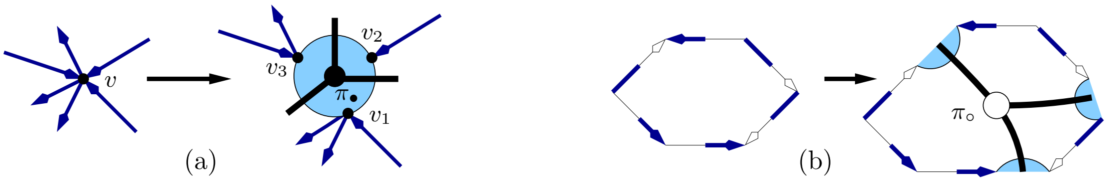

FIGURE 3. Topological representation of the *unfolding* around a vertex (a) and around a face (b).

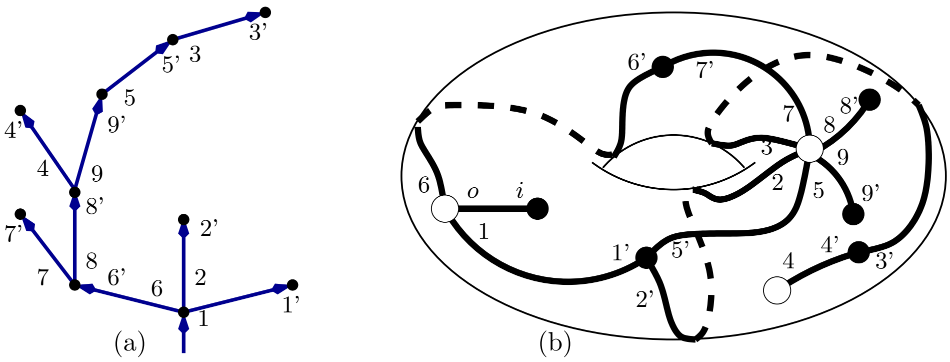

FIGURE 4. (a) The tree $\Psi_1(M, T)$. (b) The unicellular map $\Psi_2(M, T)$.

Definition 2.2. Let $M = (H, r, \sigma, \alpha)$ be a map and let T be a unicellular spanning submap. The mapping Ψ associates to the covered map (M, T) the pair $(\Psi_1(M, T), \Psi_2(M, T))$ defined by: $\Psi_1(M, T) = (H, t, \tau, \alpha)$ and $\Psi_2(M, T) = (H', i, \pi, \alpha)$ where $\tau = \tau'_{|H}$ and $t = \tau'(i)$.

The image of the covered map in Figure 1 by Ψ_1 and Ψ_2 are represented respectively in Figure 4 (a) and (b). Our main result is the following:

Theorem 2.3. *The mapping $\Psi : (M, T) \mapsto (\Psi_1(M, T), \Psi_2(M, T))$ is a bijection between covered maps of size n and genus g and pairs made of a tree of size n and a bipartite unicellular map of size $n+1$ and genus g.*

3. Enumerative corrolaries.

The immediate enumerative corrolary of Theorem 2.3 is the following.

Corollary 3.1. *The number of covered maps of size n and genus g is $S_n^g = C_n B_{n+1}^g$, where $C_n = \frac{(2n)!}{n!(n+1)!}$ is the n^{th} Catalan number and B_n^g is the number of bipartite unicellular maps with n edges.*

In [3], an expression is given for the number B_n^g of bipartite unicellular maps. In particular, it is shown there that for a given genus g the asymptotic of B_n^g is

$$B_n^g \sim_{n\to\infty} \frac{1}{\sqrt{\pi}\ g!\ 48^g} \cdot n^{3g-3/2} 4^n.$$

Using this formula, we obtain the following asymptotic result.

Proposition 3.2. *Let g be a non-negative integer. The asymptotic number of covered maps of genus g and size n is*

$$S_n^g \sim \frac{4}{\pi\ g!\ 96^g} \cdot n^{3g-3} 16^n. \tag{1}$$

Covered maps vs tree-rooted maps. As mentioned in the introduction, the notion of covered map generalise the well studied notion of tree-rooted map. In the planar case (genus 0), the two notions coincide. In the toroidal case (genus 1), a duality argument shows that exactly half of the covered maps of size n are tree-rooted maps. This property, together with the expression of B_n^1 given in [3] allows one to recover a result obtained by Lehman and Walsh:

Proposition 3.3. [4] *The number T_n^1 of tree-rooted maps of size n on the torus is*

$$T_n^1 = \frac{1}{2} A_n^1 = \frac{1}{2} C_n B_n^1 = \frac{(2n)!(2n+1)!}{12(n-2)!n!((n+1)!)^2}.$$

For genus g greater than 1, no nice relation seems to hold between the number S_n^g of covered maps of size n and the number T_n^g of tree-rooted maps of size n. However, it is proved in [1] that the asymptotic number of tree-rooted maps of genus g is

$$T_n^g \sim \frac{4}{\pi\ g!\ 48^g} \cdot n^{3g-3} 16^n. \tag{2}$$

Comparing this result with (1) shows that $S_n^g \sim 2^g T_n^g$. In other words, the probability that a covered map of genus g is a tree-rooted map tends to $1/2^g$. As an algorithmic consequence of this fact, our bijection could be used to provide an optimal coding of tree-rooted maps of genus g, using only $4 + o(1)$ bits per edge.

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