A bijection for covered maps, or a shortcut between Harer-Zagier's and Jackson's formulas.

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Abstract

We consider maps on orientable surfaces. A map is *unicellular* if it has a single face. A *covered map* is a map with a marked unicellular spanning submap. For a map of genus g, the unicellular submap can have any genus in $\{0, 1, \ldots, g\}$. Our main result is a bijection between covered maps with n edges and genus g and pairs made of a plane tree with n edges and a unicellular bipartite map of genus g with $n + 1$ edges.

In the planar case, the covered maps are maps with a marked spanning tree (a.k.a. *tree-rooted maps*) and our bijection specializes into a construction obtained by the first author in [3]. A strong connection subsists between covered maps and tree-rooted maps in genus 1 (because a covered map is either a tree-rooted map or the dual of a tree-rooted map) and we thereby obtain a bijective explanation of a formula by Lehman and Walsh on the number of tree-rooted maps of genus 1 [24]. A more surprising byproduct of our bijection is an equivalence between an enumerative formula by Harer and Zagier concerning unicellular maps of given genus and a similar formula by Jackson concerning *bipartite* unicellular maps of given genus. The equivalence is obtained by observing that covered maps can be seen as a *shuffle* of two unicellular maps, hence that our bijection gives a relations between shuffles of unicellular maps and bipartite unicellular maps.

We also show that the bijection of Bouttier, Di Francesco and Guitter [6] (which generalizes a famous bijection by Schaeffer [30]) between bipartite maps and so-called well-labelled mobiles can be described as a special case of our bijection.

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1 Introduction.

We consider maps on orientable compact surfaces of arbitrary genus. A map is said *unicellular* if it has a single face as is the case for the map represented in Figure 2. A *covered map* is a map together with a marked unicellular spanning submap. A map of genus g have spanning submaps of any genus in $\{0\ldots,g\}$. In particular, *tree-rooted maps* (maps with a marked spanning tree) are a special case of covered map since a spanning tree is a spanning unicellular submap of genus 0. A covered map of genus 2 having a unicellular spanning submap of genus 1 is represented in Figure 1(a). More details about maps and the genus of submaps are given in Section 2.

Our main result is a bijection Ψ between covered maps of genus g with n edges and pairs made of a plane tree with n edges and a bipartite unicellular map of genus g with $n+1$ edges. The image of a covered map by the mapping Ψ is represented in Figure 1(b). The bijection Ψ generalizes a construction by the first author [3] between planar tree-rooted maps with n edges and pairs of plane trees with n and $n+1$ edges respectively[1].

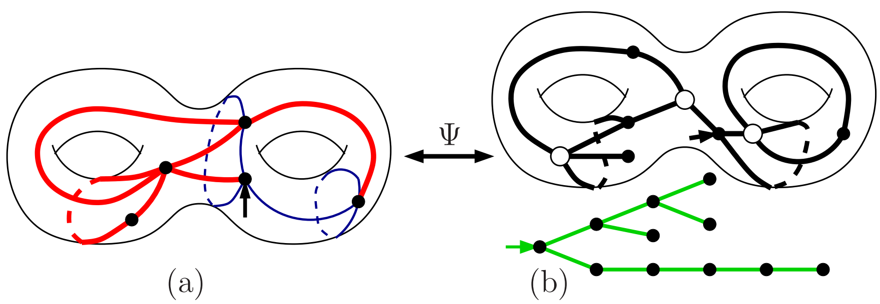

Figure 1: (a) A covered map of genus 2 (the unicellular submap of genus 1 is drawn in thick lines). (b) The image of the covered map by the bijection Ψ is made of a bipartite unicellular map of genus 2 and a plane tree.

It is an important observation that the *dual of a planar tree-rooted map is a planar tree-rooted map*, that is to say, the dual of the edges not in the spanning tree form a spanning tree of the dual map. Pushing this observation further, Mullin showed that tree-rooted maps could be encoded by a *shuffle* of two trees (one representing the spanning tree, the other representing the dual spanning tree), or more precisely as a shuffle of two parenthesis systems encoding these trees [29]. Covered maps generalize these properties: the *dual of a covered map is a covered map* and covered maps can be encoded by *shuffles* of two unicellular maps (more details are given in Section 3). We emphasize that our bijection Ψ (as the construction in [3]) is of very different nature: the result is a pair of

[1] In [3], the tree with $n+1$ edges was actually described as a non-crossing partition.

unicellular maps of a fixed size and not a shuffle.

Our bijection Ψ has the interesting property that it can be specialized in various ways in order to obtain bijections for several important classes of maps. In particular, it is shown in [4] how to specialize the bijection Ψ in order to count certain classes of triangulations and quadrangulations. Here we consider yet another specialization, namely, we will show that the bijection Ψ specializes into the bijection by Bouttier, Di Francesco and Guitter [6] (for the planar case) and its generalization to higher genus surfaces by Chapuy, Marcus and Schaeffer [11, 9]. These bijections which generalize a previous bijection by Schaeffer [30] are of fundamental importance for studying the metric properties of random maps [12, 5, 8, 7, 26] and for defining and analyzing their continuous limit, the *Brownian map* [25, 20, 22, 21].

The bijection Ψ has several enumerative corrolaries. The first corollaries concern tree-rooted maps of genus 0 and 1. In [29], Mullin used the correspondence between planar tree-rooted maps and shuffles of trees to prove that the number of planar tree-rooted maps with n edges is the product of two consecutive Catalan numbers:

$$T_0(n) = \mathrm{Cat}(n)\mathrm{Cat}(n+1), \quad \text{where } \mathrm{Cat}(n) = \frac{(2n)!}{n!(n+1)!}. \tag{1}$$

and asked for a bijection between tree-rooted maps of size n and pairs of trees of size n and $n+1$ respectively. This is precisely what our bijection Ψ gives in the planar case. This planar case was originally described in [3] as an answer to Mullin's question. It was also proved there that this specialization is isomorphic to a previous recursive bijection by Cori, Dulucq and Viennot [13].

In [24], Lehman and Walsh gave an expression for the number of tree-rooted map of genus 1 with n edges:

$$T_1(n) = \mathrm{Cat}(n)\ \frac{(2n-1)!}{12(n-1)!(n-2)!}. \tag{2}$$

Again, no bijective proof was known explaining this simple formula involving the Catalan number (and it was also noted in [24] that no clear pattern emerged for higher genera). Our bijection Ψ gives a bijective explanation to Formula (2) because a duality argument shows that exactly half of the covered maps of genus 1 are tree-rooted maps, and $\frac{(2n-1)!}{6(n-1)!(n-2)!}$ is the number of bipartite maps of genus 1 with $n+1$ edges.

Another, more surprising, enumerative corollary of our bijection is a bijective shortcut between two formulas concerning unicellular maps. In [16], Harer and Zagier proved the following formulas concerning the number $A^p(n)$ of unicellular maps with n edges and p vertices (hence genus $(n+1-p)/2$):

$$\sum_{p\geq 1} A^p(n) y^p = \frac{(2n)!}{2^n n!} \sum_{i\geq 1} 2^{i-1} \binom{n}{i-1}\binom{y}{i}. \tag{3}$$

The original proof of Harer and Zagier involved the computation of a matrix integral. Since then, a combinatorial interpretation was given by Lass [19], which was further developed into fully bijective proof in [14]. An alternative bijective approach to unicellular maps was recently given in [10]. A similar formula for the number $B^{p,q}(n)$ of bipartite unicellular maps with n edges, p white vertices and q black vertices:

$$\sum_{p,q\geq 1} B^{p,q}(n+1)y^p z^q = (n+1)! \sum_{i,j\geq 1} \binom{n}{i-1, j-1} \binom{y}{i} \binom{z}{j}, \tag{4}$$

was independently obtained by Jackson [17] and by Adrianov [1] by means of characters computations. Bijective proofs were given in [31, 28]. We show that our bijection Ψ establishes an equivalence between Formulas (3) and (4). Indeed, our bijection gives a relation between the number of shuffles of unicellular maps and the number of bipartite unicellular maps.

The paper is organized as follows. In Section 2, we recall some definitions about maps. In Section 3, we show that covered maps can be seen as shuffles of unicellular maps. In Section 4, we define the bijection Ψ between covered maps with n edges and pairs made of a plane tree with n edges and a bipartite unicellular map with $n+1$ edges. In Section 5, we explore the enumerative corollaries of our bijection. Section 6 contains the proofs of the bijectivity of Ψ. In Section 7, we give three equivalent ways of describing pairs made of a plane tree and a bipartite unicellular map and explicit the inverse mapping Ψ^{-1} for these different descriptions. In Section 8, we use one of these descriptions in order to recover the bijection of Bouttier *et al.* [6] as a specialization of Ψ. Lastly, in Section 9, we explore the properties of the bijection Ψ with respect to duality.

2 Definitions

Maps. Maps can either be defined topologically (as graphs embedded in surfaces) or combinatorially (in terms of permutations). We shall prove our results using the combinatorial definition, but resort to the topological interpretation in order to convey intuitions.

We start with the topological definition of maps. Our *surfaces* are 2-dimensional, oriented, compact and without boundaries. A *map* is a connected graph embedded in surface, considered up to homeomorphism. By *embedded*, one means drawn on the surface in such a way that the edges do not intersect and the *faces* (connected components of the complement of the graph) are simply connected. Loops and multiple edges are allowed. The *genus* of the map is the genus of the underlying surface and its *size* is its number of edges. A *planar map* is a map of genus 0. A map is *unicellular* if it has a single face. For instance, the planar unicellular maps are the *plane trees*. A map is *bipartite* if vertices can be colored in black and white in such a way that every edge join a white vertex to a black vertex. We denote by $g(M)$ the genus of a map and by $v(M)$, $f(M)$, $e(M)$ respectively its number of vertices, faces and edges. The *Euler formula* relates these quantities by

$$v(M) - e(M) + f(M) = 2 - 2g(M).$$

By removing the midpoint of an edge, one obtains two *half-edges*. Two consecutive half-edges around a vertex define a *corner*. A map is *rooted* if one half-edge is distinguished as the *root*. The vertex incident to the root is called *root-vertex*. In figures, the rooting will be indicated by an arrow pointing into the *root-corner*, that is, the corner following the root in clockwise order around the root-vertex. For instance, the root of the map in Figure 2 is the half-edge a_1.

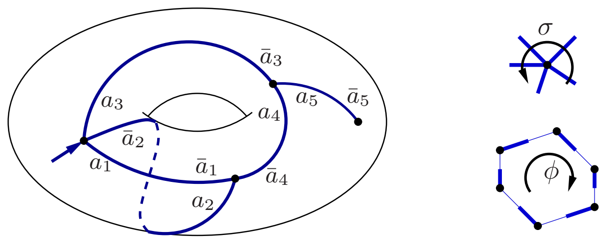

Figure 2: A unicellular map of genus 1.

Maps can also be defined in terms of permutations acting on half-edges. To obtain this equivalence, observe first that the embedding of a graph in a surface defines a cyclic order (the counterclockwise order) of the half-edges around each vertex. This gives in fact a one-to-one correspondence between maps and connected graphs together with a cyclic order of the half-edges around each vertex (see e.g. [27]). Equivalently, a *map* can be defined as a triple $M = (H, \sigma, \alpha)$ where H is a finite set whose elements are called the *half-edges*, α is an involution of H without fixed point, and σ is a permutation of H such that the group generated by σ and α acts transitively on H. This must be understood as follows: each cycle of σ describes the counterclockwise order of the half-edges around one vertex of the map, and each cycle of α describes an edge, that is, a pair of two half-edges. The transitivity assumption simply translates the fact that the graph is connected. Figure 2 shows the map $M = (H, \sigma, \alpha)$, where $H = \{a_1, \bar{a}_1, \ldots, a_5, \bar{a}_5\}$, $\sigma = (a_1, \bar{a}_2, a_3)(\bar{a}_1, a_2, \bar{a}_4)(\bar{a}_3, a_4, a_5)(\bar{a}_5)$ and $\alpha = (a_1, \bar{a}_1) \cdots (a_5, \bar{a}_5)$.

For a map $M = (H, \sigma, \alpha)$, the permutation σ is called *vertex-permutation*, the permutation α is called *edge-permutation* and the permutation $\phi = \sigma\alpha$ is called *face-permutation*. The cycles of σ, α, *phi* are called *vertices*, *edges* and *faces*. Observe that the cycles of ϕ are indeed in bijection with the faces of the map in its topological interpretation. Hence, the genus of M can be deduced from the number of cycles of σ, α and ϕ by the Euler relation. We say that a half-edge is *incident* to a vertex or a face if this edge belongs to the corresponding cycle. Again, a map is *rooted* if one of the half-edges is distinguished as the *root*; the incident vertex and face are called *root-vertex* and *root-face*.

The correspondence between topological and combinatorial map is one-to-one if combinatorial maps are considered up to *isomorphism* (or, *relabelling*). That is, two maps (H, σ, α) and (H', σ', α') are considered the same if there exists a bijection $\lambda : H \to H'$ such that $\sigma' = \lambda\sigma\lambda^{-1}$ and $\alpha' = \lambda\alpha\lambda^{-1}$ (for rooted maps, we ask furthermore that $\lambda(r) = r'$). In this article *all maps will be rooted, and considered up to isomorphism.*

We call *pseudo map* a triple $M = (H, \sigma, \alpha)$ such that α is a fixed-point free involution, but where the transitivity assumption (i.e. connectivity assumption) is not required. This can be seen as a union of maps and we still call $\phi = \sigma\alpha$ the *face-permutation*, as its cycles are indeed in correspondence with the faces of the union of maps. Lastly, we consider the case where the set of half-edges H is empty as a special case of rooted unicellular map (corresponding to the planar map with one vertex and no edge) called *empty map*.

Submaps, covered maps and motion functions. For a permutation π on a set H, we call *restriction of* π to a set $S \subseteq H$ and denote by $\pi_{|S}$ the permutation of S whose cycles are obtained from the cycles of π by erasing the elements not in S. Observe that $(\pi^{-1})_{|S} = (\pi_{|S})^{-1}$ so that we shall not use parenthesis anymore in these notations. It is sometime convenient to consider the restriction $\pi_{|S}$ as a permutation on the whole set H acting as the identity on $H \setminus S$; we shall mention this abuse of notations whenever necessary.

A spanned map is a map with a marked subset of edges. In terms of permutations, a *spanned map* is a pair (M, S), where S is a subset of half-edges stable by the edge-permutation α. The *submap defined by* S, denoted $M_{|S}$, is the pseudo map $(S, \sigma_{|S}, \alpha_{|S})$, where σ is the vertex-permutation of M. We underline that the face-permutation $\phi_S = \sigma_{|S}\alpha_{|S}$ of the pseudo-map $M_{|S}$ is not equal to $(\sigma\alpha)_{|S}$. Observe also that the genus of $M_{|S}$ can be less than the genus of M. For example, Figure 1(a) represents a submap of genus 1 of a map of genus 2. A submap $M_{|S}$ is *connecting* if it is a map containing every vertex of M, that is, S contains a half-edge in every vertex of M (except if M has a single vertex, where we authorize S to be empty) and $\sigma_{|S}$, $\alpha_{|S}$ act transitively on S. The submap represented in Figure 3 (right) is a map but is not connecting. A *covered map* is a spanned map such that the submap $M_{|S}$ is a connecting unicellular map. A *tree-rooted map* is a spanned map such that the submap $M_{|S}$ is a spanning tree, that is, a connecting plane tree.

The *motion function* of the spanned map (M, S) is the mapping θ defined on H by $\theta(h) = \phi(h) \equiv \sigma\alpha(h)$ if h is in S and $\theta(h) = \sigma(h)$ otherwise. Note that the motion function is a permutation of H since the stability of S by α implies that $\theta^{-1}(h) = \alpha\sigma^{-1}(h)$ if $\sigma^{-1}(h)$ is in S and $\theta^{-1}(h) = \sigma^{-1}(h)$ otherwise. Observe also that, given M, the set S can be recovered from the motion function θ. Topologically, the motion function is the permutation describing the *tour* of the connected components of the submap $M_{|S}$ in counterclockwise direction: we follow the border of the edges of the submap $M_{|S}$ and cross the edges not in $M_{|S}$. For instance, the submap represented in Figure 3 has motion function $\theta = (a, c, e, n, d, k, m, h, i)(b, j, l)(f, g)$.

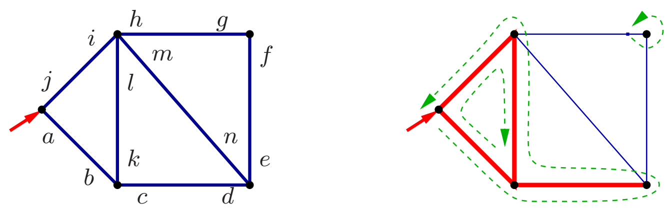

Figure 3: Motion function of the submap $M_{|S}$ defined by $S = \{a, b, c, d, i, j, k, l\}$.

Orientations. An *orientation* of a map $M = (H, \sigma, \alpha)$ is a partition $H = I \uplus O$ such that the involution α maps the set I of *ingoing* half-edges to the set O of *outgoing* half-edges. The pair $(M, (I, O))$ is an *oriented map*. A *directed path* is a sequence $h_1, h_2, \ldots, h_k$ of distinct ingoing half-edges such that h_i, $\alpha(h_{i+1})$ are incident to the same vertex (are in the same cycles of σ) for $i = 1 \ldots k-1$. A directed cycle is a directed path $h_1, \ldots, h_k$ such that h_k and $\alpha(h_0)$ are incident to the same vertex. The half-edge h_k is called the *extremity* of the directed path. An orientation is *root-connected* if for any ingoing half-edge h is the extremity of a directed path $h_1, \ldots, h_k = h$ such that $\alpha(h_1)$ is incident to the root-vertex of M.

Duality The *dual map* of a map $M = (H, \sigma, \alpha)$ is the map $M^* = (H, \phi, \alpha)$ where $\phi = \sigma\alpha$ is the face-permutation of M. The root of the dual map M^* is equal to the root of M. Observe that the genus of a map and of its dual are equal (by Euler relation) and that $M^{**} = M$. Topologically, the dual map M^* is obtained by the following two steps process: see Figure 4.

1. In each face f of M, draw a vertex v_f of M^*. For each edge e of M separating faces f and f' (which can be equal), draw the *dual edge* e^* of M^* going from v_f to $v_{f'}$ across e.

2. Flip the drawing of M^*, that is, inverse the orientation of the surface.

We now define duals of spanned maps and oriented maps. Given a subset $S \subseteq H$, we denote $\bar{S} = H \setminus S$. The *dual of a spanned map* (M, S) is the spanned map $(M^*, \bar{S})$; see Figure 4. We also say that $M_{|S}$ and $M^*_{|\bar{S}}$ are *dual submaps*. Observe that the motion functions of a spanned map (M, S) and of its dual

$(M^*, \bar{S})$ are equal. The *dual of the oriented map* $(M, (I, O))$ is $(M^*, (I, O))$. Graphically, this orientation is obtained by applying the following rule at step 1: the dual-edge e^* of an edge $e \in M$ is oriented from the left of e to the right of e; see Figure 14. Observe that duality is involutive on maps, spanned maps and oriented maps.

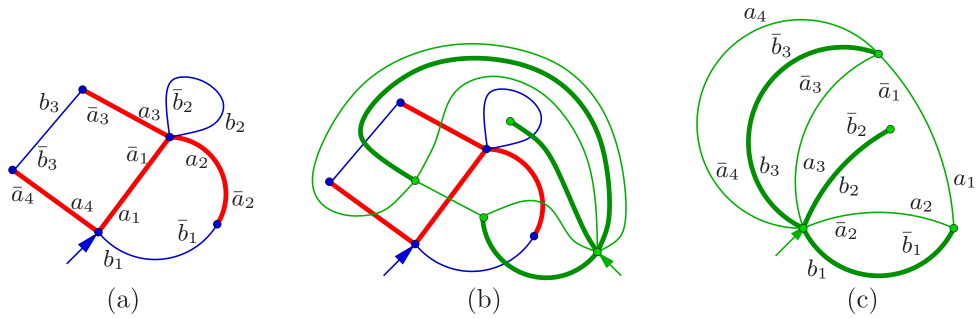

Figure 4: (a) A spanned map (the submap is indicated by thick lines). (b) Topological construction of the dual. (c) The dual covered map.

3 Covered maps as shuffles of unicellular maps.

In this Section, we establish some preliminary results about covered maps. In particular we prove that covered maps are stable by duality and explicit their decomposition as shuffles of two unicellular maps. Our first result should come as no surprise: it simply states that a spanned map (M, S) is a covered map if and only if turning around the submap $M_{|S}$ (that is following the border of its edges) starting from the root allows one to visit every half-edge of M.

Proposition 3.1. *A spanned map (M, S) is a covered map if and only if its motion function is a cyclic permutation.*

The following lemma relate the cycles of the motion function to the faces of the submap; see Figure 3 for the topological intuition.

Lemma 3.2. *Let (M, S) be a spanned map, and let σ, α and $\phi = \sigma\alpha$ be the vertex-, edge-, and face-permutations of M. The motion function θ satisfies $\theta_{|S} = \sigma_{|S}\alpha_{|S}$ and $\theta_{|\bar{S}} = \phi_{|\bar{S}}\alpha_{|\bar{S}}$. That is, the restriction $\theta_{|S}$ is the face permutation of the pseudo map $M_{|S}$, while the restriction $\theta_{|\bar{S}}$ is the face-permutation of the dual pseudo map $M^*_{|\bar{S}}$.*

Proof of Lemma 3.2. We first prove that $\theta_{|S} = \sigma_{|S}\alpha_{|S}$. Let h be in S and let $l = \theta_{|S}(h)$. By definition of restrictions, there exists a sequence $h_0 = h, h_1, h_2 \dots, h_{k+1} = l$ such that $h_1, h_2, \dots, h_k \in \bar{S}$ and $h_{i+1} = \theta(h_i)$ $i = 0 \dots k$. By definition of θ, one gets $h_1 = \sigma(\alpha(h))$ and $h_{i+1} = \sigma(h_i)$ for $i = 1 \dots k$.

Moreover, the half-edge $\alpha(h)$ is in S since h is in S which is stable by α. Thus, by definition of restriction $\alpha_{|S}(h) = \alpha(h)$ and $\sigma_{|S}(\alpha_{|S}(h)) = l$. Thus $\theta_{|S}(h) = l = \sigma_{|S}(\alpha_{|S}(h))$, that is, the permutations $\theta_{|S}$ and $\sigma_{|S}\alpha_{|S}$ coincide on h. The relation $\theta_{|\bar{S}} = \phi_{|\bar{S}}\alpha_{|\bar{S}}$ follows from the preceding point by duality (since the motion function of a spanned map and its dual are equal). □

Proof of Proposition 3.1. Suppose first that (M, S) is a covered map. Since $M_{|S}$ is connecting, each cycle of the motion function θ contains an element of S. Hence, the number of cycles of θ and $\theta_{|S}$ is the same. Moreover, by Lemma 3.2, $\theta_{|S} = \sigma_{|S}\alpha_{|S}$ is the face-permutation of $M_{|S}$. Since $M_{|S}$ is unicellular, $\theta_{|S} = \sigma_{|S}\alpha_{|S}$ is cyclic and θ is also cyclic.

Conversely, suppose that the motion function θ is cyclic. In this case, the pseudo map $M_{|S}$ has a face-permutation which is cyclic by Lemma 3.2. Hence it is a unicellular map. □

Proposition 3.1 immediately gives the following corollary concerning duality.

Corollary 3.3. *If a spanned map (M, S) is a covered map, then the dual spanned map $(M^*, \bar{S})$ is also a covered map. Moreover the genus of M is the sum of the genera of the unicellular maps $M_{|S}$ and $M^*_{|\bar{S}}$:*

$$g(M) = g(M_{|S}) + g(M^*_{|\bar{S}}).$$

Corollary 3.3 is illustrated by Figure 4.

Proof. The fact that $(M^*, \bar{S})$ is a covered map is an immediate consequence of Proposition 3.1 since the motion function of a submap and of its dual are always equal. The fact the genus add up is obtained by writing the Euler relation for the maps M, $M_{|S}$ and $M^*_{|\bar{S}}$. □

Let (M, S) be a covered map. By Lemma 3.2, the restrictions $\theta_{|S}$ and $\theta_{|\bar{S}}$ of the motion function θ correspond respectively to the face-permutations of the unicellular maps $M_{|S}$ and $M^*_{|\bar{S}}$. This inclines to say, somewhat vaguely, that *the covered map (M, S) is a shuffle of the unicellular maps $M_{|S}$ and $M^*_{|\bar{S}}$*. Making this statement precise requires introducing *codes* of unicellular maps and covered maps.

A *unicellular code* on the alphabet $A_n = \{a_1, \bar{a}_1, \ldots, a_n, \bar{a}_n\}$ is a word on A_n such that every letter of A_n appears exactly once, and for all $1 \leq i < j \leq n$, the letter a_i appears before $\bar{a}_i$ and before a_j. Let $T = (H, \sigma, \alpha)$ be a unicellular map with n edges. By definition, the face-permutation $\phi = \sigma\alpha$ is cyclic. Hence, there exists a unique way of relabelling the half-edges on the set A_n in such a way that $\alpha(a_i) = \bar{a}_i$ for all $i = 1 \ldots n$ and $\phi = (w_1, w_2, \ldots, w_{2n})$, where w_1 is the root and $w = w_1 w_2 \cdots w_{2n}$ is a unicellular code. We call w the *code* of the unicellular map T.

Topologically, the code of a unicellular map is obtained by turning around the face of the map in counterclockwise direction starting from the root and writing a_i when we discover the ith edge and writing $\bar{a}_i$ when we see this edge for the second time. For instance the code of the unicellular map in Figure 2 is $w = a_1a_2a_3a_4\bar{a}_1\bar{a}_2\bar{a}_4a_5\bar{a}_5\bar{a}_3$. We also mention that the unicellular map T is a plane tree if and only if its code w does not contain a subword of the form $a_ia_j\bar{a}_i\bar{a}_j$. In this special case, replacing all the letters $a_i, i = 1 \ldots n$ of the code w by the letter a and all the letters $\bar{a}_i, i = 1 \ldots n$ by the letter $\bar{a}$ results in no loss of information. One thereby obtains the classical bijection between plane trees and parenthesis systems on $\{a, \bar{a}\}$.

Lemma 3.4 (Folklore)**.** *The mapping which associates its code to a unicellular map is a bijection between unicellular map with n edges and unicellular code on the alphabet A_n.*

Proof. The mapping is injective since the root and the edge-permutation α and vertex-permutation $\sigma = \phi\alpha$ can be recovered from the code. It is also surjective since starting from any code one obtains a pair of permutation α, σ which indeed gives a unicellular map $T = (A_n, \alpha, \sigma)$ (the only non-obvious property is the transitivity condition, but this is granted by the fact the face-permutation $\phi = \sigma\alpha$ is cyclic). □

A word on $A_k \uplus B_l$ (where $B_l = \{b_1, \bar{b}_1, \ldots, b_l, \bar{b}_l\}$) is a *code-shuffle* if the subwords $w_{|A}$ and $w_{|B}$ made of the letters in A_k and B_l respectively are unicellular codes on A_k and B_l. Let (M, S) be a covered map, where $M = (H, \sigma, \alpha)$ and let $k = |S|/2,\ l = |\bar{S}|/2$. By Lemma 3.1, the motion function θ is cyclic. Hence, there exists a unique way of relabelling the half-edges on the set $A_k \uplus B_l$ in such a way that $S = A_k$, $\bar{S} = B_l$, $\alpha(a_i) = \bar{a}_i$ for all $i = 1 \ldots k$, $\alpha(b_i) = \bar{b}_i$ for all $i = 1 \ldots l$, and $\phi = (w_1, w_2, \ldots, w_{2n})$, where w_1 is the root of M and $w = w_1w_2 \cdots w_{2n}$ is a code-shuffle. We call w the *code* of the covered map (M, S).

Topologically, the code of a covered map (M, S) is obtained by turning around the submap $T = M_{|S}$ in counterclockwise direction starting from the root and writing a_i (resp. b_i) when we discover the ith edge in S (resp. $\bar{S}$) and writing $\bar{a}_i$ (resp. $\bar{b}_i$) when we see this edge for the second time. For instance, the code of the unicellular map in Figure 7(a) is $w = a_1b_1a_2b_2\bar{a}_2b_3\bar{a}_1\bar{b}_1a_3b_4a_4a_5\bar{b}_3\bar{a}_5\bar{b}_2\bar{a}_4\bar{b}_4\bar{a}_3$. We now state the main result of this preliminary section.

Proposition 3.5. *The mapping ϕ which associates its code to a covered map is a bijection between covered maps with n edges and code-shuffles of length $2n$. Moreover, if w is the code of the covered map (M, S), then $w_{|A}$ is the code of the unicellular map $M_{|S}$ (on the alphabet $A_{|S|/2}$) and $w_{|B}$ is the code of the dual unicellular map $M^*_{|\bar{S}}$ (on the alphabet $B_{|\bar{S}|/2}$).*

Proof. To see that ϕ is injective, observe first that the code-shuffle allows to recover the root of the map $M = (H, \sigma, \alpha)$, the subset $S = A_k$, the edge-permutation α and the motion function $\theta = (w_1, \ldots, w_{2n})$. From this, the vertex-permutation σ is deduced by $\sigma(h) = \theta\alpha(h)$ if $h \in S$ and $\sigma(h) = \theta(h)$ otherwise. We now prove that ϕ is surjective. For this, it is sufficient to prove that starting from any shuffle-code, the pair (M, S) defined as above is a covered map. First note that the permutations σ and α clearly act transitively on H since θ is cyclic, hence M is a map. Now, the fact that (M, S) is a a covered map is a consequence of Lemma 3.1 since θ is the motion function of (M, S) and is cyclic.

We now prove the second statement. Let $w_A = w'_1, \ldots, w'_{2k}$ and $w_B = w''_1, \ldots, w''_{2l}$. By definition of restrictions, $\theta_{|S} = (w'_1, \ldots, w'_{2k})$ and $\theta_{|\bar{S}} = (w''_1, \ldots, w''_{2l})$. Moreover, by Lemma 3.2, these restrictions $\theta_{|S}$ and $\theta_{|\bar{S}}$ correspond respectively to the face-permutations of $M_{|S}$ and $M^*_{|\bar{S}}$. Recall also that the root r_1 of $M_{|S}$ is $\sigma^i(r)$, where r is the root of M and i is the least integer such that $\sigma^i(r) \in S$. Equivalently, $r_1 = \theta^i(r)$ where i is the least integer such that $\theta^i(r) \in S$, hence $r_1 = w'_1$. Similarly, the root r_2 of $M^*_{|\bar{S}}$ is $\phi^j(r)$ where j is the least integer such that $\phi^j(r) \in \bar{S}$, or equivalently $r_2 = \theta^j(r)$ where j is the least integers such that $\theta^j(r) \in \bar{S}$, hence $r_1 = w''_1$. Thus, the words $w_{|A}$ and $w_{|B}$ are the codes of the unicellular maps $M_{|S}$ and $M^*_{|\bar{S}}$ respectively. □

We now explore the enumerative consequence of Proposition 3.5. Let $A_g(n)$ be the number of unicellular maps of genus g with n edges. Let $C_{g_1,g_2}(n_1, n_2)$ (resp. $C_{g_1,g_2}(n)$) be the number of covered maps (M, S) such that the unicellular maps $M_{|S}$ and $M^*_{|\bar{S}}$ have respectively n_1 and n_2 edges (resp. a total of n edges) and genus g_1 and g_2. Since there are $\binom{2n_1+2n_2}{2n_1}$ ways of shuffling unicellular codes of length $2n_1$ and $2n_2$, Proposition 3.5 gives

$$C_{g_1,g_2}(n_1, n_2) = \binom{2n_1 + 2n_2}{2n_1} A_{g_1}(n_1)A_{g_2}(n_2), \tag{5}$$

and

$$C_{g_1,g_2}(n) = \sum_{m=0}^{n} \binom{2n}{2m} A_{g_1}(m)A_{g_2}(n-m). \tag{6}$$

An alternative equation (used in Section 5) is obtained by fixing the number of vertices of $M_{|S}$ and $M^*_{|\bar{S}}$ instead of their genus. Let $A^v(g)$ be the number of unicellular maps with v vertices and n edges ($A^v(g) = A_{(n-v+1)/2}(n)$ by Euler relation and this number is 0 if $n - v + 1$ is odd). Let also $C^{v,f}(n)$ be the number of covered maps with v vertices, f faces and n edges (and genus $g = (n - v - f + 2)/2$). Proposition 3.5 gives

$$C^{v,f}(n) = \sum_{m=0}^{n} \binom{2n}{2m} A^v(m)A^f(n-m). \tag{7}$$

Equation (6) generalizes the results used by Mullin [29] and by Lehman and Walsh [24] in order to count tree-rooted maps. Indeed, the number of tree-rooted maps of genus g with n edges is

$$T_g(n) = C_{0,g}(n) = \sum_{n=0}^{m} \binom{2n}{2m} \mathrm{Cat}(m) A_g(n-m), \tag{8}$$

where $\mathrm{Cat}(m) = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number. In [29], Mullin proved Equation (1) by applying the Chu-Vandermonde identity to (8) (in the case $g = 0$). Similarly, in [24], Lehman and Walsh proved Equation (2) by applying the Chu-Vandermonde identity to (8) (in the case $g = 1$). In [2], Bender *et al.* used the asymptotic formula

$$A_g(n) \sim_{n\to\infty} \frac{n^{3g-\frac{3}{2}}}{12^g g! \sqrt{\pi}} 4^n \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

which they derived from the expressions given in [23], together with (8) in order to determine the asymptotic number of tree-rooted maps of genus g and obtained:

$$T_g(n) \sim_{n\to\infty} \frac{4}{\pi g! 96^g} n^{3g-3} 16^n.$$

Applying the same techniques as Bender *et al.* to Equation (6) gives the asymptotic number of covered map:

$$C_{g_1,g_2}(n) \sim_{n\to\infty} \binom{g_1+g_2}{g_1} \frac{4}{\pi g! 96^g} n^{3g-3} 16^n. \tag{9}$$

In particular, the the total number of covered maps of genus g with n edges satisfies:

$$C_g(n) = \sum_{h=0}^{g} C_{h,g-h}(n) \sim \frac{4}{\pi g! 48^g} n^{3g-3} 16^n. \tag{10}$$

Hence the proportion of tree-rooted maps among covered maps of genus g tends to $1/2^g$ when the size n goes to infinity. We have no simple combinatorial interpretation of this fact.

This concludes our preliminary exploration of covered maps. We now leave the world of shuffles and concentrate on the main subject of this paper, that is, the bijection Ψ between covered maps and pairs made of a tree and a unicellular bipartite map.

4 The bijection.

We now define the mapping Ψ which associates to a covered map (M, S) a pair (A, B) made of a tree $A = \Psi_1(M, S)$ and a bipartite unicellular map $B = \Psi_2(M, S)$. The mapping Ψ has two steps that we first describe in a non-formal way. The first step of the bijection associates an oriented map $(M, (I, O))$

to the covered map (M,S). For instance, Figure 7(b) represents the oriented map associated to the covered map of Figure 7(a). The second step of the bijection, which we call *unfolding*, can be seen as a way of splitting each vertex incident to $k>0$ ingoing half-edges into k vertices. The rule of this splitting process is represented in Figure 6. The map obtained after these splits is a plane tree A and the information about the splitting process is encoded into a bipartite unicellular map B called the *mobile*. The tree $A=\Psi_1(M,S)$ and the mobile $B=\Psi_2(M,S)$ are represented in Figure 9.

Step 1: Orientation Δ. The orientation step is represented in Figure 7. One starts with an covered map (M,S) and obtains an orientated map $(M,(I,O))$. Topologically, the orientation (I,O) is obtained by turning around the submap $M_{|S}$ (in counterclockwise direction starting from the root) and orient each edge of M according to the following rule:

- each edge in $M_{|S}$ is oriented in the direction it is followed for the first time during the tour,
- each edge not in $M_{|S}$ is oriented in such a way that the ingoing half-edge is crossed before the outgoing half-edge during the tour.

Let us now make definitions precise in terms of the combinatorial definition of maps. Let (M,S) be a covered map, let r be its root, and let θ be its motion function. Recall from Proposition 3.1 that θ is a cyclic permutation on the set H of half-edges. Therefore, one obtains a total order $\prec_S$, named *appearance order*, on the set H by setting $r\prec_S\theta(r)\prec_S\cdots\prec_S\theta^{|H|-1}(r)$. Topologically, the appearance order is the order in which half-edges of M appear when turning around the spanning submap $T=M_{|S}$ in counterclockwise order starting from the root. For instance, the order obtained for the spanning submap T in Figure 7(a) is $a_1\prec_S b_1\prec_S a_2\prec_S b_2\prec_S \bar{a}_2\prec_S b_3\prec_S\cdots\prec_S\bar{a}_3$. We now define the oriented map $(M,(I,O))=\Delta(M,S)$ which is represented in Figure 7(b).

Definition 4.1. Let (M,S) be a covered map with half-edge set H. The mapping Δ associates to (M,S) the oriented map $(M,(I,O))$, where the set I of ingoing half-edges contains the half-edges $h\in S$ such that $\alpha(h)\prec_S h$ and the half-edges $h\notin S$ such that $h\prec_S\alpha(h)$ (and $O=H\setminus I$).

We now characterize the image of the mapping Δ by defining left-connected orientations. Let $M=(H,\sigma,\alpha)$ be a map and let (I,O) be an orientation. Let h_0 denote the root of M. A *left-path* is a sequence $h_1,h_2,\ldots,h_k$ of ingoing half-edges such that for all $i=1\ldots k$, there exists an integer $q_i>0$ such that $h_{i-1}=\sigma^{q_i}(\alpha(h_i))$ and $\sigma^p(\alpha(h_i))\in O$ for all $p=0\ldots q_i-1$. In words, a left-path is a directed path starting from the arrow pointing the root-corner and such that no ingoing half-edges is incident to the left of the path. Clearly, for any ingoing half-edge h, there exists at most one left-path $h_1,h_2,\ldots,h_k$ whose extremity is h_k is h. We say that an oriented map $(M,(I,O))$ is *left-connected* if every ingoing half-edge is the extremity of a left-path.

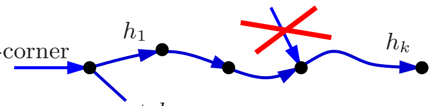

Figure 5: A left-path.

Theorem 4.2. *The mapping Δ is a bijection between covered maps and left-connected maps.*

The proof of Theorem 4.2 and of the following lemma are postponed to Section 6.

Lemma 4.3. *If $(M,(I,O))$ is a left-connected map with root r, then every non-root vertex of M is incident to a half-edge in I and every non-root face is incident to a half-edge in O.*

Remark on the planar case: It is shown in [3, Prop. 3] that the mapping Δ is a bijection between planar covered maps (i.e. tree-rooted maps) and (planar) oriented maps which are *root-connected* (there exists a directed path from the root-vertex to any other vertex) and *minimal* (no directed cycle is oriented in clockwise direction when the map is drawn in the plane with the root-face being the infinite-face). Thus, in the planar case the left-connected orientations are the minimal root-connected orientations. However, giving a direct proof of this fact would lead us too far from our main subject.

Step 2: Unfolding Λ. The unfolding step is represented in Figures 8 and 9. One starts with a left-connected map $(M,(I,O))$ and obtains two maps $A = \Lambda_1(M,(I,O))$ and $B = \Lambda_2(M,(I,O))$. The map A is a plane tree and the map B is a bipartite unicellular map (with *black* and *white* vertices) called *mobile*. Let us start with the topological description of this step. Let v be a vertex of the oriented map $(M,(I,O))$ and let $h_1,\ldots,h_d$ be the incident half-edges in counterclockwise order around v (here it is convenient to think of the arrow pointing the root-corner as an ingoing half-edge). If the vertex v is incident to $k>0$ ingoing half-edges, say $h_{i_1},h_{i_2},\ldots,h_{i_k}=h_d$, then the vertex v of M will be split into k vertices $v_1,v_2,\ldots,v_k$ of the tree A. The splitting rule is represented in Figure 6: for $j=1\ldots k$, the vertex v_j of the trees A is incident to the half-edges $h_{i_{j-1}+1},h_{i_{j-1}+2},\ldots,h_{i_j}$.

Observe that the splitting of the vertex v can be written conveniently in terms of permutations. Indeed, seeing the vertex v as the cycle $(h_1,\ldots,h_d)$ of the vertex-permutation σ and the vertices $v_1=(h_1,\ldots h_{i_1})$, $\ldots$, $v_k=(h_{i_{k-1}+1},\ldots,h_{i_k})$ as cycles of the vertex-permutation τ of the tree A gives the following relation between v and the product of cycles $v'=v_1v_2\ldots,v_k$ (these are both permutations on $\{h_1,\ldots,h_k\}$)

$$v = v'\pi_\circ,$$

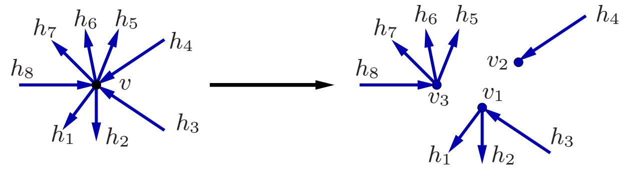

Figure 6: Splitting of a vertex v incident to 3 ingoing half-edges h_4, h_5, h_8.

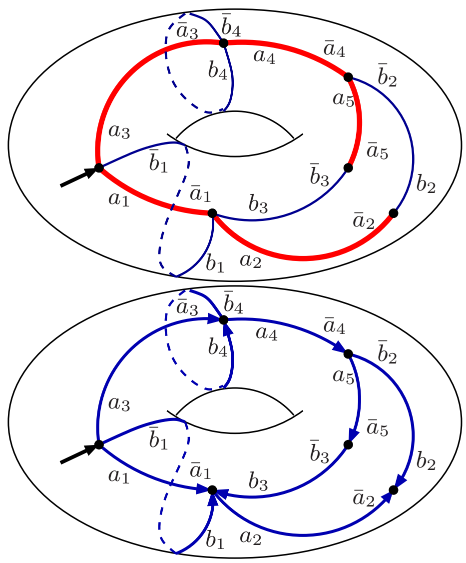

Figure 7: (a) A covered map of genus 1 (the unicellular submap is indicated by thick lines) (b) The associated oriented map.

where $\pi_\circ$ is the permutation such that $\pi_\circ(h) = h$ if $h \in O$ and $\pi_\circ(h_{i_j}) = h_{i_{j+1}}$ for $j = 1, \ldots, l$. Hence, $v' = v\pi_\circ^{-1}$, where $\pi_\circ = v_{|I}$ (with the convention that the restriction $v_{|I}$ acts as the identity on O). The cycle $(h_{i_1}, h_{i_2}, \ldots, h_{i_l})$ of $\pi_\circ$ will represent one of the (white) vertices of the bipartite unicellular map B. This white vertex is represented in Figure 8(a).

We now describe the unfolding step in more details. Let r be the root of the map $M = (H, \sigma, \alpha)$ and let $\phi = \sigma\alpha$ be its face-permutation . We consider two new *half-edges* i and o not in H and define $H' = H \cup \{i, o\}$, $I' = I \cup \{i\}$ and $O' = O \cup \{o\}$ (the half-edge i should be thought as this half-edge pointing to the root-corner, while o should be thought as its dual). We define the involution α' on H' by setting $\alpha'(i) = o$ and $\alpha'(h) = \alpha(h)$ for all $h \in H$. We also define σ' as the permutation on H' obtained from σ by inserting the new half-edge i just before the root r in the cycle of σ containing r and creating a cycle made

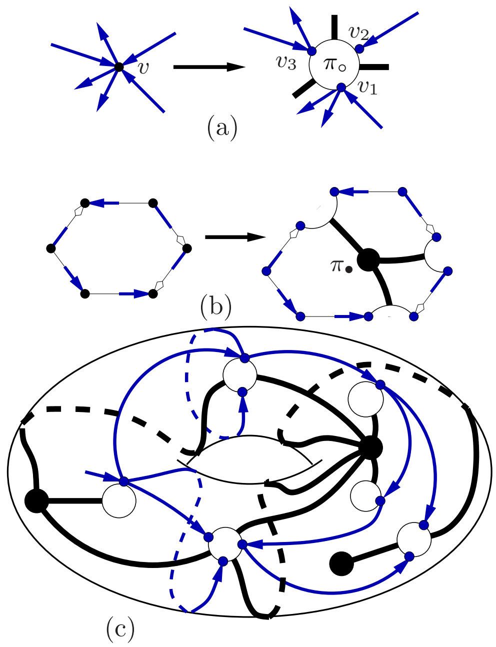

Figure 8: Representation of the unfolding: (a) around one vertex; (b) around one face; (c) on the map of Figure 7.

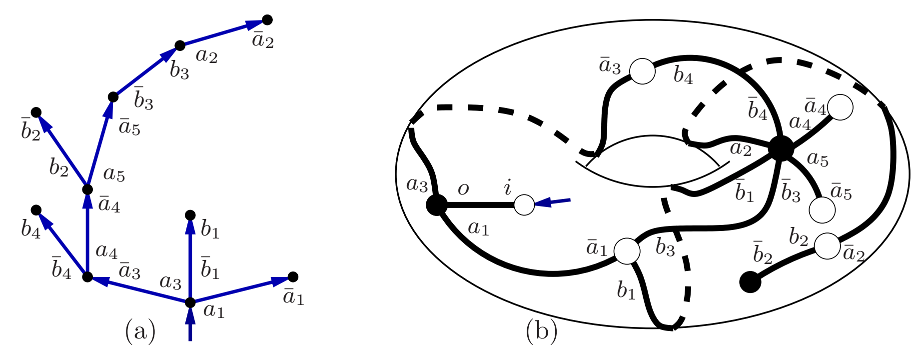

Figure 9: (a) The tree $\Psi_1(M, S)$. (b) The unicellular map $\Psi_2(M, S)$.

of o alone (that is, $\sigma'(o) = o$). Similarly we define ϕ' as the permutation on H' obtained from ϕ by inserting the new half-edge o just before r in the cycle of ϕ containing r and creating a cycle made of i alone. Recall that $\phi = \sigma\alpha$ and observe that $\phi' = (i, o)\sigma'\alpha'$. We consider the restrictions

$$\pi_\circ = \sigma'_{|I'} \quad \text{and} \quad \pi_\bullet = \phi'_{|O'} \ . \tag{11}$$

In the example of Figure 7, one gets $\pi_\circ = (i)(\bar{a}_1, b_1, b_3)(\bar{a}_2, b_2)(\bar{a}_3, b_4)(b_3)(\bar{a}_4)$ and $\pi_\bullet = (o, a_1, a_3)(\bar{b}_1, a_2, \bar{b}_4, a_4, a_5, \bar{b}_3)(\bar{b}_2)$. We now define the permutation π and τ' on H', and a permutation τ on H by setting

$$\pi = \pi_\circ \pi_\bullet^{-1}, \quad \tau' = \sigma' \pi_\circ^{-1} \quad \text{and} \quad \tau = \tau'_{|H} \ , \tag{12}$$

where a slight abuse of notation is done by considering that $\pi_\circ = \sigma'_{|I'}$ acts as the identity on O' and that $\pi_\bullet = \phi'_{|O'}$ acts as the identity on I'. It is easily seen that $\tau'(o) = o$. On the other hand, we will show (Lemma 6.9) that the half-edge i is not alone in its cycle of τ'. Hence, the half-edge $t = \tau'(i)$ is distinct from i and o. We now consider the pseudo maps $A = (H, \tau, \alpha)$ with root $t = \tau'(i)$ and $B = (H', \pi, \alpha')$ with root i.

Definition 4.4. We denote by Λ the mapping which to a left-connected map $(M, (I, O))$ associates the pair (A, B). We also denote $\Psi = \Lambda \circ \Delta$. Lastly if (M, S) denotes the covered map such that $(M, (I, O)) = \Delta(M, S)$, we denote $\Psi_1(M, S) = \Lambda_1(M, (I, O)) = A$ and $\Psi_2(M, S) = \Lambda_2(M, (I, O)) = B$.

The images (A, B) of the covered map in Figure 7(a) by the mappings Ψ_1 and Ψ_2 are represented respectively in Figure 9(a) and (b). In terms of permutations, one gets $A = (H, \tau, \alpha)$ and $B = (H', \pi, \alpha')$, where

$$\tau = (a_1, \bar{b}_1, \bar{a}_3)(\bar{a}_1)(b_1)(\bar{a}_3, a_4, \bar{b}_4)(a_4)(\bar{a}_4, a_5, b_2)(\bar{a}_5, \bar{b}_3)(b_3, a_2)(\bar{a}_2)(\bar{b}_2)(b_4)$$

and

$$\pi = (i)(\bar{a}_1, b_1, b_3)(\bar{a}_2, b_2)(\bar{a}_3, b_4)(b_3)(\bar{a}_4)(o, a_3, a_1)(\bar{b}3, a_5, a_4, \bar{b}_4, a_2, \bar{b}1)(\bar{b}_2).$$

Our main result is the following theorem which will be proved in Section 6.

Theorem 4.5. *The mapping $\Psi = \Lambda \circ \Delta$ which to a covered map (M, S) associates the pair $(\Psi_1(M, S), \Psi_2(M, S))$ is a bijection between covered maps of size n and genus g and pairs made of a plane tree $\Psi_1(M, S)$ of size n and a bipartite unicellular map $\Psi_2(M, S)$ of size $n+1$ and genus g. Moreover by coloring the vertices of the bipartite map $\Psi_2(M, S)$ in two colors, say white and black, with the root-vertex being white, one gets $v(M)$ white vertices and $f(M)$ black vertices.*

Remark (topological intuition). From Figures 8(a) and (b), the reader should see that the *mobile* $B = \Lambda_2(M, (I, O))$ has white vertices (the cycles of $\pi_\circ$ made of half-edges in I') corresponding to the vertices of M and black vertices (the cycles of $\pi_\bullet^{-1}$ made of half-edges in O') corresponding to the faces of M. The topological intuition that the pseudo map $A = \Psi_1(M, S)$ is connected is that left-paths are preserved during the unfolding step. From that, counting vertices and edges show that A is a plane tree. The topological intuition that the mobile B is a unicellular map comes from the fact that A can reach every white corners of B (without crossing its edges). Indeed, this implies that the pseudo-map B has no contractible cycles. From this, a counting argument involving the Euler relation for pseudo-maps shows that B is a unicellular map of genus $g(M)$.

5 Enumerative corollaries and a shortcut between Harer-Zagier and Jackson formulas.

Recall the notations of Section 3: $A_g(n)$, $B_g(n)$, $C_g(n)$ are respectively the number of general unicellular maps, bipartite unicellular maps, and covered maps with n edges and genus g. Similarly $A^v(n)$, $B^{v,f}(n)$, $C^{v,f}(n)$ are respectively the number of general unicellular maps with v vertices, bipartite unicellular maps with v white and f black vertices, and covered maps with v vertices and f faces having n edges.

The first direct consequence of Theorem 4.5 is:

Theorem 5.1. *The numbers $C_g(n)$ and $C^{v,f}(n)$ of covered maps, and the numbers $B_g(n)$ and $B^{v,f}(n)$ of bipartite maps are related by:*

$$C_g(n) = \mathrm{Cat}(n)\, B_g(n+1) \qquad (13)$$

$$C^{v,f}(n) = \mathrm{Cat}(n)\, B^{v,f}(n+1) \qquad (14)$$

where $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Using known closed-form expressions for the numbers $B_g(n)$ (see [15]), we obtain the following expressions for the numbers of covered maps of small genus:

$$C_0(n) = \mathrm{Cat}(n)\mathrm{Cat}(n+1), \qquad C_1(n) = \mathrm{Cat}(n)\frac{(2n-1)!}{6(n-2)!(n-1)!}.$$

We now examine the special case of the torus (genus 1). By Lemma 3.3, a covered map on the torus is either a tree-rooted map (the submap has genus 0, that is, is a spanning tree) or the dual of a tree-rooted map. Since duality is involutive, *exactly half* of toroidal covered maps of given size are tree-rooted maps. This gives the first bijective proof to the following result:

Corollary 5.2 (Lehman and Walsh [24]). *The number of tree-rooted maps with n edges on the torus is:*

$$T_1(n) = \frac{1}{2}C_1(n) = \frac{(2n)!(2n-1)!}{12(n+1)!n!(n-1)!(n-2)!}.$$

Another enumerative byproduct of our bijection is a relation between the numbers of general and bipartite unicellular maps. Indeed, by comparing the expression of $C^{v,f}(n)$ obtained by the shuffle approach (Equation (7)) with the one of Theorem 5.1, we obtain the following:

Theorem 5.3. *The numbers of bipartite and monochromatic unicellular maps are related by the formula:*

$$B^{v,f}(n+1) = \sum_{n_1+n_2=n} \frac{n!(n+1)!}{(2n_1)!(2n_2)!} A^v(n_1)A^f(n_2) \qquad (15)$$

In terms of generating series, the Harer-Zagier formula (3) *implies the Jackson-Adrianov formula* (4).

Proof. The first statement is obtained by comparing Equations (7) and (14). We now show how to retrieve (4) from (3). One has:

$$
\begin{aligned}
&\sum_{p,q\geq 1} B^{p,q}(n+1) y^p z^q \\
\overset{Eq.~(15)}{=}\quad & \sum_{p,q\geq 1} \sum_{n_1+n_2=n} \frac{n!(n+1)!}{(2n_1)!(2n_2)!} A^p(n_1) A^q(n_2) y^p z^q \\
=\quad & \sum_{n_1+n_2=n} \frac{n!(n+1)!}{(2n_1)!(2n_2)!} \left(\sum_{p\geq 1} A^p(n_1) y^p\right) \left(\sum_{q\geq 1} A^q(n_2) z^q\right) \\
\overset{Eq.~(3)}{=}\quad & \sum_{n_1+n_2=n} \frac{n!(n+1)!}{2^n n_1! n_2!} \sum_{i,j\geq 1} 2^{i+j-2} \binom{n_1}{i-1} \binom{n_2}{j-1} \binom{y}{i} \binom{z}{j} \\
=\quad & \sum_{i,j\geq 1} \frac{2^{i+j-n-2} n!(n+1)!}{(i-1)!(j-1)!} \binom{y}{i} \binom{z}{j} \sum_{\substack{n_1+n_2=n \\ n_1\geq i-1,\ n_2\geq j-1}} \frac{1}{(n_1-i+1)!(n_2-j+1)!}
\end{aligned}
$$

where the second and fourth equalities just correspond to rearrangements of the summations. Moreover, the inner sum in the last equation is equal to $2^{n-i-j-2}/(n-i-j+2)!$ by Newton's binomial theorem. This gives Jackson-Adrianov formula. $\square$

Remark. Connoisseurs know that Harer-Zagier and Jackson's formulas can be interpreted in terms of unicellular maps with colored vertices (see e.g. [18, sec. 3.2.7]). For those readers, we point out that the equivalence between Harer-Zagier and Jackson's formulas can be seen in terms of colorings as well: Ψ is a bijection between shuffles of two unicellular maps with vertices colored using all colors respectively in $\{1,\ldots,i\}$ and $\{1,\ldots,j\}$ and pairs made of a plane tree and a unicellular map with black and white vertices colored using all colors respectively in $\{1,\ldots,i\}$ and $\{1,\ldots,j\}$.

6 Proofs and inverse bijection.

This section is devoted to the proof of Theorems 4.2 and 4.5 concerning respectively the orientation and unfolding steps of the bijection Ψ.

6.1 Proofs concerning the orientation step.

In this Subsection, we prove Theorem 4.2 about the orientation step Δ and and define the inverse mapping Γ. Given an oriented map $(M,(I,O))$ with vertex-permutation σ and face-permutation ϕ, the *backward function* β is defined by $\beta(h)=\sigma(h)$ if $h\in O$ and $\beta(h)=\phi(h)$ otherwise. We point out that the backward function is not a permutation, since $\beta(h)=\beta(\alpha(h))$ for any half-edge h.

Lemma 6.1. *Let $(M,(I,O))$ be an oriented map with root h_0 and let β be the backward function. The oriented map $(M,(I,O))$ is left-connected if and only if for any half-edge, there exists an integer $q>0$ such that $\beta^q(h)=h_0$.*

Proof. Suppose first that $(M,(I,O))$ is left-connected. Let h be a half-edge. If h is ingoing, then it is the extremity of a left path $h_1,\ldots,h_k=h$. By definition of left-paths, there exist positive integers $q_1,\ldots,q_k$ such that $h_{i-1}=\beta^{q_i}(h_i)$ for all $i=1\ldots k$. Hence, $\beta^q(h)=h_0$ for $q=q_1+\cdots+q_k$. Now, if h is outgoing, $\beta(h)=\beta(\alpha(h))$, hence there exists $q>0$ such that $h_0=\beta^q(\alpha(h))=\beta^q(h)$.

Suppose conversely that for any half-edge h, there exists an integer $q>0$ such that $\beta^q(h)=h_0$. In this case, for any ingoing half-edge h, the sequence $h_1,h_2,\ldots,h_k=h$ of ingoing half-edges appearing (in this order) in the sequence $\beta^{q-1}(h),\beta^{q-2}(h),\ldots,\beta(h),h$ is a left-path. Hence $(M,(I,O))$ is left-connected. □

Proposition 6.2. *The image of any covered map by the mapping Δ is left-connected.*

Proof. Let (M,S) be a covered map, where the map $M=(H,\sigma,\alpha)$ has root r, and let $(M,(I,O))$ be its image by Δ. Our strategy is to prove that for any half-edge h such that $\beta(h)\neq r$, one has $h\prec_S\beta(h)$. This will clearly prove that the sequence $\beta(h),\beta^2(h),\beta^3(h)\ldots$ must contain the root r, and by Lemma 6.1, that $(M,(I,O))$ left-connected.

We distinguish four cases, depending on the fact that h is in I or O, and in S or $\bar{S}$. In these four cases, we denote h' the half-edge $\theta^{-1}(\beta(h))$, where θ is the motion function of (M,S). Observe that $h'\prec_S\beta(h)$ since by hypothesis $\beta(h)\neq r$.
Case 1: h is in O and in $\bar{S}$. In this case, one has $\beta(h)=\sigma(h)=\theta(h)$, hence $h'=h$. Thus $h=h'\prec_S\beta(h)$.
Case 2: h is in O and in S. In this case, one has $\beta(h)=\sigma(h)=\theta(\alpha(h))$, hence $h'=\alpha(h)$. Moreover, by definition of Δ, one has $h\prec_S\alpha(h)$ thus $h\prec_S h'\prec_S\beta(h)$.
Case 3: h is in I and in $\bar{S}$. In this case, one has $\beta(h)=\sigma(\alpha(h))=\theta(\alpha(h))$, hence $h'=\alpha(h)$. Moreover, by definition of Δ, one has $h\prec_S\alpha(h)$, thus $h\prec_S h'\prec_S\beta(h)$.
Case 4: h is in I and in S. In this case, one has $\beta(h)=\sigma\alpha(h)=\theta(h)$, hence $h'=h$. Thus, $h=h'\prec_S\beta(h)$. □

We will now define a mapping Γ that we will prove to be the inverse of Δ. Let us first give the intuition behind the injectivity of Δ by considering a covered map (M,S) with motion function θ and its image $(M,(I,O))$ by Δ. Observe from the definition of Δ that the root r of M is in S if and only if it is in O. Thus, it is possible to know from the orientation (I,O) whether r belongs to S or not, and thereby deduce the next half-edge $h_1=\theta(r)$ around the submap $M_{|S}$. The same reasoning will allow to determine, from the orientation (I,O), whether the half-edge h_1 belongs to S and deduce the next half-edge $h_2=\theta(h_1)$ around $M_{|S}$ and so on… This should convince the reader that the mapping Δ

is injective and highlight the definition of Γ given below.

It is convenient to define Γ as a procedure which given an oriented map $(M,(I,O))$ with root r returns a subset S of half-edges. This procedure visit some half-edges of M starting from the root r, and decide at each step whether the current half-edge h belongs to the set S or not.

Definition 6.3. The mapping Γ associates to an oriented map $(M,(I,O))$ the spanned map obtained by the following procedure.
Initialization: Set $S=\emptyset$, $R=\emptyset$ and set the current half-edge h to be the root r.
Core:
- If $h \notin S \cup R$ do:
  If h is in O then add h and $\alpha(h)$ to S; otherwise add h and $\alpha(h)$ to R.
- Set the the current half-edge h to be $\sigma\alpha(h)$ if h in S and $\sigma(h)$ otherwise.

Repeat until the current half-edge h returns to be r.
End: Return the spanned map (M,S).

We first prove the termination of the procedure Γ.

Lemma 6.4. *For any oriented map $(M,(I,O))$, the procedure Γ terminates and returns a spanned map (M,S). Moreover, the list of all successive current half-edges visited by the procedure is the cycle containing the root r of the motion function θ associated to (M,S).*

Proof. We first prove that the procedure Γ terminates. Observe that, at any step of the procedure, the sets S and R are disjoint and stable by α. Moreover, these sets are both increasing, hence they are constant after a while, equal to some sets S_∞ and R_∞ which are disjoint. Let θ_∞ be the motion function of the submap $M_{|S_\infty}$. Then, at each core step of the procedure, the current half-edge h becomes the half-edge $\theta_\infty(h)$. Indeed, if at the current step h is in S, then h is in S_∞ (since the set S cannot decrease) so that $\theta_\infty(h)=\sigma\alpha(h)$, while if h is in R, then h is in R_∞ (since the set R cannot decrease), hence it is not in S_∞ so that $\theta_\infty(h)=\sigma(h)$.

Hence the sequence of all successive current half-edges form a cycle of the *permutation* θ_∞. Since the procedure starts with h equal to the root r, it follows that r is reached a second time, and that the procedure terminates. Finally, the spanned map returned by the algorithm is (M,S_∞), which concludes the proof. □

Proposition 6.5. *The image of a left-connected map by the mapping Γ is a covered map.*

Proof. Let $(M,(I,O))$ be a left-connected map. We denote by r the root of $M=(H,\sigma,\alpha)$, by β the backward function of $(M,(I,O))$, and by θ the motion function of $(M,S)=\Gamma((M,(I,O)))$. Let K be the set of half-edges contained in the cycle of the motion function θ containing the root r. In order to prove the proposition, it suffices to prove that $K=H$. Indeed, in this case the motion function θ is cyclic which implies that (M, S) is a covered map by Proposition 3.1.

Moreover, since $(M, (I, O))$ is left-connected, for any half-edge h in H, there exists a positive integer q such that $\beta^q(h)$ is the root r which belongs to K. Hence, it suffices to prove that *any half-edge h such that $\beta(h)$ is in K, is also in K.*

Let h be an half-edge such that $\beta(h)$ is in K and let $h' = \theta^{-1}(\beta(h))$. Observe that, by definition of K, the half-edge h' is in K. We now distinguish four cases, depending on the fact that h is in I or O, and in S or $\bar{S}$.

Case 1: h is in O and in $\bar{S}$. In this case, one has $\beta(h) = \sigma(h) = \theta(h)$, hence $h' = h$. Thus $h = h'$ is in K.

Case 2: h is in O and in S. In this case, by definition of Γ, the half-edge h was the current half-edge when it was added to S. Thus, by Lemma 6.4, the half-edge h is in K.

Case 3: h is in I and in $\bar{S}$. In this case, one has $\beta(h) = \sigma(\alpha(h)) = \theta(\alpha(h))$, hence $h' = \alpha(h)$. Since h' is in K, Lemma 6.4 ensures that it was the current half-edge at a certain step of the procedure Γ. Hence, since $h' = \alpha(h)$ is in O but not in S, it means that h and h' were added to the set R at a step of the procedure Γ, such that h was the current half-edge. Thus, by Lemma 6.4, the half-edge h is in K.

Case 4: h is in I and in S. In this case, one has $\beta(h) = \sigma\alpha(h) = \theta(h)$, hence $h' = h$. Thus $h = h'$ is in K. □

We now complete the proof of Theorem 4.2.

Proposition 6.6. *The mappings Δ and Γ are inverse bijections between covered maps and left-connected maps.*

Proof. We first prove that the mapping $\Delta \circ \Gamma$ is the identity on left-connected maps. Observe that this composition is well-defined by Proposition 6.5. Let $(M, (I, O))$ be a left-connected map and let (M, S) be its image by Γ. We want to prove that $(M, (\tilde{I}, \tilde{O})) \equiv \Delta(M, S)$ is equal to $(M, (I, O))$. For that, it suffices to show that any half-edge h such that $h \prec_S \alpha(h)$ is in O if and only if it is in $\tilde{O}$. Let h be such a half-edge. By definition of Δ, it follows that h is in $\tilde{O}$ if and only if h is in S. Now, by Lemma 6.4, the sequence $h_1 = r, h_2, \ldots, h_{2n}$ of current half-edge visited during the procedure Γ satisfies $h_1 \prec_S h_2 \prec_S \cdots \prec_S h_{2n}$, hence h is visited before $\alpha(h)$ during the procedure Γ. Hence, by definition of Γ, the half-edge h is in O if and only if h is in S.

We now prove that the mapping $\Gamma \circ \Delta$ is the identity on covered maps. Observe that this composition is well-defined and returns a covered map by Propositions 6.2 and 6.5. Let (M, S) be a covered map with root r and let $(M, (I, O))$ be its image by Δ. Let also $(M, S') = \Gamma(M, (I, O))$ and let θ and θ' be respectively the motion function of (M, S) and (M, S'). In order to prove that $S = S'$ it suffices to prove that $\theta = \theta'$ (indeed, the set S and S' are completely determined by θ and θ'). Suppose now that $\theta \neq \theta'$ and consider the smallest integer $k \geq 0$ such that $\theta^{k+1}(r) \neq {\theta'}^{k+1}(r)$ (such an integer exists since θ and θ' are cyclic). Observe that for all $0 \leq j < k$, the half-edge $\theta^j(r) = {\theta'}^j(r)$ is in S if and only if it is in S' (since $\theta^{j+1}(r) = {\theta'}^{j+1}(r)$). On the other hand, the half-edge $h = \theta^k(r) = {\theta'}^k(r)$ is in the symmetric difference of S and S' (since $\theta^{k+1}(r) \neq {\theta'}^{k+1}(r)$). This implies that $h \prec_S \alpha(h)$ and $h \prec_{S'} \alpha(h)$. Since $h \prec_S \alpha(h)$, the definition of Δ shows that the half-edge h is in S if and only if h is in O. Since $h \prec_{S'} \alpha(h)$, Lemma 6.4 proves that h is the current half-edge before $\alpha(h)$ in the procedure Γ on $(M, (I, O))$. Hence, by definition of Γ, the half-edge h is in S' if and only if h is in O. This proves that $h = \theta^k(r)$ is not in the symmetric difference of S and S', a contradiction. □

Before leaving the world of left-connected maps, we prove Lemma 4.3.

Proof of Lemma 4.3. If a cycle of the vertex-permutation σ contains no edge in I, then $\beta(h) = \sigma(h)$ for every half-edge h in this cycle. By Lemma 6.1, this implies that the root r belongs to this cycle. Similarly, if a cycle of the face-permutation ϕ contains no edge in O, then $\beta(h) = \phi(h)$ for every half-edge h in this cycle. By Lemma 6.1, this implies that the root r belongs to this cycle. □

6.2 Proofs concerning the unfolding step.

In this Subsection we prove Theorem 4.5. We fix a left-connected map $(M, (I, O))$, where the map $M = (H, \sigma, \alpha)$ has n edges, genus g, root r and face-permutation ϕ. We denote $A = (H, \tau, \alpha) = \Lambda_1(M, (I, O))$ and $B = (H', \pi, \alpha') = \Lambda_2(M, (I, O))$ and adopt the notation of Section 4 for the sets H', I', O' and the permutations σ' ϕ', τ', $\pi_\circ$ and $\pi_\bullet$.

Lemma 6.7. *The permutations τ, α act transitively on H, thus A is a map.*

As mentioned above, the intuition behind the connectivity of A is that left-paths are preserved by the unfolding. This can be formalized as follows.

Proof. Let β be the backward function of the oriented map $(M, (I, O))$ defined by $\beta(h) = \sigma(h)$ if h is in O and $\beta(h) = \sigma\alpha(h)$ otherwise. Let $\tilde{\beta}$ be the backward function of the oriented map $(A, (I, O))$ defined by $\tilde{\beta}(h) = \tau(h)$ if h is in O and $\tilde{\beta}(h) = \tau\alpha(h)$ otherwise.

We first prove that $\beta(h) = \tilde{\beta}(h)$ for any half-edge $h \in H$ such that $\beta(h) \neq r$. If h is in O (and $\beta(h) \neq r$), then

$$\beta(h) \equiv \sigma(h) = \sigma'(h) = \tau'(h) = \tau(h) \equiv \tilde{\beta}(h),$$

since the permutations σ' and τ' coincide on O'. If h is in I (and $\beta(h) \neq r$), then

$$\beta(h) \equiv \sigma(\alpha(h)) = \sigma'(\alpha(h)) = \tau'(\alpha(h)) = \tau(\alpha(h)) \equiv \tilde{\beta}(h),$$

since again the permutations σ' and τ' coincide on O'.

Since the oriented map $(M, (I, O))$ is left-connected, Lemma 6.1 ensures that for any half-edge h, there exists an integer $q > 0$ such that $\beta^q(h) = r$. Taking the least such integer q, and using the preceding point shows that for any half-edge h, there exists an integer $q > 0$ such that $\tilde{\beta}^{q-1}(h) = \beta^{q-1}(h) = l$, where

l is a half-edge such that $\beta(l) = r$. Moreover, the relation $\beta(l) = r$ shows that l is either equal to $u = \sigma^{-1}(h)$ or to $\alpha(u)$. Therefore, any half-edge h can be sent to one of the half-edges $u, \alpha(u)$ by applying the function $\tilde{\beta}$, hence by acting with the permutations τ and α. This proves the lemma. □

We call *root-to-leaves* orientation of a plane tree the unique orientation such that each non-root vertex is incident to exactly one ingoing half-edge.

Proposition 6.8. *The map A is a tree. Moreover, (I, O) is the root-to-leaves orientation of A.*

We start with an easy lemma.

Lemma 6.9. *The half-edge $u = \sigma^{-1}(r)$ is in O and $\tau'(u) = i$. In particular, the half-edge i is not alone in its cycle of τ'.*

Proof. Consider the backward function β of the oriented map $(M, (I, O))$. Since, $(M, (I, O))$ is left-connected, Lemma 6.1 ensures that there exists a half-edge h such that $\beta(h) = r$. Since $\beta(h) = \beta(\alpha(h))$ we can take h in O and get $h = \sigma^{-1}(r) = u$. This proves that $u = \sigma^{-1}(r)$ is in O. It is then obvious from the definition of τ' that $\tau'(u) = i$. □

Proof of Proposition 6.8. Recall that $n = |H|/2$ denotes the number of edges of M, hence of A. We first prove that the map A has (at least) $n+1$ vertices. By construction, the permutation $\tau' = \sigma'\pi_\circ^{-1} = \sigma'\sigma'^{-1}_{|I}$ has at most one element of I' in each of its cycle. Hence it has at least $n+1$ cycles beside the cycle made of o alone. Moreover, by Lemma 6.9, the half-edge i is not alone in its cycle of τ', thus the vertex-permutation $\tau = \tau'_{|H}$ has at least $n+1$ cycles.

The (connected) map A has n edges and (at least) $n+1$ vertices hence it is a tree. Moreover, each non-root vertex of A is incident to exactly one half-edge in I. Thus, (I, O) is the root-to-leaves orientation of A. □

We prove a last easy lemma about the tree $A = (H, \tau, \alpha)$.

Lemma 6.10. *The permutation $\tau' = \sigma'\pi_\circ^{-1}$ is obtained from the vertex-permutation τ by inserting the half-edge i before the root t of A in the cycle of τ containing t and creating a cycle made of o alone. The permutation $\varphi' = (i, o)\tau'\alpha'$ is obtained from the face-permutation $\varphi = \tau\alpha$ by inserting the half-edge o before t in the cycle of φ containing t and creating a cycle made of i alone.*

Proof. By definition, $\tau'(i) = t$ and $\tau'_{|H} = \tau$. Moreover, it is easy to check that $\tau'(o) = \sigma'\pi_\circ^{-1}(o) = o$. This proves the statement about τ'. We now denote $u = \tau'^{-1}(i)$. By definition, $\varphi(\alpha(u)) = \tau(u) = t$ and one can check $\varphi'(\alpha(u)) = o$, $\varphi'(o) = t$, $\varphi'(i) = i$, and $\varphi'(h) = \varphi(h)$ for all $h \notin \{i, o, \alpha(u)\}$. This proves the statement about φ'. □

We will now prove that the mobile $B = (H', \pi, \alpha') = \Lambda_2(M, (I, O))$ is a unicellular bipartite map. We introduce some notations for the half-edges in H. From Proposition 6.8, the tree $(A, (I, O))$ is oriented from root to leaves.

In particular, the root t of A is outgoing. We denote $o_1 = t, o_2, \ldots, o_n$ the outgoing half-edges as appearing during a counterclockwise tour around the tree A, that is to say, $\varphi_{|O} = (o_1, o_2, \ldots, o_n)$ where $\varphi = \tau\alpha$ is the face-permutation of A. This labelling is indicated in Figure 10(a). Observe that Lemma 6.10 implies ${\varphi'}_{|O'} = (o, o_1, o_2, \ldots, o_n)$. For $j = 1, \ldots, n$, we denote $i_j = \alpha(o_j)$, so that $\alpha'{\varphi'}_{|O}\alpha' = (i, i_1, i_2, \ldots, i_n)$.

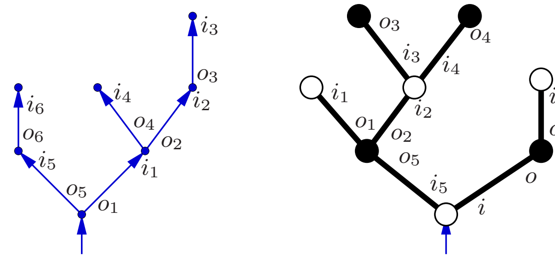

Figure 10: A pair (A, B) coherently labelled on the alphabet $\{i, i_1, \ldots, i_6, o, o_1, \ldots, o_6\}$: the face-permutation φ of the tree A satisfies $\varphi'_{|O'} = (o, o_1, \ldots, o_6)$, while the face-permutation ψ of the mobile B satisfies $\psi_{|I'} = (i, i_1, \ldots, i_6)$.

Proposition 6.11. *The mobile $B = \Lambda_2(M, (I, O))$ is a bipartite unicellular map of genus $g(M)$. Moreover, if we color the vertices of B in two colors, say white and black, with the root-vertex being white, then it has $v(M)$ white vertices and $f(M)$ black vertices. Lastly, the half-edges incident to white vertices are $i, i_1, i_2, \ldots, i_n$ and appear in this order during a clockwise tour around B, that is to say, $\psi^{-1}_{|I'} = (i, i_1, \ldots, i_n) = \alpha'{\varphi'}_{|O}\alpha'$, where $\psi = \pi\alpha'$ is the face-permutation of B and $\varphi' = (i, o)\tau'\alpha'$.*

Lemma 6.12. *The permutation $\psi = \pi\alpha'$ and $\varphi' = (i, o)\tau'\alpha'$ are related by $\psi^{-1}_{|I'} = (i, i_1, \ldots, i_n) = \alpha'{\varphi'}_{|O'}\alpha'$.*

Proof. Since the involution α' maps I' to O', one gets $\alpha'{\varphi'}_{|O'}\alpha' = (\alpha'\varphi'\alpha')_{|I'}$. Hence, $\alpha'{\varphi'}_{|O'}\alpha' = (\alpha'(i, o)\tau')_{|I'} \equiv (\alpha'(i, o)\sigma'{\sigma'}^{-1}_{|I'})_{|I'}$. Consider now a half-edge h in I'. By definition of restrictions, one gets

$$\alpha'{\varphi'}_{|O'}\alpha'(h) = (\alpha'(i, o)\sigma')_{|I'}{\sigma'}^{-1}_{|I'}(h),$$

since ${\sigma'}^{-1}_{|I'}$ acts as the identity on O'. We now determine $\psi^{-1}_{|I'}(h)$. Observe that the sets I' and O' are stable by the permutation $\pi = \pi_\circ\pi_\bullet^{-1}$. Thus the permutation $\psi^{-1} \equiv \alpha'\pi^{-1}$ maps I' to O'. This gives

$$\psi^{-1}_{|I'}(h) = \alpha'\pi^{-1}\alpha'\pi^{-1}(h) = \alpha'\pi_\bullet\alpha'\pi_\circ^{-1}(h) = \alpha'\phi'_{|O'}\alpha'\sigma^{-1}_{|I'}(h),$$

since $\pi_\bullet \equiv \phi'_{|O'} = \pi^{-1}_{|O'}$ and $\pi_\circ \equiv \sigma'_{|I'} = \pi_{|I'}$. Moreover, $\alpha'\phi'_{|O'}\alpha' = (\alpha'\phi'\alpha')_{|I'} = (\alpha'(i,o)\sigma')_{|I'}$ since $\phi' = (i,o)\sigma'\alpha'$ (and the sets I' and O' are exchanged by α'). Thus, for any half-edge h in I',

$$\psi^{-1}_{|I'}(h) = (\alpha'(i,o)\sigma')_{|I'}\sigma^{-1}_{|I'}(h)$$

which concludes the proof. □

Proof of Proposition 6.11. By Lemma 6.12, every half-edge in I' belongs to the same cycle C of the permutation $\psi = \pi\alpha'$. Now, if h is a half-edge in O', $\psi(h) = \pi\alpha(h)$ is in I' (since I' is stable by π) hence it belongs to the cycle C. Thus, the permutation ψ is cyclic. Hence, the permutations π and α act transitively on H', that is, B is a map. Moreover, its face-permutation ψ is cyclic, that is, B is unicellular.

Moreover, since the sets I' and O' are stable by the vertex-permutation π and exchanged by the edge-permutation α', the map B is bipartite. Let us therefore consider the bipartite coloring where the vertices incident to half-edges in I' are white while the vertices incident to the half-edges in O' are black. By Lemma 4.3, each of the cycles of σ' except the cycle made of o alone contains at least one half-edge in I'. Therefore, the number of cycles of the permutation $\pi_\circ = \sigma'_{|I'}$ on I' is the number $v(M)$ of cycles of the vertex-permutation σ. Similarly, the number of cycles of the permutation $\pi_\bullet = \phi'_{|O'}$ on O' is the number $f(M)$ of cycles of the face-permutation ϕ. Thus, the map B has $v(M)$ white vertices and $f(M)$ black vertices. Now, Euler relation gives

$$2g(B) = 2+e(B)-f(B)-v(B) = 2+(e(M)+1)-1-(v(M)+f(M)) = 2g(M).$$

Thus, the genus of B is $g(M)$. This concludes the proof of Proposition 6.11. □

Topological description of the folding step (Figure 11). We now define a mapping Ω, the *folding step*, which we will prove to be the inverse of the unfolding step Λ. Before defining Ω in terms of permutations, let us explain the topological interpretation of Proposition 6.11. We denote by $v_0, v_1, \ldots, v_n$ the vertices of A *in counterclockwise order around* A (starting from the root-corner) and by $c_0, c_1, \ldots, c_n$ the *first corners* of these vertices; see Figure 11(a). Equivalently, v_0 is the root-vertex and c_0 is the root-corner, while for $j = 1 \ldots n$, v_j is the vertex incident to the ingoing half-edge i_j and c_j is the corner following i_j in counterclockwise order around v_j; see Figure 10. We also denote by $x_0, \ldots, x_n$ the white corners of B *in clockwise order around* B (starting from the root-corner x_0); see Figure 11(a). By Proposition 6.11, for $j = 0 \ldots n$, x_j is the corner of B following the half-edge i_j in clockwise order around the incident vertex. Therefore this proposition indicates how the folding step $\Omega = \Lambda^{-1}$ should be defined topologically: for $j = 0, \ldots, n$ the first-corner c_j of the vertex v_j (the jth vertex of A in counterclockwise direction) is glued to the corner x_j (the jth white corner of B in clockwise direction). This gives a map containing edges of both A and B that we call *partially folded map* which is represented in Figure 11(b). The oriented map $(M, (I, O)) = \Omega(A, B)$ is obtained from the partially folded map by keeping the half-edges of A with their cyclic ordering around the white vertices of B (while the edges and black vertices of B are deleted).

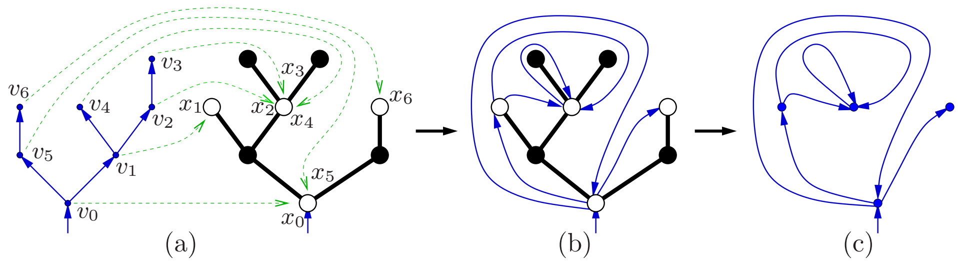

Figure 11: Topological representation of the folding step. Figure (b) represents the partially folded map.

We now defines the mapping Ω in terms of permutations. Let $\tilde{A} = (H, \tilde{\tau}, \alpha)$ be a rooted plane tree with $n = |H|/2$ edges, and $\tilde{B} = (H', \tilde{\pi}, \alpha')$ be a rooted bipartite unicellular map with $n+1 = |H'|/2$ edges. We consider the usual black-and-white coloring of $\tilde{B}$ (with the root-vertex being white). The pair $(\tilde{A}, \tilde{B})$ is said *coherently labelled* if the following conditions are satisfied:

(i) $H' = H \cup \{i, o\}$, where i is the root of M and $o = \alpha'(i)$.

(ii) $\alpha = \alpha'_H$.

(iii) The root-to-leaves orientation (I, O) of $\tilde{A}$ is such that the half-edges in $I' = I \cup \{i\}$ are incident to white vertices of $\tilde{B}$, while half-edges in $O' = O \cup \{o\}$ are incident to black vertices of B.

(iv) If the half-edges $o_1, o_2, \dots, o_n$ in O appear in this order in counterclockwise direction around $\tilde{A}$ with o_1 being the root of $\tilde{A}$, then the half-edges $i, i_1, \dots, i_n$ defined by $i_j = \alpha(o_j)$ for $j = 1 \dots n$ appear in this order in clockwise direction around $\tilde{B}$. Equivalently, $\psi_{|I'}^{-1} = (i, i_1, \dots, i_n) = \alpha' \varphi'_{|O} \alpha'$, where $\psi = \pi\alpha'$ is the face-permutation of B and $\varphi' = (i, o)\tau'\alpha'$.

For example, the pair (A, B) represented in Figure 10 is coherently labelled. Observe that Proposition 6.11 precisely states that the image by Λ of a left-connected map $(M, (I, O))$ is coherently labelled. We now prove (the somewhat obvious fact) that any pair $(\tilde{A}, \tilde{B})$ can be relabelled coherently.

Lemma 6.13. *Let $\tilde{A} = (H, \tilde{\tau}, \alpha)$ be a rooted plane tree with $n = |H|/2$ edges, and $\tilde{B} = (H', \tilde{\pi}, \alpha')$ be a rooted bipartite unicellular map with $n+1 = |H'|/2$ edges. Then, there is a unique way of relabelling $\tilde{B}$ in such a way that the pair $(\tilde{A}, \tilde{B})$ is coherently labelled.*

Proof. We denote by (I, O) the root-to-leaves labelling of A. We denote by I' and O' respectively the set of half-edges incident to white and black vertices of $\tilde{B}$ and observe that these sets are exchanged by α' (since $\tilde{B}$ is bipartite). We denote $\tilde{\varphi} = \tilde{\tau}\alpha$ and $\tilde{\psi} = \tilde{\pi}\alpha'$ are the face-permutation of $\tilde{A}$ and $\tilde{B}$ respectively. Lastly, we denote $(o_1, o_2, \ldots, o_n)$ the cycle $\tilde{\varphi}_{|O}$ with o_1 being the root of $\tilde{A}$ and $(i', i'_1, \ldots, i'_n)$ the cycle $\tilde{\psi}^{-1}_{|I'}$ with i' being the root of $\tilde{B}$. Now we consider the relabelling of $\tilde{B}$ given by the bijection λ from H' to $H \cup \{o, i\}$ (where i, o are half-edges not in H) given by $\lambda(i') = i$, $\lambda(\alpha(i')) = o$ and for all $j = 1 \ldots n$, $\lambda(i'_j) = \alpha(o_j)$ and $\lambda(\alpha(i'_j)) = o_j$. Clearly λ is a bijection and it is the unique bijection making the pair $(\tilde{A}, \tilde{B})$ coherently labelled. □

We denote by P_n the set of pairs $(\tilde{A}, \tilde{B})$ made of a tree of size n and a unicellular bipartite map of size $n+1$. We now consider such a pair $(\tilde{A}, \tilde{B})$, where $\tilde{A} = (H, \tilde{\tau}, \alpha)$ and $\tilde{B} = (H', \tilde{\pi}, \alpha')$. By Lemma 6.13, we can assume that the pair $(\tilde{A}, \tilde{B})$ is coherently labelled and we adopt the notations i, o, I, O, I', O' introduced in the conditions (i- iv) (in particular, (I, O) is the root-to-leaves orientation of A). We then define $\tilde{\tau}'$ as the permutation on H' obtained from τ by inserting i before the root t of $\tilde{A}$ in the cycle containing it and creating a cycle made of o alone. We also define the permutations $\tilde{\pi}_\circ$ and $\tilde{\sigma}'$ on H' and the permutation $\tilde{\sigma}$ on H by

$$\tilde{\pi}_\circ = \tilde{\pi}_{|I'}, \quad \tilde{\sigma}' = \tilde{\tau}'\tilde{\pi}_\circ \quad \text{and} \quad \tilde{\sigma} = \tilde{\sigma}'_{|H}\ , \tag{16}$$

(where a slight abuse of notation is done by considering $\tilde{\pi}_\circ$ as a permutation on H' acting as the identity on O'). With these notations, we define $\Omega(\tilde{A}, \tilde{B}) = (\tilde{M}, (I, O))$, where $\tilde{M} = (H, \tilde{\sigma}, \alpha)$.

We now complete the proof of Theorem 4.5 by proving the following proposition.

Proposition 6.14. *The mappings Λ and Ω are inverse bijections between left-connected maps of size n and pairs in P_n.*

Proof. • We first prove that the mapping $\Omega \circ \Lambda$ is the identity on left-connected maps.
Let $(M, (I, O))$ be a left-connected map, where $M = (H, \sigma, \alpha)$. Let $(A, B) = \Lambda(M, (I, O))$, where $A = (H, \tau, \alpha)$ and $B = (H', \pi, \alpha')$ (recall that (A, B) is coherently labelled by Proposition 6.11). Let also $(\tilde{M}, (\tilde{I}, \tilde{O})) = \Omega(A, B)$, where $\tilde{M} = (H, \tilde{\sigma}, \alpha)$. We want to prove that $(I, O) = (\tilde{I}, \tilde{O})$ and $M = \tilde{M}$ (or equivalently, $\sigma = \tilde{\sigma}$).

By definition of Ω, $(\tilde{I}, \tilde{O})$ is the root-to-leaves orientation of A. Moreover, by Proposition 6.8, (I, O) is also the root-to-leaves orientation of A. Hence, $(I, O) = (\tilde{I}, \tilde{O})$.

By definition, $\sigma = \sigma'_{|H}$, where $\sigma' = \tau'\pi_\circ = \tau'\pi_{|I'}$ (see (12)). Similarly, $\tilde{\sigma} = \tilde{\sigma}'_{|H}$ and $\tilde{\sigma}' = \tilde{\tau}'\tilde{\pi}_\circ = \tilde{\tau}'\pi_{|I'}$ (see (16)). Moreover, Lemma 6.10 ensures that $\tau' = \tilde{\tau}'$ (since the permutations τ' and τ'' are obtained from τ by the same procedure). Thus, $\sigma' = \tilde{\sigma}'$ and $\sigma = \tilde{\sigma}$.

• We now prove that the mapping $\Lambda \circ \Omega$ is the identity on P_n.
We must first prove that this mapping is well-defined, that is, the image of any pair $(\tilde{A}, \tilde{B}) \in P_n$ by Ω is a left-connected map. Let us denote $\tilde{A} = (H, \tilde{\tau}, \alpha)$ and $\tilde{B} = (H', \tilde{\pi}, \alpha')$. By Lemma 6.13, we can assume that the pair $(\tilde{A}, \tilde{B})$ is coherently labelled and we adopt the notations i, o, I, O, I', O' of conditions (i-iv) and the notations $\tilde{\pi}_\circ$, $\tilde{\sigma}'$, $\tilde{\sigma}$ introduced in (16). Lastly, we denote $\Omega(\tilde{A}, \tilde{B}) = (\tilde{M}, (I, O))$, where $\tilde{M} = (H, \tilde{\sigma}, \alpha)$.

Let $\tilde{\beta}$ be the backward function of the tree $\tilde{A}$ defined on H by $\tilde{\beta}(h) = \tilde{\tau}(h)$ if h is in O and $\tilde{\beta}(h) = \tilde{\tau}\alpha(h)$ otherwise. Let β be defined on H by $\beta(h) = \tilde{\sigma}(h)$ if h is in O and $\beta(h) = \tilde{\sigma}\alpha(h)$ otherwise. Let t be the root of $\tilde{A}$ and let $u = \tau^{-1}(t)$. It is easy to show (as is done in the proof of Lemma 6.7) that $\beta(h) = \tilde{\beta}(h)$ as soon as $\tilde{\beta}(h) \neq t$. The tree $(A, (I, O))$ is oriented from-root-to leaves, hence it is left-connected. Thus, by Lemma 6.1, for any half-edge $h \in H$ there exists an integer $q > 0$ such that $\beta^q(h) = t$. By taking the least such integer q, one gets $\tilde{\beta}^{q-1}(h) = \beta^{q-1}(h) \in \{u, \alpha(u)\}$. Since u is in O, one gets $\tilde{\beta}(\alpha(u)) = \tilde{\beta}(u) = \tilde{\sigma}(u) = r$. Hence, $\tilde{\beta}^q(h) = r$. By Lemma 6.1, this implies that $(\tilde{M}, (I, O))$ is left-connected.

We now study the restrictions $\tilde{\pi}_\circ \equiv \tilde{\pi}_{|I'}$ and $\tilde{\pi}_\bullet \equiv \tilde{\pi}^{-1}_{|O'}$. Recall that the map $\tilde{B}$ is bipartite with white vertices incident to half-edges in I' and black vertices incident to half-edges in O'. Hence, $\tilde{\pi} = \tilde{\pi}_\circ \tilde{\pi}_\bullet^{-1}$.

We first prove that the permutation $\tilde{\pi}_\circ$ is equal to $\tilde{\sigma}'_{|I'}$. We consider a half-edge h in I'. By definition of restrictions, $\tilde{\sigma}'_{|I'}(h) = \tilde{\sigma}'^{k}_{|I'}(h)$ for a positive integer k such that for all $0 < j < k$, the half-edge $\tilde{\sigma}'^{j}_{|I'}(h)$ is in O'. Moreover, the permutations $\tilde{\tau}'$ and $\tilde{\sigma}'$ coincide on O'. Thus, $\tilde{\sigma}'_{|I'}(h) = \tilde{\tau}'^{k-1}(\tilde{\sigma}'(h))$. By (16), $\tilde{\sigma}' = \tilde{\tau}'\tilde{\pi}_\circ$, hence $\tilde{\sigma}'_{|I'}(h) = \tilde{\tau}'^{k}(\tilde{\pi}_\circ(h))$. Therefore, $\tilde{\sigma}'_{|I'}(h)$ is a half-edge in I' contained in the cycle of $\tilde{\tau}'$ containing the half-edge $\pi_\circ(h)$ (which is in I'). Since (I, O) is the root-to-leaves orientation of $\tilde{A}$, every cycle of $\tilde{\tau}'$ contains exactly one half-edge in I' (except for the cycle made of o alone). Thus, $\tilde{\sigma}'_{|I'}(h) = \tilde{\pi}_\circ(h)$.

We now prove that the permutation $\tilde{\pi}_\bullet$ is equal to $\tilde{\phi}'_{|O'}$, where $\tilde{\phi}' = (i, o)\tilde{\sigma}'\alpha'$. We consider the face-permutation $\tilde{\psi} = \tilde{\pi}\alpha'$ of $\tilde{B}$. Since $\tilde{\pi} = \tilde{\pi}_\circ\tilde{\pi}_\bullet^{-1}$ one gets $\tilde{\psi}^{-1} = \alpha'\tilde{\pi}_\bullet\tilde{\pi}_\circ^{-1}$, hence $\tilde{\psi}^{-1}_{|I'} = \alpha'\tilde{\pi}_\bullet\alpha'\tilde{\pi}_\circ^{-1}$ and finally, $\tilde{\pi}_\bullet = \alpha'\tilde{\psi}^{-1}_{|I'}\tilde{\pi}_\circ\alpha'$. We now use the property (iv) of the coherently labelled pair $(\tilde{A}, \tilde{B})$. This property reads $\tilde{\psi}^{-1}_{|I'} = \alpha'\tilde{\varphi}'_{|O'}\alpha'$, where $\tilde{\varphi}' = (i, o)\tilde{\tau}'\alpha'$. Thus, $\tilde{\pi}_\bullet = \tilde{\varphi}'_{|O'}\alpha'\tilde{\pi}_\circ\alpha'$. We now consider a half-edge h in O'. By definition of restrictions, $\tilde{\pi}_\bullet(h) = \tilde{\varphi}'^{k}(\alpha'\tilde{\pi}_\circ\alpha'(h))$, where k is the least positive integer k such that $\tilde{\varphi}'^{k}(\alpha'\tilde{\pi}_\circ\alpha'(h))$ is in O'. Moreover, the permutations $\tilde{\varphi}' = (i, o)\tilde{\tau}'\alpha'$ and $\tilde{\phi}' = (i, o)\tilde{\sigma}'\alpha'$ coincide on I' (since $\tilde{\sigma}'$ and $\tilde{\tau}'$ coincide on O'). Thus, $\tilde{\pi}_\bullet(h) = \tilde{\phi}'^{k-1}(\tilde{\varphi}'\alpha'\tilde{\pi}_\circ\alpha'(h))$, where k is the least positive integer k such that $\tilde{\phi}'^{k-1}(\tilde{\varphi}'\alpha'\tilde{\pi}_\circ\alpha'(h))$ is in O'. Moreover, $\tilde{\varphi}'\alpha'\tilde{\pi}_\circ\alpha'(h) = (i, o)\tilde{\sigma}'\alpha'(h) \equiv \tilde{\phi}'(h)$ since $\tilde{\varphi}' = (i, o)\tilde{\tau}'\alpha'$ and $\tilde{\pi}_\circ = \tilde{\tau}'^{-1}\tilde{\sigma}'$

by (16). Thus, $\tilde{\pi}_\bullet(h) = \tilde{\phi}'^k(h) = \tilde{\phi}'_{|O'}(h)$.

Given that $\tilde{\pi}_\circ \equiv \tilde{\pi}_{|I'} = \tilde{\sigma}'_{|I'}$ and $\tilde{\pi}_\bullet \equiv \tilde{\pi}_{|O'} = \tilde{\phi}'_{|O'}$, it is clear from the definition of the mapping Λ that $\Lambda(\tilde{M}, (I, O)) = (\tilde{A}, \tilde{B})$. This concludes the proof of Proposition 6.14. □

7 Alternative descriptions of the unfolding step

In this section we present two alternative ways of encoding the pairs (A, B) made of a tree and a mobile. We also give the description of the folding and unfolding steps in terms of these encodings. These alternative descriptions are particularly useful for studying the specializations of the bijection Ψ. In particular we will use them in the next section in order to prove that Ψ specializes to a classical bijection by Bouttier *et al* [6]. These descriptions are also used in the planar case in [4] for handling specializations of Ψ allowing for a bijective counting of certain classes of triangulations and quadrangulations.

We first define the degree-code and height-code of a tree. Let A be a (rooted plane) tree with n edges. Let $v_0, v_1, \dots, v_n$ be the vertices in counterclockwise order of appearance around the tree (with v_0 being the root). The *height-code* of the tree A is the sequence $c_0, \dots, c_n$, where c_j is the *height* of the vertex v_j (the number of edges on the path from v_0 to v_j). The *degree-code* (or *Lukasiewicz code*) is the sequence $d_0, \dots, d_n$, where d_j is the number of children of v_j. The height- and degree-code for the tree A represented in Figure 12 are respectively 0123212 and 2210010. It is well-known that the height-code or degree-code both determine the tree A. Moreover, a sequence of non-negative integers $c_0, \dots, c_n$ is a height-code if and only if

$$c_0 = 0 \quad \text{and} \quad \forall i < n,\ 0 < c_{i+1} \leq c_i + 1,$$

and a sequence of non-negative integers $d_0, \dots, d_n$ is a degree-code if and only if

$$\sum_{i=0}^{n} (d_i - 1) = -1 \quad \text{and} \quad \forall k < n,\ \sum_{i=0}^{k} (d_i - 1) \geq 0.$$

We can now give alternative descriptions of a pair (A, B) by encoding the tree A as some *decorations* added to the mobile B (the decorations corresponding either to the height- or degree-code of A). We consider the usual black and white coloring of the mobile B (with the root-vertex being white). We say that the mobile B is *corner-labelled* if a non-negative number called *label* is attributed to each of the $n + 1$ white corners. The mobile B is *corner-well-labelled* if the root-corner has label 0, all other corners have positive labels and the labels do not increase by more than 1 from a corner to the next one in clockwise direction around B. Equivalently, B is corner-well-labelled if the sequence of corners encountered in clockwise order around the mobile starting from the root-corner is the height-code of a tree. A corner-well-labelled mobile is shown in Figure 12(b). We now consider mobiles with *buds*, that is, dangling half-edges. A *blossoming mobile* is a mobile B together with some outgoing buds glued in each white corners. The *sequence of buds* encountered in clockwise order around the mobile is the sequence $d_0, d_1, \ldots, d_n$ where d_i is the number of buds in the ith corner of B (in clockwise order starting from the root). The blossoming mobile is *balanced* if its sequence of buds is the degree-code of a tree. A balanced blossoming mobile is shown in Figure 12(c). Since both the height- and degree-code (made of $n+1$ integers) determine a plane tree (with n edges) the following result is obvious.

Lemma 7.1. *The three following sets are in bijection:*

- *pairs (A, B) made of a plane tree A and a mobile B with respectively n and $n+1$ edges,*
- *corner-well-labelled mobiles with $n+1$ edges,*
- *balanced blossoming mobiles with $n+1$ edges.*

The correspondences between the three sets considered in Lemma 7.1 are represented in Figure 12 (top part). By Theorem 4.5, these sets are all in bijection with left-connected maps (and covered maps) with n edges. In the rest of this section we describe the folding and unfolding step in terms of corner-well-labelled mobiles and balanced blossoming mobiles.

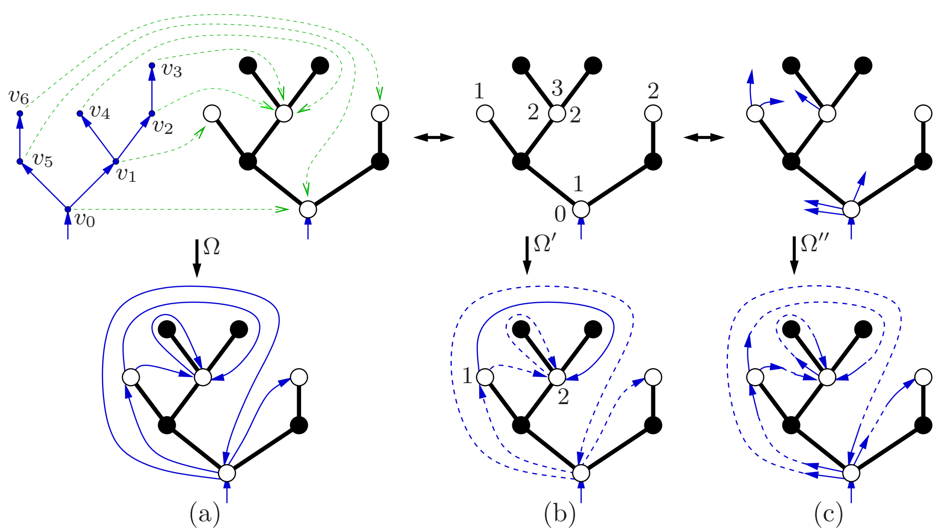

Figure 12: Three equivalent representations of a pair (A, B) and descriptions of the folding step.

Let (A, B) be a pair made of a plane tree A and a mobile B with respectively n and $n+1$ edges. Recall from Section 6 the topological description of the *folding* *step* $\Omega = \Lambda^{-1}$ for (A, B): if the vertices of the tree A are denoted $v_0, v_1, \ldots, v_n$ in counterclockwise order around A and the white corners of B are denoted $x_0, x_1, \ldots, x_n$ in clockwise order around B, then the *partially folded* map N is obtained by gluing the first-corner of the vertex v_j to the corner x_j (see Figure 11). The oriented map $(M, (I, O)) = \Omega(A, B)$ is then obtained from the partially folded map N by deleting the edges and black vertices of B.

This description implies that endowing A with its root-to-leaf orientation (for which the vertex v_j is incident to a unique ingoing half-edge i_j) results in having exactly one ingoing half-edge of A in each white corner of B (except in the root-corner); see Figure 8(a). More precisely, around a white vertex of N, one has in clockwise order between the half-edges of B defining the corner x_j: first the ingoing half-edge i_j of A (or the arrow pointing the root-corner if $j = 0$), and then the outgoing half-edges of A leading to the children of v_j.

Description of the folding step in terms of corner-labelled mobiles. Let (B, ℓ) be a corner-well-labelled mobile, where ℓ is the function associating a label to each white corner of B. For $j = 1 \ldots n$, we denote by x'_j be the last corner of B having label $\ell(x_j) - 1$ appearing before x_j in clockwise direction around B. Because (B, ℓ) is well-labelled, the corner x'_j always exists and appears between the root-corner and the corner x_j in clockwise order around B. The *partially folded* map N' associated to (B, ℓ) is defined as the map obtained from B by adding n edges sequentially: for $j = 1 \ldots n$ a directed edge e_j of N' is created from the corner x'_j to the corner x_j. More precisely, at each step the newly created edge is between the corner x_j and the corner of N' inside the corner x'_j of B which is incident to the root-face. The result is represented in Figure 12(b).

Clearly, the procedure Ω' described above is well-defined if and only if at each step j there exists a unique corner of N' incident to the root-face inside the corner x'_j of B. The fact that there is at most one such corner is clear by induction (and this is the corner of N' following the ingoing half-edge inside the corner x'_j). The fact that there is such a corner is also obtained by using the following induction hypothesis: after step j the white corners of B appearing before x_j and containing a corner of N' incident to the root-face are, in clockwise order around B, the last corners of B before x_j with respective labels $0, 1, \ldots, \ell(x_j)$. It remains to prove that the procedure Ω' is indeed equivalent to the folding step Ω.

Proposition 7.2. *If (B, ℓ) is the corner-well-labelled mobile corresponding with the pair (A, B) (i.e. $\ell(w_0)\ell(w_1)\ldots\ell(w_n)$ is the height-code of A), then the partially folded maps N and N' coincide.*

Proof. It suffices to show that the edges $e_1, \ldots, e_n$ created by the procedure Ω' are the edges that would have been obtained by gluing the tree A on the mobile B. Let j be in $\{1 \ldots n\}$ and let u_j be the parent of v_j in A. The edge of A joining v''_j and v_j gives an edge e''_j of N from a corner x''_j to the corner x_j of the mobile B. We need to show that $x''_j = x'_j$. First observe that $\ell(x''_j) = \ell(x_j) - 1$ because the labels $\ell(x_j)$ and $\ell(x''_j)$ correspond to the height of v_j and v''_j in A. Moreover, the vertex v''_j (which is the parent of v_j) is the last vertex with height $\ell(x_j) - 1$

appearing before v_j in counterclockwise order around A. Thus, by definition of N, the corner x''_j is the last corner with label $\ell(x_j) - 1$ appearing before x_j in clockwise order around B, that is, $x''_j = x_j$. This shows that the edges e_i and e''_i are incident to the same corners of B. Moreover, for both N and N' the clockwise order of the half-edges inside a white corner x_j of B coincide with the clockwise order of appearance of their other half around B. Thus, $N = N'$. □

Description of the folding step in terms of blossoming mobiles. Let $\vec{B}$ be a balanced blossoming mobile. Let $\vec{B}'$ be the *fully blossoming* mobile with ingoing and outgoing buds obtained from $\vec{B}$ by inserting an ingoing bud in each white corner of $\vec{B}$ following an edge of B (and not a outgoing bud) in clockwise order around the white vertex ($\vec{B}'$ is represented in solid lines in the bottom part of Figure 12(c)). Because the blossoming mobile $\vec{B}$ is balanced, the sequence of outgoing and ingoing buds in clockwise order around the $\vec{B}'$ (starting from the root-corner) is a parenthesis system (if outgoing and ingoing buds are seen respectively as opening and closing parentheses). Hence, there is a unique way of pairing each outgoing bud to an ingoing bud following it without creating any crossings. The *partially folded* map N' associated to the blossoming mobile $\bar{B}$ is the map obtained from $\vec{B}'$ by performing these pairings. The result is represented in Figure 12(c).

Proposition 7.3. *If $\vec{B}$ is the blossoming mobile associated with the pair (A, B) (i.e. the sequence of buds of $\bar{B}$ is the degree-code of A), then the partially folded maps N and N' coincide.*

Proof. It suffices to show that the paired edges of N' are the edges that would have been obtained by gluing the tree A on the mobile B. First observe that around a white vertex of N', one has in clockwise order between the half-edges of B defining the corner x_j: first an ingoing half-edge and then d_j outgoing half-edges, where d_j is the number of children of v_j in A (because the sequence of buds of $\vec{B}$ is the degree-code of A). Thus, $\vec{B}'$ is the map obtained from N by cutting each edge of A at their midpoint. Moreover, for both N and N' the clockwise order of the half-edges inside a white corner of B coincide with the clockwise order of appearance of their other half around B. Thus, $N = N'$. □

8 Link with the bijection of Bouttier, Di Francesco and Guitter.

In [6] Bouttier, Di Francesco and Guitter defined a bijection between bipartite maps and vertex-well-labelled mobiles[2] (see definition below). The goal of this section is to show the bijection of Bouttier *et al.* can be obtained as a specialization of the *unfolding mapping* $\Lambda' = \Omega'^{-1}$ associating a corner-labelled mobile

[2]Strictly speaking, the bijection in [6] only describes the planar case. But is was explained in [9] how to extend it to higher genera.

to a left-connected map (Figure 12(b)).

We first recall some definitions. The *distance* between two vertices of a map is the minimum number of edges on paths between them. We denote by $d(v)$ the distance of a vertex v from the root-vertex. Clearly, any pair of adjacent vertices u, v satisfies $|d(u) - d(v)| \leq 1$. An orientation is *geodesic* if any edge with origin u and end v satisfies $d(u) \leq d(v)$ (i.e. edges are oriented away from the root-vertex). For a bipartite map any pair of adjacent vertices u, v satisfies $|d(u) - d(v)| = 1$ (since every cycle has even length), hence there is a unique geodesic orientation. The geodesic orientation is indicated in Figure 13(b).

A *vertex-well-labelled mobile* is a corner-well-labelled mobile such that the labels coincide around each white vertices, that is, any two corners incident to the same vertex have the same label. An example is given in Figure 13(c). Observe that vertex-well-labelled mobile are equivalently defined as mobiles with a label $\ell(v)$ associated to each vertex v satisfying:

- the root-vertex has label 0 and degree 1, while other white vertices have positive labels,
- the increase between the labels of two consecutive white vertices in clockwise order around a black vertex is at most 1.

Proposition 8.1. *The geodesic orientation of a bipartite map is left-connected. Moreover, the unfolding mapping Λ' induces a bijection between the set of bipartite maps (with n edges and genus g) endowed with their geodesic orientation and the set of vertex-well-labelled mobiles (with $n+1$ edges and genus g). This induced bijection is exactly the bijection described by Bouttier* et al. *in [6].*

Proof. Let $(M, (I, O))$ be a bipartite map endowed with its geodesic orientation. We first prove that the geodesic orientation is left-connected by using Lemma 6.1 concerning the backward function β. Clearly, for any half-edge h incident to a non-root vertex v, there exists an integer $p > 0$ such that the half-edge $\beta^p(h)$ is incident to a vertex u satisfying $d(u) = d(v) - 1$ (because there are ingoing edges incident to v, and they all join v to a vertex u satisfying the property). Thus, there exists $q > 0$ such that the half-edge $\beta^q(h)$ is incident to the root-vertex. Moreover, for any half-edge h' incident to the root-vertex there exists an integer $r > 0$ such that the half-edge $\beta^r(h')$ is the root (because the root-vertex is only incident to outgoing half-edges). Therefore, by Lemma 6.1, the geodesic orientation is left-connected. We now show that the corner-labelled mobile $(B, \ell) = \Lambda'(M, (I, O))$ is vertex-well-labelled. Let v be a vertex of M and let $v_1, \ldots, v_k$ be the vertices of the tree $A = \Lambda_1(M, (I, O))$ resulting from unfolding the vertex v. Clearly, any directed path from the root-vertex to v in M has length $d(v)$. Hence, for all $i \in \{1, \ldots, k\}$ every directed paths from the root-vertex to v_i in A has length $d(v)$. Hence, the label ℓ of every corner of the white vertex v of the mobile B is equal to $d(v)$. Thus, the corner-labelled mobile (B, ℓ) is vertex-well-labelled.

Conversely, let (B, ℓ) be a vertex-well-labelled mobile, let $(M, (I, O)) = \Omega'(B, \ell)$ be the corresponding left-connected map. We want to prove that M

is bipartite and (I, O) is the geodesic orientation. By definition of the folding Ω' any edge of M goes from a white vertex u to a white vertex v satisfying $\ell(v) = \ell(u) + 1$. Hence, reasoning on the parity of labels shows that M is bipartite. In order to prove that (I, O) is the geodesic orientation, it suffices to prove that the label function ℓ is equal to the distance function d. Let v be a non-root vertex. On one hand, one gets $d(v) \geq \ell(v)$ from the fact that labels cannot decrease by more than one when following an edge of M (hence the root-vertex cannot be reached by following less than $\ell(v)$ edges). On the other hand, one gets $d(v) \leq \ell(v)$ from the fact that any non-root vertex of M is adjacent to a vertex having a smaller label (by definition of the folding step Ω'). Thus $d = \ell$ and the orientation is geodesic. □

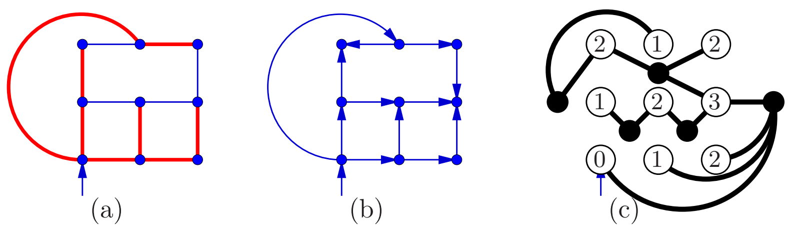

Figure 13: (a) The rightmost BFS tree. (b) The geodesic orientation. (c) The associated vertex-well-labelled mobile.

In the remaining of this section, we complete the picture by characterizing the unicellular submap of a bipartite map which corresponds to the geodesic orientation (by the orientation step Δ). A spanning tree is said *BFS* (for *Breadth-First-Search*) if for any vertex v, the distance $d(v)$ is equal to the height in the spanning tree.

Definition 8.2. The *rightmost BFS tree* is the spanning tree T obtained by the following procedure:
Initialization: Set every vertex to be *alive*. Set the tree T as the tree containing the root-vertex of M and no edge.
Core: Consider the alive vertex v which has been in the tree T for the longest time and set it dead. Inspect the half-edges incident to v in counterclockwise order (starting from the root if v is the root-vertex, and starting from the half-edge following the edge of T leading v to its parent otherwise) and whenever a half-edge leads to a vertex not in the tree T add this vertex and the edge to T. Repeat until all vertices are dead.
End: Return the spanning tree T.

The rightmost BFS tree is indicated in Figure 13(a). We omit the proof of the following easy result.

Lemma 8.3. *The procedure terminates and returns a BFS spanning tree. Moreover, the order in which the set of half-edges incident to vertices at a given dis-*

tance from the root-vertex are inspected coincide with the order of appearance during the counterclockwise tour of the tree.

Proposition 8.4. *Let (M,S) be a bipartite covered map, and let $(M,(I,O)) = \Delta(M,S)$ be the associated left-connected map. The orientation (I,O) is geodesic if and only if $M_{|S}$ is the rightmost BFS tree of M.*

Proof. We suppose that $M_{|S}$ is the rightmost BFS tree and consider the orientation $(M,(I,O)) = \Delta(M,S)$. We need to prove that (I,O) is the geodesic orientation. Let e be an edge in $M_{|S}$. By definition of Δ, the edge e is oriented from parent to child. Since the tree $M_{|S}$ is BFS, this orientation coincide with the geodesic orientation of e. Let now e be an edge not in $M_{|S}$ with origin u and end v. We want to prove that $d(v) = d(u) + 1$. Suppose the contrary: $d(u) = d(v) + 1$. Let v' be the parent of u and let e' be the edge from v' to u. Let h and h' be respectively the half-edges of e and e' incident to v and v'. By definition of Δ, the edge e is oriented in such a way that the ingoing half-edge h is encountered before the outgoing half-edge during the counterclockwise tour of $M_{|S}$. This implies that h is encountered before h' during the counterclockwise tour of $M_{|S}$. By Lemma 8.3, this implies that the half-edge h is inspected before h' during the procedure constructing the rightmost BFS. Hence, when the half-edge h is inspected, the vertex u is not in the tree T and should be added together with the edge e. We reach a contradiction. Thus, we have shown that the orientation (I,O) associated to the rightmost BFS is the geodesic orientation of M. □

9 Duality.

Recall from Section 3 that the dual of a covered map is a covered map. In this section, we explore the properties of the bijection Ψ with respect to duality. Throughout this section, we consider a covered map (M,S), where the map $M = (H,\sigma,\alpha)$ has root r and face-permutation $\phi = \sigma\alpha$. We denote $(M,(I,O)) = \Delta(M,S)$ and $(A,B) = \Psi(M,S)$.

Lemma 9.1 (Duality at the orientation step)**.** *The oriented map associated to the dual covered map is the dual oriented map, that is to say, $\Delta(M^*,\bar{S}) = (M^*,(O,I))$.*

Lemma 9.1 is illustrated in Figure 14.

Proof. Recall that the the submaps $M_{|S}$ and $M^*_{|\bar{S}}$ have the same motion functions, hence define the same appearance order on H. Thus, Lemma 9.1 immediately follows from the definition of the mapping Δ. □

We now explore the properties of the unfolding step with respect to duality. We denote $A = (H,\tau,\alpha) = \Psi_1(M,S)$ and $B = (H',\pi,\alpha') = \Psi_2(M,S)$, where H' stands for $H \cup \{i,o\}$ and i is the root of the mobile B. We also denote by

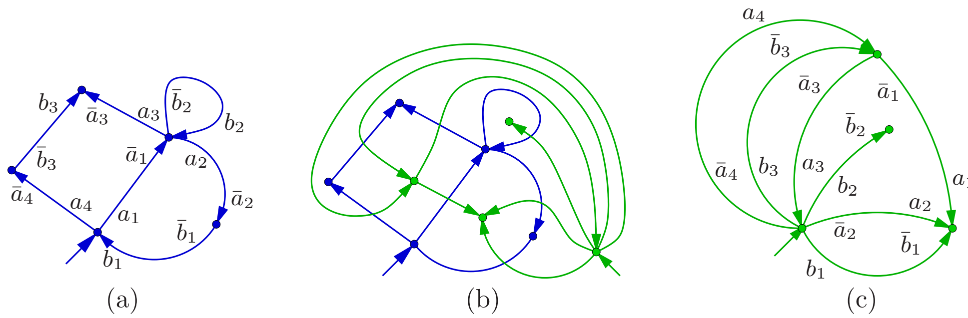

Figure 14: (a) The oriented map $(M,(I,O)) = \Delta(M,S)$ associated to the covered map represented in Figure 4(a). (b) Topological construction of the dual: each oriented edge of M is crossed by the the dual oriented edge of M^* *from left to right*. (c) The oriented map $(M^*,(O,I))$.

$A^\star = (H,\tau^\star,\alpha) = \Psi_1(M^*,\bar{S})$ and $B^\star = (H',\pi^\star,\alpha') = \Psi_2(M^*,\bar{S})$, the plane tree and mobile associated to the dual covered map $(M^*,\bar{S})$. We shall prove the existence of two independent mappings Υ and Ξ such that $A^\star = \Upsilon(A)$ and $B^\star = \Xi(B)$. In words, the duality acts *component-wise* on the plane tree and the mobile.

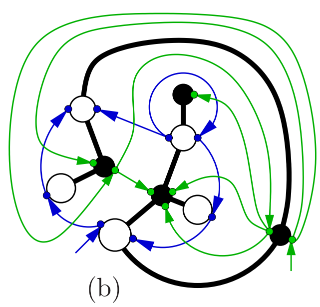

Figure 15: Simultaneous unfolding of the oriented map of Figure 14 and of its dual.

Proposition 9.2 (Duality and the mobile)**.** *Let* (M,S) *be a covered map and let* $(M^*,\bar{S})$ *be the dual covered map. If the mobile* $B = \Psi_2(M,S)$ *is denoted* (H',π,α') *and has root* i*, then the mobile* $B^\star = \Psi_2(M^*,\bar{S})$ *is the map* (H',π^{-1},α') *with root* $o = \alpha(i)$*.*

Proposition 9.2 is illustrated in Figure 16. It implies that the mobile $B^\star$ is entirely determined by the mobile B.

Proof. The map M has vertex-permutation σ and face-permutation ϕ, while the map M^* has vertex-permutation $\sigma^\star = \phi$ and face-permutation $\phi^\star = \sigma$. We denote $\Delta(M,S) = (M,(I,O))$, so that $B = \Lambda(M,(I,O))$ and $B = \Lambda(M^*,(I^\star,O^\star))$, where $I^\star = O$ and $O^\star = I$ by Lemma 9.1. We adopt the notations i, o, I', O', σ', ϕ', $\pi_\circ$, $\pi_\bullet$, π of Section 4 for defining B and adopt the corresponding notations $i^\star$, $o^\star$, $I'^\star$, $O'^\star$, $\sigma'^\star$, $\phi'^\star$, $\pi_\circ{}^\star$, $\pi_\bullet{}^\star$, $\pi^\star$ for defining $B^\star$. We choose $i^\star = o$ and $o^\star = i$, so that $I'^\star \equiv I^\star \cup \{i^\star\} = O'$, $O'^\star \equiv O^\star \cup \{o^\star\} = I'$, $\sigma'^\star = \phi'$ and $\phi'^\star = \sigma'$. From this, it follows that $\pi_\circ^\star \equiv \sigma'^\star{}_{|I'^\star} = \phi'{}_{|O'} = \pi_\bullet$ and $\pi_\bullet^\star \equiv \phi'^\star_{|O'^\star} = \sigma'{}_{|I'} = \pi_\circ$ and finally $\pi^\star \equiv \pi_\bullet^\star \pi_\circ^{\star\,-1} = \pi_\circ \pi_\bullet^{-1} = \pi^{-1}$. Lastly, the root of $B^\star$ is $i^\star = o$. □

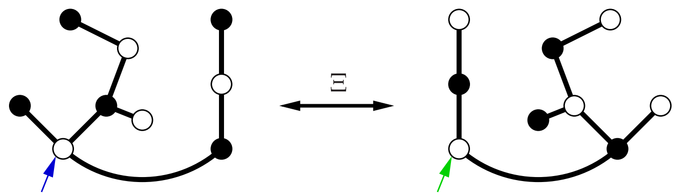

Figure 16: The mapping Ξ between the mobile $B = \Psi_2(M,S)$ associated to the covered map (M,S) of Figure 14 and the mobile $B^\star = \Psi_2(M^*,\bar{S})$ associated to the dual covered map.

We now explicit the relation between the trees A and $A^\star$ in terms of their codes.

Proposition 9.3 (Duality and the tree). *If the height-code of $A = \Psi_1(M,S)$ is $c_0,\dots,c_n$, then the degree-code of $A^\star = \Psi_1(M^*,\bar{S})$ is $d_0,\dots,d_n$, where $d_{n-j} = c_j + 1 - c_{j+1}$ for $j = 1,\dots,n-1$ and $d_0 = c_n$.*

Recall that a tree is completely determined by its height-code or by its degree-code. Hence, Proposition 9.3 shows that the tree $A^\star$ is entirely determined by the tree A. Observe that the mapping $A \mapsto A^\star$ is an involution since duality of covered map is an involution. A topological version of this mapping is illustrated in Figure 17(b), where the two trees A and $A^\star$ are represented simultaneously in the way they interlace around the mobile's face. The rest of this section is devoted to the proof of Proposition 9.3.

We denote by t the root of the tree $A = (H,\tau,\alpha)$ and by $t^\star$ the root of the tree $A^\star = (H,\tau^\star,\alpha)$. We also adopt the notations σ', ϕ', $\pi_\circ$, $\pi_\bullet$, π, τ' of Section 4 for the tree A and adopt the corresponding notations $\sigma'^\star$, $\phi'^\star$, $\pi_\circ{}^\star$, $\pi_\bullet{}^\star$, $\pi^\star$, $\tau'^\star$ for the tree $A^\star$. Lastly, we denote $\varphi = \tau\alpha$, $\varphi' = (i,o)\tau'\alpha'$, $\varphi^\star = \tau^\star\alpha$ and $\varphi'^\star = (i,o)\tau'^\star\alpha'$. Recall that Lemma 6.10 describes the (simple) link existing between the permutations τ and τ' and between φ and φ'.

Lemma 9.4. *The permutations φ' and $\varphi'^\star$ are related by $\varphi'{}_{|O'} = (\alpha'\varphi'^\star\alpha')^{-1}_{|O'}$.*

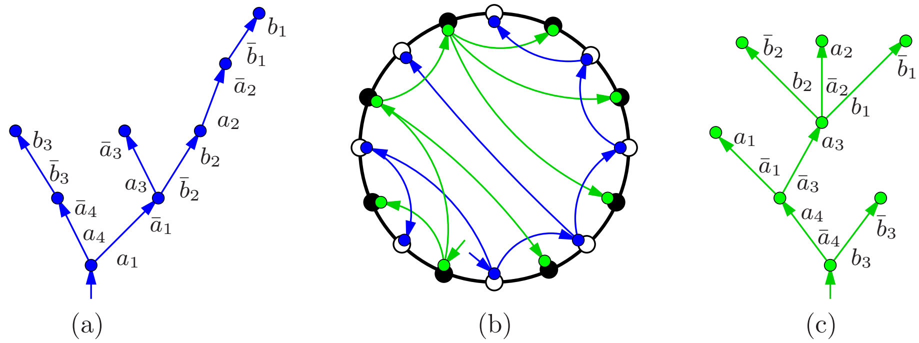

Figure 17: (a) The plane tree $A = \Psi_1(M, S)$ associated to the covered map of Figure 14. (b) Topological construction of the tree $A^\star = \Psi_1(M^*, \bar{S})$: in the mobile face, the trees A and $A^\star$ interlace in such a way that each edge of A is crossed by exactly one edge of $A^\star$. (c) The plane tree $A^\star$.

For the example in Figure 17, one gets $\varphi'_{|O'} = (o, a_1, \bar{b}_2, a_2, \bar{b}_1, a_3, a_4, \bar{b}_3)$ and $\varphi'^{\star}_{|I'} = (b_3, \bar{a}_4, \bar{a}_3, b_1, \bar{a}_2, b_2, \bar{a}_1, i)$.

Proof. By Proposition 6.11, the face-permutation $\psi = \pi\alpha$ of the mobile B satisfies $\varphi'_{|O'} = \alpha'\psi^{-1}_{|I'}\alpha'$. The same property applied to the mobile $B^\star$ gives $\varphi'^{\star}_{|I'} = \alpha'\psi^{\star}{}^{-1}_{|O'}\alpha'$, where $\psi^\star = \pi^\star\alpha'$ is the face-permutation of B. This gives

$$(\alpha'\varphi'^{\star}\alpha')^{-1}_{|O'} = \alpha'(\varphi'^{\star})^{-1}_{|I'}\alpha' = \psi^{\star}{}_{|O'}.$$

Moreover, by Proposition 9.2, $\pi^\star = \pi^{-1}$, so that $\psi^\star = \pi^{-1}\alpha' = \alpha'\psi^{-1}\alpha$. Hence,

$$(\alpha'\varphi'^{\star}\alpha')^{-1}_{|O'} = \psi^{\star}{}_{|O'} = (\alpha'\psi^{-1}\alpha)_{|O'} = \alpha'\psi^{-1}_{|I'}\alpha' = \varphi'_{|O'}.$$

□

Lemma 9.5. *The permutations φ' and $\tau'^\star$ are related by $\tau'^\star = \varphi'\varphi'^{-1}_{|O'}$.*

Proof. By definition, $\tau'^\star = \sigma'^\star\pi_\circ^{\star\,-1} = \phi'\phi'^{-1}_{|O'}$, where $\phi' = (i, o)\sigma'\alpha'$. We want to prove $\phi'\phi'^{-1}_{|O'} = \varphi'\varphi'^{-1}_{|O'}$, or equivalently, $\phi'_{|O'}\phi'^{-1} = \varphi'_{|O'}\varphi'^{-1}$ (by taking the inverse). Observe that the permutations $\phi' = (i, o)\sigma'\alpha'$ and $\varphi' = (i, o)\tau'\alpha'$ coincide on I' (since σ' and τ' coincide on O'). We now consider a half-edge h in H'. Suppose first that $\phi'^{-1}(h)$ is in I'. In this case, $\phi'^{-1}(h) = \varphi'^{-1}(h)$ (since ϕ' and φ' coincide on I'), hence $\phi'_{|O'}\phi'^{-1}(h) = \phi'^{-1}(h) = \varphi'^{-1}(h) = \varphi'_{|O'}\varphi'^{-1}(h)$. Suppose now that $\phi'^{-1}(h)$ is in O'. Observe that $\varphi'^{-1}(h)$ is also in O' (since ϕ' and φ' coincide on I'). Moreover, by definition of reductions, $\phi'_{|O'}\phi'^{-1}(h) = \phi'^k(h)$, where $k \geq 0$ is such that $\phi'^k(h) \in O'$ and $\phi'^j(h) \in I'$ for all $0 \leq j < k$. Since ϕ' and φ' coincide on I', we get $\phi'^j(h) = \varphi'^j(h)$

for $0 \leq j \leq k$. Thus, $\phi'_{|O'}\phi'^{-1}(h) = \varphi'^{k}(h)$ and where $k \geq 0$ is such that $\varphi'^{k}(h) \in O'$ and $\varphi'^{j}(h) \in I'$ for all $0 \leq j < k$. Hence, by definition of reductions, $\phi'_{|O'}\phi'^{-1}(h) = \varphi'_{|O'}\varphi'^{-1}(h)$. □

Proof of Proposition 9.3.
We denote by $o_0 = o, o_1, \ldots, o_n$ the half-edges in O' in such a way that $\varphi'_{|O'} = (o, o_1, \ldots, o_n)$ and we denote $i_j = \alpha(o_j)$ for $j = 0 \ldots n$. We denote by $v_0, v_1, \ldots, v_n$ the vertices of A in counterclockwise order around A. By definition (and because (I, O) is the root-to-leaves orientation of A), this means that v_j is incident to the ingoing half-edge i_j for $j = 1 \ldots n$. Therefore, the height-code of A is $c_0 c_1 \cdots c_n$, where $c_0 = 0$ and for $j = 0 \ldots n-1$, $c_{j+1} = c_j + 1 - \delta_j$ where δ_j is the number of half-edges in I between the half-edge o_j and o_{j+1} in the face-permutation φ (hence, also in the permutation φ'). Equivalently, for $j = 0 \ldots n-1$, $\delta_j \equiv c_j + 1 - c_{j+1}$ is the number of half-edges in I' in the cycle of the permutation $\varphi'\varphi'^{-1}_{|O'}$ containing o_j. We also denote $\delta_n = c_n$ and observe that this is the number of half-edges in I' in the cycle of the permutation $\varphi'\varphi'^{-1}_{|O'}$ containing o_n.

We now consider degree-code $d_0 d_1 \cdots d_n$ of $A^\star$ and want to prove that $\delta_j = d_{n-j}$ for $j = 0 \ldots n$. Let $v_0^\star, v_1^\star, \ldots, v_n^\star$ be the vertices of $A^\star$ in counterclockwise order around $A^\star$. By Lemma 9.1, the root-to-leaves orientation of $A^\star$ is (O, I) and by Lemma 9.4, $\varphi'^{\star}_{|I'} = (i, i_n, \ldots, i_2, i_1)$. Therefore, for $j = 1 \ldots n-1$ the vertex $v_{n-j}^\star$ of $A^\star$ is incident to the half-edge o_j. Thus, for $j = 0 \ldots n-1$, the number of children d_{n-j} of $v_{n-j}^\star$ is the number of half-edges in I' in the cycle of the vertex-permutation $\tau^\star$ containing o_j (hence, also in the permutation $\tau'^\star$). For $j = n$ also, we observe that d_{n-j} is the number of half-edges in I' in the cycle of the permutation $\tau'^\star$ containing o_j. By Lemma 9.5, $\tau'^\star = \varphi'\varphi'^{-1}_{|O'}$, hence $\delta_j = d_{n-j}$ for $j = 0 \ldots n$. This concludes the proof of Proposition 9.3. □

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