

A new combinatorial identity for unicellular maps, via a direct bijective approach

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Abstract

A unicellular map, or one-face map, is a graph embedded in an orientable surface such that its complement is a topological disk. In this paper, we give a new viewpoint to the structure of these objects, by describing a decomposition of any unicellular map into a unicellular map of smaller genus. This gives a new combinatorial identity for the number $\epsilon_g(n)$ of unicellular maps with n edges and genus g . Unlike the Harer-Zagier recurrence formula, this identity is recursive in only one parameter (the genus).

Iterating the construction gives an explicit bijection between unicellular maps and plane trees with distinguished vertices, which gives a combinatorial explanation (and proof) of the fact that $\epsilon_g(n)$ is the product of the n -th Catalan number by a polynomial in n . The combinatorial interpretation also gives a new and simple formula for this polynomial. Variants of the problem are considered, like bipartite unicellular maps, or unicellular maps with only cubic vertices.

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Mathematics Subject Classification: 05A19, 05A15, 05C30, 05A10.

1 Introduction.

A *unicellular map* is a graph embedded in a compact orientable surface, in such a way that its complement is a topological polygon. Equivalently, a unicellular map can be viewed as a polygon, with an even number of edges, in which edges have been glued pairwise in order to create an orientable surface (to ensure orientability, one must glue each pair of edges in opposite directions, in any cyclic orientation of the polygon). The number of handles of this surface is called the *genus* of the map.

These objects are recurrent in combinatorics, and have been considered in many different contexts. The numbers of unicellular maps of given size and genus appear in random matrix theory as the moments of the Gaussian Unitary Ensemble (see [LZ04]). In the study of characters of the symmetric group, unicellular maps appear as factorisations of cyclic permutations [Jac87, Zag95].

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According to the context, unicellular maps are also called *one-face maps*, *polygon gluings*, or *one-border ribbon graphs*. Sometimes, their duals, *one-vertex maps*, are considered. The most famous example of unicellular maps is the class of *plane* unicellular maps, that is the class of plane trees, enumerated by the Catalan numbers.

The first result in the enumeration of unicellular maps in positive genus was obtained by Lehman and Walsh [WL72]. Using a direct recursive method, relying on formal power series, they expressed the number $\epsilon_g(n)$ of unicellular maps with n edges on a surface of genus g as follows:

$$\epsilon_g(n) = \sum_{\gamma \vdash g} \frac{(n+1) \dots (n+2-2g-l(\gamma))}{2^{2g} \prod_i c_i! (2i+1)^{c_i}} \text{Cat}(n), \quad (1)$$

where the sum is taken over partitions γ of g , c_i is the number of parts equal to i in γ , $l(\gamma)$ is the total number of parts, and $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number. This formula has been extended by other authors ([GS98]).

Later, Harer and Zagier [HZ86], via matrix integrals techniques, obtained the two following identities¹, known respectively as the Harer-Zagier *recurrence* and the Harer-Zagier *formula*:

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2), \quad (2)$$

$$\sum_{g \geq 0} \epsilon_g(n) x^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{x}{i}. \quad (3)$$

Formula (3) has been reproved by several authors, by various techniques. A combinatorial interpretation of this formula was given by Lass [Las01], and the first bijective proof was given by Goulden and Nica [GN05]. Generalizations were given for bicolored, or multicolored maps [Jac87, Adr97, SV08].

The purpose of this paper is to give a new approach to the enumeration of unicellular maps of fixed genus, at a level which is much more combinatorial than what existed before. Indeed, until now no bijective proof (or combinatorial interpretation) of Formulas (1) and (2) are known. As for Formula (3), its bijective proof given in [GN05] is concerned with *colored* unicellular maps, that is to say, with unicellular maps whose vertices are arbitrarily colored with x colors for some integer x , but whose genus is not fixed. Counting these colored unicellular maps for all x , which amounts to putting a weight x^{n+1-2g} on each unicellular map of genus g with n edges, is *analytically* equivalent to enumerating unicellular maps by genus, as the number of maps of fixed genus can be extracted by selecting the correct exponent of x in Formula (3). However, the genus itself does not appear explicitly in the construction.

On the contrary, this article is concerned with the structure of unicellular maps themselves, at fixed genus, and with no additional coloring. We investigate in details the way the unique face of such a map intertwines with itself in order to create the handles of the surface. We show that, in each unicellular map of genus g , there are $2g$ special "places", which we call *trisections*, that concentrate, in some sense, the handles of the surface. Each of these places can be used to *slice* the map to a unicellular map of lower genus. Conversely, we show that a unicellular map of genus g can always be obtained in $2g$ different ways by gluing vertices together in a map of lower genus.

¹Here and in the rest of the paper, the quantity $\binom{n}{p}$ is defined as the number of p -element subsets of $\{1, 2, \dots, n\}$, and in particular $\binom{n}{p} = 0$ if $p > n$.

In terms of formulas, this leads us to the new combinatorial identity:

$$2g \cdot \epsilon_g(n) = \binom{n+1-2(g-1)}{3} \epsilon_{g-1}(n) + \binom{n+1-2(g-2)}{5} \epsilon_{g-2}(n) + \cdots + \binom{n+1}{2g+1} \epsilon_0(n) \quad (4)$$

$$= \sum_{p=0}^{g-1} \binom{n+1-2p}{2g-2p+1} \epsilon_p(n). \quad (5)$$

The main advantage of this identity is that it is recursive *only in the genus*: the size n is fixed. For a given g , this enables one to compute directly the formula giving $\epsilon_g(n)$, by iteration. From the combinatorial viewpoint, this enables one to construct maps of fixed genus and size very easily.

When iterated, our bijection shows that all unicellular maps can be obtained in a canonical way from plane trees by successive gluings of vertices, hence giving the first explanation to the fact that $\epsilon_g(n)$ is the product of a polynomial $R_g(n)$ by the n -th Catalan number. More precisely, we obtain the formula $\epsilon_g(n) = R_g(n)\text{Cat}(n)$ with:

$$R_g(n) = \sum_{0=g_0 < g_1 < \cdots < g_r = g} \prod_{i=1}^r \frac{1}{2g_i} \binom{n+1-2g_{i-1}}{2(g_i - g_{i-1}) + 1}, \quad (6)$$

which comes with a clear combinatorial interpretation. This interpretation gives the answer to questions asked by Zagier [LZ04, p159].

In the paper [Cha10], we presented a less powerful bijection, that worked only for an asymptotically dominating subset of all unicellular maps. The bijection presented here is really a generalization of the bijection of [Cha10], in the sense that it coincides with it when specialized to maps in the dominating set. However, new difficulties and structures appear in the general case, and there is an important gap between the combinatorial results in [Cha10] and the ones of this paper.

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2 Unicellular maps.

2.1 Permutations and ribbon graphs.

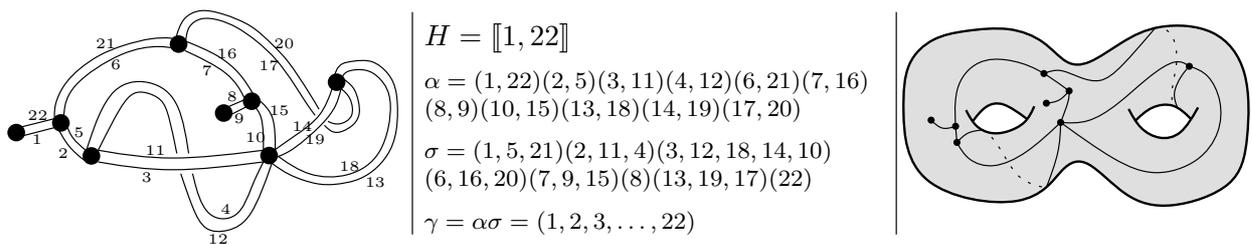


Figure 1: A unicellular map with 11 edges, 8 vertices, and genus 2: (a) ribbon graph; (b) permutations; (c) topological embedding.

Rather than talking about topological embeddings of graphs, we work with a combinatorial definition of unicellular maps in terms of permutations. In this paper, permutations will always be multiplied from right to left.

Definition 1. A *unicellular map* \mathbf{m} of size n is a triple $\mathbf{m} = (H, \alpha, \sigma)$, where H is a set of cardinality $2n$, α is an involution of H without fixed points, and σ is a permutation of H such that $\gamma = \alpha\sigma$ has only one cycle. The elements of H are called the *half-edges* of \mathbf{m} . The cycles of α and σ are called the *edges* and the *vertices* of \mathbf{m} , respectively, and the permutation γ is called the *face* of \mathbf{m} .

The graph-theoretic terminology used in Definition 1 comes from the correspondence between unicellular maps as we just defined them and ribbon graphs, which we now describe. First, in this paper, the word *graph* will always be used in the meaning of multigraph, i.e. we allow loops and multiple edges. Given a unicellular map $\mathbf{m} = (H, \sigma, \alpha)$, its associated graph G is the graph whose edges are given by the cycles of α , vertices by the cycles of σ , and the natural incidence relation $v \sim e$ if v and e share an element. Moreover, we draw each edge of G as a *ribbon*, where each side of the ribbon represents one half-edge; we decide which half-edge corresponds to which side of the ribbon by the convention that, if a half-edge h belongs to a cycle e of α and v of σ , then h is the right-hand side of the ribbon corresponding to e , when considered entering v . Furthermore, we draw the graph G in such a way that around each vertex v , the *counterclockwise ordering* of the half-edges belonging to the cycle v is given by that cycle: we obtain a graphical object called the *ribbon graph* associated to \mathbf{m} , as in Figure 1(a). Note that the unique cycle of the permutation $\gamma = \alpha\sigma$ is interpreted as the sequence of half-edges visited when *making the tour* of the graph, keeping the graph on its left.

A *rooted* unicellular map is a unicellular map carrying a distinguished half-edge r , called the root. These maps are considered up to relabellings of H preserving the root, i.e. two rooted unicellular maps \mathbf{m} and \mathbf{m}' are considered the same if there exists a permutation $\pi : H \rightarrow H'$, such that $\pi(r) = r'$, $\alpha = \pi^{-1}\alpha'\pi$, and $\sigma = \pi^{-1}\sigma'\pi$. In this paper, *all* unicellular maps will be rooted, even if not stated explicitly.

Given a unicellular map \mathbf{m} with root r and face $\gamma = \alpha\sigma$, we define the linear order $<_{\mathbf{m}}$ on H by setting:

$$r <_{\mathbf{m}} \gamma(r) <_{\mathbf{m}} \gamma^2(r) <_{\mathbf{m}} \dots <_{\mathbf{m}} \gamma^{2n-1}(r).$$

In other words, if we relabel the half-edge set H by elements of $\llbracket 1, 2n \rrbracket$ in such a way that the root is 1 and the tour of the face is given by the permutation $(1, \dots, 2n)$, the order $<_{\mathbf{m}}$ is the natural order on the integers. However, since in this article we are going to consider maps with a fixed half-edge set, but a changing permutation γ , it is more convenient (and prudent) to define the order $<_{\mathbf{m}}$ in this way.

Unicellular maps can also be interpreted as graphs embedded in a topological surface, in such a way that the complement of the graph is homeomorphic to a disk. If considered up to homeomorphism, and suitably rooted, these objects are in bijection with ribbon graphs. See [MT01], or the example of Figure 1(c). The *genus* of a unicellular map is the genus, or number of handles, of the corresponding surface. If a unicellular map of genus g has n edges and v vertices, then Euler's characteristic formula states that

$$v = n + 1 - 2g.$$

From a combinatorial point of view, this formula can also be taken as a definition of the genus.

2.2 The gluing operation.

We let $\mathbf{m} = (H, \alpha, \sigma)$ be a unicellular map of genus g , and $a_1 <_{\mathbf{m}} a_2 <_{\mathbf{m}} a_3$ be three half-edges of \mathbf{m} belonging to three distinct vertices. Each half-edge a_i belongs to some vertex $v_i = (a_i, h_i^1, \dots, h_i^{m_i})$,

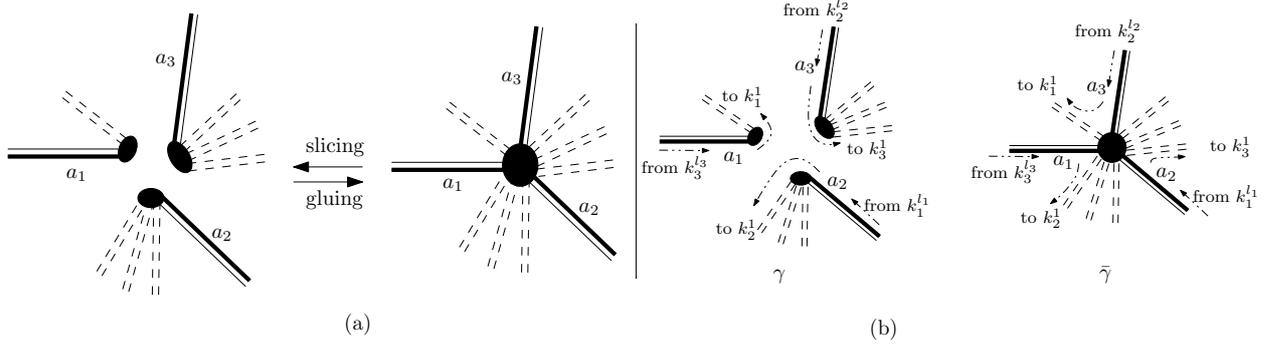


Figure 2: (a) The gluing and slicing operations. (b) The "proof" of Lemma 1.

for some $m_i \geq 0$. We define the permutation

$$\bar{v} := (a_1, h_2^1, \dots, h_2^{m_2}, a_2, h_3^1, \dots, h_3^{m_3}, a_3, h_1^1, \dots, h_1^{m_1}),$$

and we let $\bar{\sigma}$ be the permutation of H obtained by deleting the cycles v_1 , v_2 , and v_3 , and replacing them by \bar{v} . The transformation mapping σ to $\bar{\sigma}$ is interpreted combinatorially as the *gluing* of the three half-edges a_1, a_2, a_3 , as shown in Figure 2(a). We have:

Lemma 1. *The map $\bar{\mathfrak{m}} := (H, \alpha, \bar{\sigma})$ is a unicellular map of genus $g + 1$. Moreover, if we let*

$$\gamma = \alpha\sigma = (a_1, k_1^1, \dots, k_1^{l_1}, a_2, k_2^1, \dots, k_2^{l_2}, a_3, k_3^1, \dots, k_3^{l_3})$$

be the face permutation of \mathfrak{m} , then the face permutation of $\bar{\mathfrak{m}}$ is given by:

$$\bar{\gamma} = (a_1, k_2^1, \dots, k_2^{l_2}, a_3, k_1^1, \dots, k_1^{l_1}, a_2, k_3^1, \dots, k_3^{l_3})$$

Proof. In order to prove that $\bar{\mathfrak{m}}$ is a well-defined unicellular map, it suffices to check that its face is given by the long cycle $\bar{\gamma}$ given in the lemma. To check that this is true, it is enough to notice that the only half-edges whose image is not the same in γ and in $\bar{\gamma}$ are the three half-edges a_1, a_2, a_3 , and that by construction $\bar{\gamma}(a_i) = \alpha\bar{\sigma}(a_i) = \alpha\sigma(a_{i+1}) = \gamma(a_{i+1})$. For a more "visual" explanation, see Figure 2(b).

Now, by construction, $\bar{\mathfrak{m}}$ has two less vertices than \mathfrak{m} , and the same number of edges, so from Euler's formula it has genus $g + 1$ (the gluing operation has created a new "handle"). \square

2.3 Locating the intertwinings of the map, and the slicing operation.

In this paper, we will show that all unicellular maps of genus $g + 1$ can be obtained from unicellular maps of genus g by the gluing operation defined above, and that in some sense this operation can be performed *in a canonical way*. Our first step in this direction is to observe that, in the map $\bar{\mathfrak{m}}$ obtained after the gluing operation, the new vertex \bar{v} is not *any* vertex of the map: it satisfies some very special properties. Namely, in the unicellular map $\bar{\mathfrak{m}}$, the three half-edges a_1, a_2, a_3 appear in that order counterclockwise around the vertex \bar{v} , whereas they appear in the inverse order in the face $\bar{\gamma}$. Note that this is very different from what we observe in the planar case: the tour of a plane tree (performed with edges on the left) necessarily visits the different half-edges around each vertex in *counterclockwise order*, as illustrated in Figure 3. This leads us to the intuition (which we will

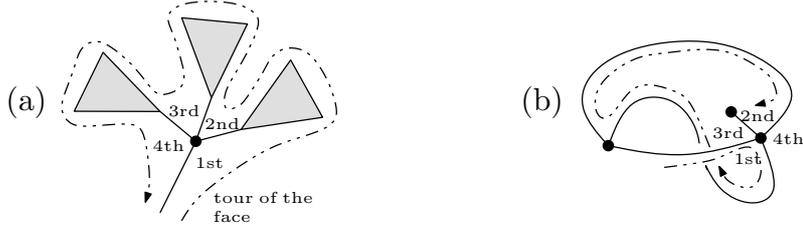


Figure 3: (a) In a plane tree, the tour of the face always visits the half-edges around one vertex in *counterclockwise* order; (b) in positive genus (here in genus 1), things can be different.

soon make more precise) that, in a map of positive genus, those vertices where the vertex-order does not coincide with the face-order hide some “intertwining” (some handle) of the map, and that they may be used to inverse the gluing operation.

We let $\bar{\mathbf{m}} = (H, \alpha, \bar{\sigma})$ be a map of genus $g + 1$, and three half-edges a_1, a_2, a_3 belong to a same vertex \bar{v} of $\bar{\mathbf{m}}$. We say that a_1, a_2, a_3 are *intertwined* if they do not appear in the same order in $\bar{\gamma} = \alpha\bar{\sigma}$ and in $\bar{\sigma}$. In this case, we write $\bar{v} = (a_1, h_2^1, \dots, h_2^{m_2}, a_2, h_3^1, \dots, h_3^{m_3}, a_3, h_1^1, \dots, h_1^{m_1})$, and we let σ be the permutation of H obtained from $\bar{\sigma}$ by replacing the cycle \bar{v} by the product $(a_1, h_1^1, \dots, h_1^{m_1})(a_2, h_2^1, \dots, h_2^{m_2})(a_3, h_3^1, \dots, h_3^{m_3})$. The transformation mapping $\bar{\gamma}$ to γ can be interpreted as the *slicing* of the vertex \bar{v} , as in Figure 2(a).

Lemma 2. *The triple $\mathbf{m} = (H, \alpha, \sigma)$ obtained after the slicing of the three intertwined half-edges a_1, a_2, a_3 is a unicellular map of genus g . If we let*

$$\bar{\gamma} = (a_1, k_2^1, \dots, k_2^{l_2}, a_3, k_1^1, \dots, k_1^{l_1}, a_2, k_3^1, \dots, k_3^{l_3})$$

be the unique face of $\bar{\mathbf{m}}$, then the unique face of \mathbf{m} is given by:

$$\gamma = \alpha\sigma = (a_1, k_1^1, \dots, k_1^{l_1}, a_2, k_2^1, \dots, k_2^{l_2}, a_3, k_3^1, \dots, k_3^{l_3}).$$

The gluing and slicing operations are inverse one to the other.

Proof. The proof is the same as in Lemma 1: it is sufficient to check the expression given for γ in terms of $\bar{\gamma}$, which is easily done by checking the images of a_1, a_2, a_3 . \square

A priori it is not obvious that the slicing operation results in a connected graph, but the previous lemma shows that it indeed does, since the underlying graph of a unicellular map is always connected. Of course this property would not necessarily hold if the three half-edges a_1, a_2, a_3 were not intertwined in the original map $\bar{\mathbf{m}}$. It is possible to show (but of no use for the present paper) that the slicing operation applied to three non-intertwined half-edges in a unicellular map of genus $g + 1$ results either in a disconnected map, or in a map of genus $g - 1$ with three faces.

2.4 Around one vertex: up-steps, down-steps, and trisections.

Let $\mathbf{m} = (H, \alpha, \sigma)$ be a map of face permutation $\gamma = \alpha\sigma$. For each vertex v of \mathbf{m} , we let $\min_{\mathbf{m}}(v)$ be the minimal half-edge belonging to v , for the order $<_{\mathbf{m}}$. Equivalently, $\min_{\mathbf{m}}(v)$ is the first half-edge from which one reaches v during the tour of the map, starting from the root. Given a half-edge $h \in H$, we note $V(h)$ the unique vertex it belongs to (i.e. the cycle of σ containing it).

Definition 2. We say that a half-edge $h \in H$ is an *up-step* if $h <_{\mathfrak{m}} \sigma(h)$, and that it is a *down-step* if $\sigma(h) \leq_{\mathfrak{m}} h$. A down-step h is called a *trisection* if $\sigma(h) \neq \min_{\mathfrak{m}} V(h)$, i.e. if $\sigma(h)$ is not the minimum half-edge inside its vertex.

The fact that, in a plane tree, the vertex-order and the face-order always coincide (Figure 3(a)) implies that trisections are specific to the non-planar case: there are no trisections in a plane tree. This observation is actually a very special case of the following lemma, which is the cornerstone of this paper:

Lemma 3 (The trisection lemma). *Let \mathfrak{m} be a unicellular map of genus g . Then \mathfrak{m} has exactly $2g$ trisections.*

Proof. We let $\mathfrak{m} = (H, \alpha, \sigma)$, and $\gamma = \alpha\sigma$. We let n_+ and n_- denote the number of up-steps and down-steps in \mathfrak{m} , respectively. Then, we have $n_- + n_+ = 2n$, where n is the number of edges of \mathfrak{m} . Now, let i be a half-edge of \mathfrak{m} , and $j = \sigma^{-1}\alpha\sigma(i)$. Note that we have $\sigma(j) = \gamma(i)$, and $\gamma(j) = \sigma(i)$. Graphically, i and j lie in two "opposite" corners of the same edge, as shown in Figure 4. On the picture, it seems clear that if the tour of the map visits i before $\sigma(i)$, then it necessarily visits $\sigma(j)$ before j (except if the root is one of these four half-edges) so that, roughly, there must be almost the same number of up-steps and down-steps. More precisely, let us distinguish three cases.

First, assume that i is an up-step. Then we have $i <_{\mathfrak{m}} \sigma(i) = \gamma(j)$. Now, by definition of the total order $<_{\mathfrak{m}}$, $i <_{\mathfrak{m}} \gamma(j)$ implies that $\gamma(i) \leq_{\mathfrak{m}} \gamma(j)$. Hence, $\sigma(j) \leq_{\mathfrak{m}} \gamma(j)$, which, by definition of $<_{\mathfrak{m}}$ again, implies that $\sigma(j) \leq_{\mathfrak{m}} j$ (here, we have used that $\sigma(j) \neq \gamma(j)$ since α has no fixed point). Hence, if i is an up-step, then j is a down-step.

Second, assume that i is a down-step, and that $\gamma(j)$ is not equal to the root of \mathfrak{m} . In this case, we have $j <_{\mathfrak{m}} \gamma(j)$, and $\gamma(j) = \sigma(i) \leq_{\mathfrak{m}} i = \sigma(j)$. Hence $j <_{\mathfrak{m}} \sigma(j)$, and j is an up-step.

The third and last case is when i is a down step, and $\gamma(j)$ is the root r of \mathfrak{m} . In this case, j is the maximum element of H for the order $<_{\mathfrak{m}}$, so that it is necessarily a down-step.

Combining the three cases we have proved that each edge of \mathfrak{m} (more precisely, each cycle of $\sigma^{-1}\alpha\sigma$) is associated to one up-step and one down-step, except the edge containing the root which is associated to two down-steps. Consequently, there are exactly two more down-steps than up-steps in the map \mathfrak{m} , i.e.: $n_- = n_+ + 2$. Recalling that $n_- + n_+ = 2n$, this gives $n_- = n + 1$.

Finally, each vertex of \mathfrak{m} carries exactly one down-step which is not a trisection (its minimal half-edge). Hence, the total number of trisections equals $n_- - v$, where v is the number of vertices of \mathfrak{m} . Since from Euler's characteristic formula, v equals $n + 1 - 2g$, the lemma is proved. \square

3 Making the gluing operation injective.

We have defined above an operation that glues a triple of half-edges, and increases the genus of a map. In this section, we explain that, if we restrict it to *certain types* of triples of half-edges, this operation can be made invertible.

3.1 A diagram representation of vertices.

We first describe a graphical way to visualize the order $<_{\mathfrak{m}}$ restricted to the half-edges belonging to a given vertex. Let v be a vertex of \mathfrak{m} , with a distinguished half-edge h . We write $v = (u_0, u_1, \dots, u_m)$, with $u_0 = h$. We now consider a grid with $m + 1$ columns and $2n$ rows. Each row represents an

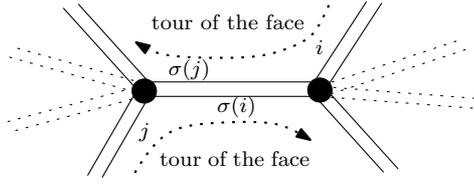


Figure 4: The main argument in the proof of the trisection lemma: the tour of the face visits i before $\sigma(i)$ if and only if it visits $\sigma(j)$ before j , unless $\sigma(i)$ or $\sigma(j)$ is the root of the map.

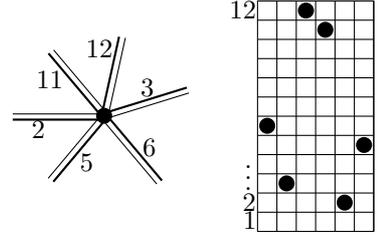


Figure 5: A vertex $(6, 3, 12, 11, 2, 5)$ in a map with 12-half-edges, and its diagram representation (the distinguished half-edge is 6).

element of H , and the rows are ordered from the bottom to the top by the total order $<_m$ (for example the lowest row represents the root). Now, for each i , inside the i -th column, we plot a point at the height corresponding to the half-edge u_i . We say that the diagram thus obtained is *the diagram representation of v , starting from h* . In other words, if we identify $\llbracket 1, 2n \rrbracket$ with H via the order $<_m$, the diagram representation of v is the graphical representation of the sequence of labels appearing around the vertex v . If one changes the distinguished half-edge h , the diagram representation of v is changed by a circular permutation of its columns. Figure 5 gives an example of such a diagram (where the permutation γ is equal to $\gamma = (1, 2, 3, \dots)$).

The gluing operation is easily visualised on diagrams. We let as before $a_1 <_m a_2 <_m a_3$ be three half-edges belonging to distinct vertices in a unicellular map \mathbf{m} , and we let $\Delta_1, \Delta_2, \Delta_3$ be their corresponding diagrams, with a_i in the first column of Δ_i . We now consider the three horizontal rows corresponding to a_1, a_2 , and a_3 : they separate each diagram Δ_i into four blocks (some of which may be empty). We give a name to each of these blocks: A_i, B_i, C_i, D_i , from bottom to top, as in Figure 6(a).

We now attach $\Delta_2, \Delta_3, \Delta_1$ together, from left to right, and we rearrange the three columns containing a_1, a_2, a_3 so that these half-edges appear in that order: we obtain a new diagram (Figure 6(b)), whose columns represent the order of the half-edges around the vertex \bar{v} . But the *rows* of that diagram are still ordered according to the order $<_m$. In order to obtain the diagram representing \bar{v} in the new map $\bar{\mathbf{m}}$, we have to rearrange the rows according to $<_{\bar{\mathbf{m}}}$. We let A be the union of the three blocks A_i (and similarly, we define B, C , and D). We know that the face permutation of \mathbf{m} has the form $\gamma = (w_A, a_1, w_B, a_2, w_C, a_3, w_D)$, where for each block $F \in \{A, B, C, D\}$, the word w_F consists of the list of the rows appearing in the block F , read from bottom to top. Now, from the expression of $\bar{\gamma}$ given in Lemma 1, the permutation $\bar{\gamma}$ is given by $\bar{\gamma} = (w_A, a_1, w_C, a_3, w_B, a_2, w_D)$. In terms of diagrams, this means that the diagram representing \bar{v} in the new map $\bar{\mathbf{m}}$ can be obtained by *swapping* the block B with the block C , and the row corresponding to a_2 with the one corresponding to a_3 : see Figure 6(c). To sum up, we have:

Lemma 4. *The diagram of the vertex \bar{v} in the map $\bar{\mathbf{m}}$ is obtained from the three diagrams $\Delta_1, \Delta_2, \Delta_3$ by the following operations, as represented in Figure 6:*

- Attach the diagrams $\Delta_2, \Delta_3, \Delta_1$ in that order from left to right, and rearrange the columns containing a_1, a_2, a_3 , so that they appear in that order from left to right.
- Exchange the blocks B and C , and swap the rows containing a_2 and a_3 .

Note that, when read from right to left, Figure 6 gives the way to obtain the diagrams of the

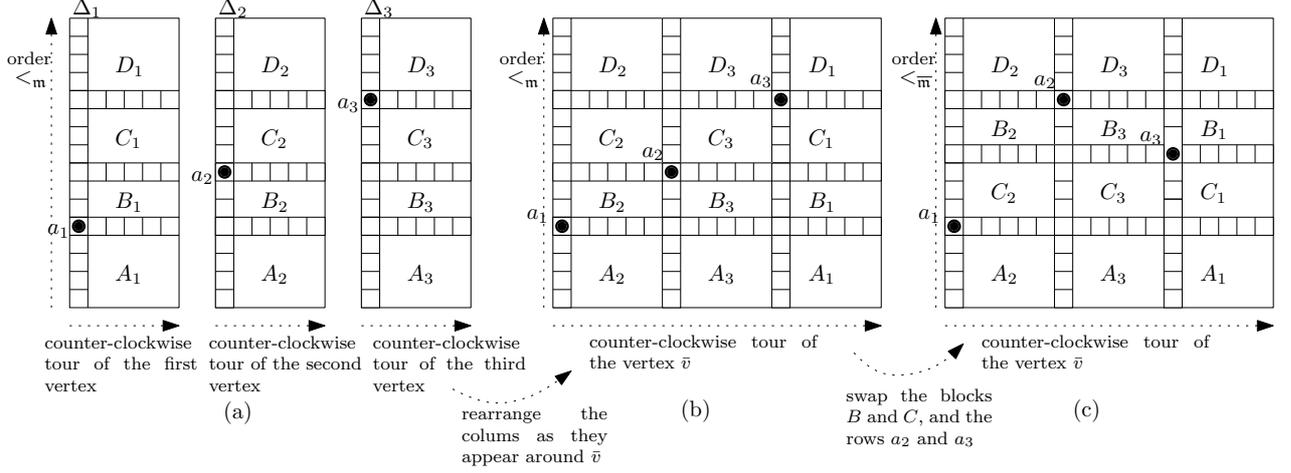


Figure 6: The gluing operation visualized on diagrams. (a) the diagrams before gluing; (b) a temporary diagram, where the columns represent the counterclockwise turn around \bar{v} , but the rows still represent the original permutation γ ; (c) the final diagram of the new vertex in the new map, where the rows represent the permutation $\bar{\gamma}$.

three vertices resulting from the *slicing* operation of three intertwined half-edges a_1, a_2, a_3 in the map $\bar{\mathbf{m}}$.

Remark 1. *The slicing operation does not change the order $\leq_{\bar{\mathbf{m}}}$ for half-edges which appear strictly between the root and the half-edge a_1 . Precisely if $w_1 <_{\bar{\mathbf{m}}} w_2 <_{\bar{\mathbf{m}}} \dots <_{\bar{\mathbf{m}}} w_r$ are elements of H such that $w_r <_{\bar{\mathbf{m}}} a_1$, then Lemma 2 (or, more visually, Figure 6) implies that we have :*

$$w_1 <_{\mathbf{m}} w_2 <_{\mathbf{m}} \dots <_{\mathbf{m}} w_r <_{\mathbf{m}} a_1$$

in the map \mathbf{m} . The converse statement is also true.

3.2 Gluing three vertices: trisections of type I.

In this section, we let v_1, v_2, v_3 be three distinct vertices in the map \mathbf{m} . We let $a_i := \min_{\mathbf{m}} v_i$, and, up to re-arranging the three vertices, we may assume (and we do) that $a_1 <_{\mathbf{m}} a_2 <_{\mathbf{m}} a_3$. We let $\Delta_1, \Delta_2, \Delta_3$ be the three corresponding diagrams. Since in each diagram the distinguished half-edge is the minimum in its vertex, note that the blocks $A_1, A_2, B_2, A_3, B_3, C_3$ do not contain any point. We say that they are *empty*, and we write: $A_1 = A_2 = B_2 = A_3 = B_3 = C_3 = \emptyset$.

We now glue the three half-edges a_1, a_2, a_3 in \mathbf{m} : we obtain a new unicellular map $\bar{\mathbf{m}}$, with a new vertex \bar{v} resulting from the gluing. Now, let τ be the element preceding a_3 around \bar{v} in the map $\bar{\mathbf{m}}$. Since $A_3 = B_3 = C_3 = \emptyset$, we have either $\tau \in D_3$ or $\tau = a_2$, so that in both cases $a_3 <_{\bar{\mathbf{m}}} \tau$. Moreover, a_3 is not the minimum inside its vertex (the minimum is a_1). Hence, τ is a *trisection* of the map $\bar{\mathbf{m}}$. We let $\Phi(\mathbf{m}, v_1, v_2, v_3) = (\bar{\mathbf{m}}, \tau)$ be the pair formed by the new map $\bar{\mathbf{m}}$ and the newly created trisection τ .

It is clear that given $(\bar{\mathbf{m}}, \tau)$, we can inverse the gluing operation. Indeed, it is easy to recover the three half-edges a_1 (the minimum of the vertex), a_3 (the one that follows τ), and a_2 (note that, since B_2 and B_3 are empty, a_2 is the smallest half-edge on the left of a_3 which is greater than a_3).

Once a_1, a_2, a_3 are recovered, it is easy to recover the map \mathfrak{m} by *slicing* \bar{v} at those three half-edges. This gives:

Lemma 5. *The mapping Φ , defined on the set of unicellular maps with three distinguished vertices, is injective.*

It is now natural to study the image of the mapping Φ : in particular, can we obtain all pairs $(\bar{\mathfrak{m}}, \tau)$ in this way? In order to answer this question, we need the following definition (see Figure 7):

Definition 3. Let $\bar{\mathfrak{m}} = (H, \alpha, \bar{\sigma})$ be a map of genus $g + 1$, and τ be a trisection of $\bar{\mathfrak{m}}$. We let $\bar{v} = V(\tau)$, $b_1 = \min_{\bar{\mathfrak{m}}}(\bar{v})$, and we let Δ be the diagram representation of \bar{v} , starting from the half-edge b_1 . We let $b_3 = \sigma(\tau)$ be the half-edge following τ around \bar{v} , and we let b_2 be the minimum half-edge among those which appear before b_3 around \bar{v} and which are greater than b_3 for the order $<_{\bar{\mathfrak{m}}}$. Note that b_2 is well-defined because τ is one of these half-edges. The rows and columns containing b_1, b_2, b_3 split the diagram Δ into twelve blocks, five of which are necessarily empty, as in Figure 7. We let K be second-from-left and second-from-bottom block. We say that τ is a *trisection of type I* if K is empty, and that τ is a *trisection of type II* otherwise.

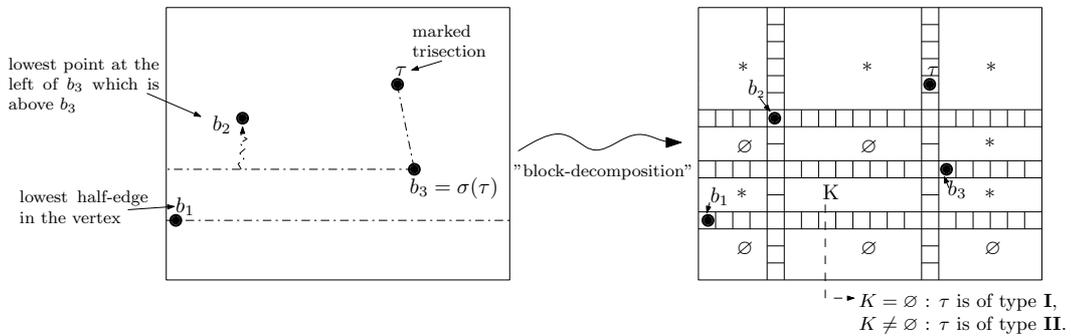


Figure 7: Trisections of type I and II.

The following proposition is the half way to our main result:

Proposition 1. *The mapping Φ is a bijection between the set $\mathcal{U}_g^3(n)$ of unicellular maps of genus g with n edges and three distinguished vertices, and the set $\mathcal{D}_{g+1}^I(n)$ of unicellular maps of genus $g + 1$ with n edges and a distinguished trisection of type I.*

Proof. We already know that Φ is injective.

We let \mathfrak{m} be a unicellular map of genus g with three distinguished vertices v_1, v_2, v_3 , and $\bar{\mathfrak{m}}$ be the map obtained, as above, by the gluing of M by the half-edges $a_1 = \min_{\mathfrak{m}} v_1$, $a_2 = \min_{\mathfrak{m}} v_2$, $a_3 = \min_{\mathfrak{m}} v_3$ (we assume again that $a_1 <_{\mathfrak{m}} a_2 <_{\mathfrak{m}} a_3$). We let $\bar{\Delta}$ be the diagram representation of the new vertex \bar{v} obtained from the gluing in the map $\bar{\mathfrak{m}}$, and we use the same notations for the blocks as in Section 3.1. We also let $\tau = \sigma^{-1}(a_3)$ be the created trisection, and we use the notations of Definition 3 with respect to the trisection τ , so that $b_3 = a_3$. Then, since $a_1 = \min_{\bar{\mathfrak{m}}} \bar{v}$, we have $a_1 = b_1$, and since the blocks B_2, B_3 , are empty, we have $b_2 = a_2$. Hence, the block C_3 of Figure 6(c) coincides with the block K of Figure 7. Since C_3 is empty, τ is a trisection of type I. Therefore the image of Φ is included in $\mathcal{D}_{g+1}^I(n)$.

Conversely, let $\bar{\mathbf{m}} = (H, \alpha, \bar{\sigma})$ be a map of genus $g + 1$, and τ be a trisection of type **I** in $\bar{\mathbf{m}}$. We let b_1, b_2, b_3 and K be as in Definition 3. First, since $b_1 <_{\bar{\mathbf{m}}} b_3 <_{\bar{\mathbf{m}}} b_2$, these half-edges are intertwined, and we know that the slicing of $\bar{\mathbf{m}}$ by these half-edges creates a well-defined unicellular map \mathbf{m} of genus g (Lemma 2). Now, if we compare Figures 7 and 6, we see that the result of the slicing is a triple of vertices v_1, v_2, v_3 , such that each half-edge b_i is the minimum in the vertex v_i : indeed, the blocks A_1, A_2, A_3, B_2, B_3 are empty by construction, and the block $C_3 = K$ is empty since τ is a trisection of type **I**. Hence we have $\Phi(\mathbf{m}, v_1, v_2, v_3) = (\bar{\mathbf{m}}, \tau)$, so that the image of Φ exactly equals the set $\mathcal{D}_{g+1}^I(n)$. \square

3.3 Trisections of type **II**.

Of course, it would be nice to have a similar result for trisections of type **II**. Let $\bar{\mathbf{m}} = (H, \alpha, \bar{\sigma})$ be a map of genus $g + 1$ with a distinguished trisection τ of type **II**. We let b_1, b_2, b_3 and K be as in Definition 3 and Figure 7, and we let \mathbf{m} be the result of the slicing of $\bar{\mathbf{m}}$ at the three half-edges b_1, b_2, b_3 . If we use the notations of Figure 6, with $a_i = b_i$, we see that we obtain three vertices, of diagrams $\Delta_1, \Delta_2, \Delta_3$, such that $A_1 = A_2 = B_2 = A_3 = B_3 = \emptyset$. Hence, we know that $a_1 = \min_{\mathbf{m}}(v_1)$, that $a_2 = \min_{\mathbf{m}}(v_2)$, and that $a_2 < \min_{\mathbf{m}}(v_3)$. Note that, in contrast to what happened in the previous section, the block $C_3 = K$ is *not* empty, therefore a_3 is *not* the minimum inside its vertex.

Now, we claim that τ is still a trisection in the map \mathbf{m} . Indeed, by construction, we know that τ belongs to D_3 (since, by definition of a trisection, it must be above a_3 in the map $\bar{\mathbf{m}}$, and since B_3 is empty). Hence we still have $a_3 <_{\mathbf{m}} \tau$ in the map \mathbf{m} . Moreover, we have clearly $\sigma(\tau) = a_3$ in \mathbf{m} (since τ is the rightmost point in the blocks $C_3 \cup D_3$), and it follows that τ is a trisection in \mathbf{m} .

We let $\Gamma(\bar{\mathbf{m}}, \tau) = (\mathbf{m}, v_1, v_2, \tau)$ be the 4-tuple consisting of the new map \mathbf{m} , the first two vertices v_1 and v_2 obtained from the slicing, and the trisection τ . It is clear that Γ is injective: given $(\bar{\mathbf{m}}, v_1, v_2, \tau)$, one can reconstruct the map $\bar{\mathbf{m}}$ by letting $a_1 = \min v_1$, $a_2 = \min v_2$, and $a_3 = \sigma(\tau)$, and by gluing back together the three half-edges a_1, a_2, a_3 .

Conversely, we introduce the following notation:

Notation 1. We let $\mathcal{V}_g(n)$ be the set of 4-tuples $(\mathbf{m}, v_1, v_2, \tau)$, where \mathbf{m} is a unicellular map of genus g with n edges, and where v_1, v_2 , and τ are respectively two vertices and a trisection of M such that:

$$\min_{\mathbf{m}} v_1 <_{\mathbf{m}} \min_{\mathbf{m}} v_2 <_{\mathbf{m}} \min_{\mathbf{m}} V(\tau). \quad (7)$$

Given $(\mathbf{m}, v_1, v_2, \tau) \in \mathcal{V}_g(n)$, we let $\bar{\mathbf{m}}$ be the map obtained from the gluing of the three half-edges $\min v_1, \min v_2$, and $\sigma(\tau)$, and we let $\Psi(\mathbf{m}, v_1, v_2, \tau) := (\bar{\mathbf{m}}, \tau)$.

We can now state the following proposition, that completes Proposition 1:

Proposition 2. *The mapping Ψ is a bijection between the set $\mathcal{V}_g(n)$ of unicellular maps of genus g with n edges and a distinguished triple (v_1, v_2, τ) satisfying Condition (7), and the set $\mathcal{D}_{g+1}^{II}(n)$ of unicellular maps of genus $g + 1$ with n edges and a distinguished trisection of type **II**.*

Proof. In the discussion above, we have already given a mapping $\Gamma : \mathcal{D}_{g+1}^{II}(n) \rightarrow \mathcal{V}_g(n)$, such that $\Psi \circ \Gamma$ is the identity on $\mathcal{D}_{g+1}^{II}(n)$.

Conversely, let $(\mathbf{m}, v_1, v_2, \tau) \in \mathcal{V}_g(n)$, and let $a_1 = \min v_1$, $a_2 = \min v_2$, and $a_3 = \sigma(\tau)$. By definition, we know that $a_2 < \min V(\tau)$, so that in the diagram representation of the three vertices

$v_1, v_2, V(\tau)$ (Figure 6(a)) we know that the blocks A_1, A_2, A_3, B_2, B_3 are empty. Moreover, since τ is a trisection, a_3 is not the minimum inside its vertex, so the block C_3 is *not* empty. Hence, comparing Figures 6(c) and 7, and observing once again that the blocks C_3 and K coincide, we see that after the gluing, τ is a trisection of type **II** in the new map $\bar{\mathbf{m}}$. Moreover, since the slicing and gluing operations are inverse one to each other, it is clear that $\Gamma(\bar{\mathbf{m}}, \tau) = (\mathbf{m}, v_1, v_2, \tau)$. Hence, $\Gamma \circ \Psi$ is the identity, and the proposition is proved. \square

4 Iterating the bijection.

Clearly Proposition 1 looks nicer than its counterpart Proposition 2: in the first one, one only asks to distinguish three vertices in a map of lower genus, whereas in the second one, the distinguished triple must satisfy a nontrivial constraint (Condition (7)). In this section we will work a little more in order to get rid of this difficulty. We start by introducing the following notations.

Notation 2. We let $\mathcal{U}_g^k(n)$ be the set of unicellular maps of genus g with n edges and k distinct distinguished vertices. Note that for $k = 3$ this is coherent with the previous notation.

Notation 3. We let $\mathcal{D}_g(n) = \mathcal{D}_g^I(n) \cup \mathcal{D}_g^{II}(n)$ be the set of unicellular maps of genus g with n edges and a distinguished trisection.

4.1 Examples: genera 1 and 2.

Note that the set $\mathcal{V}_0(n)$ is empty, since there are no trisections in a plane tree. Hence, from Proposition 2, there are no trisections of type **II** in a map of genus 1 (i.e. $\mathcal{D}_1^{II}(n) = \emptyset$). Proposition 1 therefore implies:

Corollary 1. *The set $\mathcal{D}_1(n)$ of unicellular maps of genus 1 with n edges and a distinguished trisection is in bijection with the set $U_0^3(n)$ of rooted plane trees with n edges and three distinguished vertices.*

Since from the trisection lemma (Lemma 3) each unicellular map of genus 1 has exactly 2 trisections, we obtain that the number $\epsilon_1(n)$ of rooted unicellular maps of genus 1 with n edges satisfies:

$$2 \cdot \epsilon_1(n) = \binom{n+1}{3} \text{Cat}(n),$$

which gives a clear combinatorial proof of the formula $\epsilon_1(n) = \frac{(n+1)n(n-1)}{12} \text{Cat}(n)$ [WL72].

We now consider the case of genus 2. Let \mathbf{m} be a unicellular map of genus 2, and τ be a trisection of \mathbf{m} . If τ is of type **I**, we know that we can use the mapping Φ^{-1} , and obtain a unicellular map of genus 1, with three distinguished vertices.

Similarly, if τ is of type **II**, we can apply the mapping Ψ^{-1} , and we are left with a unicellular map \mathbf{m}' of genus 1, and a marked triple (v_1, v_2, τ) , such that $\min_{\mathbf{m}'} v_1 <_{\mathbf{m}'} \min_{\mathbf{m}'} v_2 <_{\mathbf{m}'} \min_{\mathbf{m}'} V(\tau)$. From now on, we use the more compact notation: $v_1 <_{\mathbf{m}'} v_2 <_{\mathbf{m}'} V(\tau)$, i.e. we do not write the min's anymore. The map (\mathbf{m}', τ) is a unicellular map of genus 1 with a distinguished trisection: therefore we can apply the mapping Φ^{-1} to (\mathbf{m}', τ) . We obtain a plane tree \mathbf{m}'' , with three distinguished vertices v_3, v_4, v_5 inherited from the slicing of τ in \mathbf{m}' ; up to renaming the vertices we can assume that

$v_3 <_{\mathbf{m}''} v_4 <_{\mathbf{m}''} v_5$. Note that in \mathbf{m}'' we also have the two marked vertices v_1 and v_2 inherited from the slicing of τ in \mathbf{m} . Moreover the fact that $v_1 <_{\mathbf{m}'} v_2 <_{\mathbf{m}'} V(\tau)$ in \mathbf{m}' implies that $v_1 <_{\mathbf{m}''} v_2 <_{\mathbf{m}''} v_3$ in \mathbf{m}'' , as follows from Remark 1. Hence, we are left with a plane tree \mathbf{m}'' , with *five* distinguished vertices $v_1 <_{\mathbf{m}''} v_2 <_{\mathbf{m}''} v_3 <_{\mathbf{m}''} v_4 <_{\mathbf{m}''} v_5$. Conversely, given such a 5-tuple of vertices, it is always possible to glue the three last ones together by the mapping Φ to obtain a triple (v_1, v_2, τ) satisfying Condition (7), and then to apply the mapping Ψ to obtain a map of genus 2 with a marked trisection of type **II**. This gives:

Corollary 2. *The set $\mathcal{D}_2^{II}(n)$ is in bijection with the set $\mathcal{U}_0^5(n)$ of plane trees with five distinguished vertices. The set $\mathcal{D}_2(n)$ of unicellular maps of genus 2 with one marked trisection is in bijection with the set $\mathcal{U}_1^3(n) \cup \mathcal{U}_0^5(n)$.*

Now, from Euler's formula, a unicellular map of genus 1 with n edges has $n - 1$ vertices, so that $|\mathcal{U}_1^3(n)| = \binom{n-1}{3} \epsilon_1(n)$. Moreover, since from the trisection lemma each unicellular map of genus 2 has 4 trisections, we obtain the following formula for the number $\epsilon_2(n)$ of unicellular maps of genus 2 with n edges:

$$4 \cdot \epsilon_2(n) = \binom{n-1}{3} \epsilon_1(n) + \binom{n+1}{5} \text{Cat}(n),$$

from which it follows that

$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440} \text{Cat}(n).$$

4.2 The general case, and our main theorem.

In the general case, we will work as in the example of genus 2: starting with a map and a distinguished trisection, we apply recursively the opening operation on trisections of type **II**, and we stop when we have opened the first encountered trisection of type **I**. We start with the description of the inverse procedure, which goes as follows.

We let $p \geq 0$ and $q \geq 1$ be two integers, and $(\mathbf{m}, v_*) = (\mathbf{m}, v_1, \dots, v_{2q+1})$ be an element of $\mathcal{U}_p^{2q+1}(n)$. Up to renumbering the vertices, we can assume that $v_1 <_{\mathbf{m}} v_2 <_{\mathbf{m}} \dots <_{\mathbf{m}} v_{2q+1}$. We consider the following procedure:

Algorithm 1 (construction of the mapping Λ).

i. Glue the three last vertices $v_{2q-1}, v_{2q}, v_{2q+1}$ together, via the mapping Φ , in order to obtain a new map \mathbf{m}_1 of genus $p + 1$ with a distinguished trisection τ of type **I**.

ii. for i from 1 to $q - 1$ **do:**

Let $(v_{2q-2i-1}, v_{2q-2i}, \tau)$ be the triple consisting of the last two vertices which have not been used until now, and the trisection τ . Apply the mapping Ψ to that triple, in order to obtain a new map \mathbf{m}_{i+1} of genus $p + i + 1$, with a distinguished trisection τ of type **II**.

end for.

We let $\Lambda(\mathbf{m}, v_*) := (\mathbf{m}_q, \tau)$ be the map with a distinguished trisection obtained at the end of this procedure. Note that if $q = 1$, the distinguished trisection is of type **I**, and that it is of type **II** otherwise.

As in the case of genus 2, we have the following theorem:

Theorem 1 (Our main result). *The application Λ defines a bijection:*

$$\Lambda : \bigsqcup_{p=0}^{g-1} \mathcal{U}_p^{2g-2p+1}(n) \longrightarrow \mathcal{D}_g(n).$$

In other words, all unicellular maps of genus g with a distinguished trisection can be obtained in a canonical way by starting with a map of a lower genus with an odd number of distinguished vertices, and then applying once the mapping Φ , and a certain number of times the mapping Ψ .

Note that in the statement of the theorem the disjoint-union sign is used only to emphasize the fact that the sets whose union is taken are disjoint. This sign will always have this meaning in the rest of the paper.

Given a map with a marked trisection (\mathbf{m}, τ) , the converse application consists in slicing recursively the trisection τ while it is of type **II**, then slicing *once* the obtained trisection of type **I**, and remembering all the vertices resulting from the successive slicings. More formally, we have the following proposition:

Proposition 3. *Let $(\mathbf{m}, \tau) \in \mathcal{D}_g(n)$. We define the pair $\Xi(\mathbf{m}, \tau)$ by the following procedure:*

1. We let $\mathbf{m}_0 := \mathbf{m}$ and $i := 0$.
2. **If τ is of type **II** in \mathbf{m}_i , we let $(\mathbf{m}_{i+1}, v_{2i+1}, v_{2i+2}) := \Psi^{-1}(\mathbf{m}_i, \tau)$. Then we let $i := i + 1$ and we return to step 2.**
Else, τ is of type **I in \mathbf{m}_i and we go to step 3.**
3. Let $(\mathbf{m}_{i+1}, v_{2i+1}, v_{2i+2}, v_{2i+3}) := \Phi^{-1}(\mathbf{m}_i, \tau)$. We end the procedure and we let

$$\Xi(\mathbf{m}, \tau) := (\mathbf{m}_{i+1}, v_1, v_2, \dots, v_{2i+3}).$$

Then the mapping

$$\Xi : \mathcal{D}_g(n) \longrightarrow \bigsqcup_{p=0}^{g-1} \mathcal{U}_p^{2g-2p+1}(n).$$

is a bijection, which is the inverse bijection of Λ .

Proof of Theorem 1 and Proposition 3. First, the mapping Ξ is well defined. Indeed, by definition of a trisection of type **II**, we know by induction that each time we enter steps 2 and 3, τ is a trisection of the map \mathbf{m}_i . Moreover, since the genus of the maps \mathbf{m}_i decreases with i , we necessarily reach step 3, and the procedure stops.

Then, the mapping Λ is clearly injective, since the mappings Ψ and Φ are.

Finally, to prove at the same time that Ξ is injective and that it is the inverse mapping of Λ , it is enough to show that the vertices v_i produced by the procedure defining Ξ satisfy $v_1 <_{\mathbf{m}} v_2 <_{\mathbf{m}} \dots <_{\mathbf{m}} v_{2q+1}$. Indeed, after that it will be clear by construction that $\Lambda \circ \Xi = \Xi \circ \Lambda = Id$. Now, we deduce from Remark 1 and by an induction on i that after the i th passage in step 2 in the definition of Ξ , we have $v_1 <_{\mathbf{m}_{i+1}} v_2 <_{\mathbf{m}_{i+1}} \dots <_{\mathbf{m}_{i+1}} v_{2i+2}$. The same remark shows that at the end of step 3, we have $v_1 <_{\mathbf{m}_{i+1}} v_2 <_{\mathbf{m}_{i+1}} \dots <_{\mathbf{m}_{i+1}} v_{2i+3}$, which concludes the proof. \square

5 Enumerative corollaries.

5.1 A combinatorial identity

Using the trisection lemma (Lemma 3), Euler's formula, and Theorem 1, we obtain the following new identity (stated in the introduction as Formula (4)):

Theorem 2. *The number $\epsilon_g(n)$ of rooted unicellular maps of genus g with n edges satisfies the following combinatorial identity:*

$$2g \cdot \epsilon_g(n) = \binom{n+1-2(g-1)}{3} \epsilon_{g-1}(n) + \binom{n+1-2(g-2)}{5} \epsilon_{g-2}(n) + \cdots + \binom{n+1}{2g+1} \epsilon_0(n).$$

Note that this identity is recursive only in the genus (the number of edges n is fixed). For that reason, it enables one to compute easily, for a fixed g , the closed formula giving $\epsilon_g(n)$ by a simple iteration (as we did for genera 1 and 2).

5.2 The polynomials $R_g(n)$

Theorem 1 implies by induction that all unicellular maps of genus g with n edges can be obtained from a plane tree with n edges, by successively gluing vertices together. From the enumeration viewpoint, we obtain the first bijective proof that the numbers $\epsilon_g(n)$ are the product of a polynomial and a Catalan number:

Corollary 3. *The number $\epsilon_g(n)$ of unicellular maps of genus g with n edges equals:*

$$\epsilon_g(n) = R_g(n) \text{Cat}(n),$$

where R_g is the polynomial of degree $3g$ defined by the formula:

$$R_g(n) = \sum_{0=g_0 < g_1 < \cdots < g_r = g} \prod_{i=1}^r \frac{1}{2g_i} \binom{n+1-2g_{i-1}}{2(g_i - g_{i-1}) + 1}.$$

Proof. The statement directly comes from an iteration of the bijection of Theorem 1. More precisely, the formula given here for $R_g(n)$ reads as follows. In order to generate a unicellular map of genus g , we start with a plane tree with n edges, and we apply a certain number of times (say r) the mapping Λ to create unicellular maps of increasing genera. In the formula, $g_1 < \cdots < g_r = g$ are the genera of the maps produced by the successive applications of Λ . Now, in order to increase the genus from g_{i-1} to g_i , we have to choose $2(g_i - g_{i-1}) + 1$ vertices in a unicellular map of genus g_{i-1} , which gives the binomial in the formula. The factor $1/(2g_i)$ is here to compensate the multiplicity in the construction coming from the trisection lemma (Lemma 3). \square

From Theorem 2 and the fact that $\text{Cat}(n)$ is asymptotically equivalent to $\frac{1}{\sqrt{\pi}} n^{-\frac{3}{2}} 4^n$, one obtains easily the asymptotic behaviour of $\epsilon_g(n)$, already computed in [BCR88]:

Corollary 4 ([BCR88]). *The polynomial $R_g(n)$ has degree $3g$ and leading coefficient $\frac{1}{12^g g!}$. When n tends to infinity, one has:*

$$\epsilon_g(n) \sim \frac{1}{12^g g! \sqrt{\pi}} n^{3g - \frac{3}{2}} 4^n.$$

Our construction also answers a question of Zagier concerning the interpretation of a property of the polynomials R_g :

Corollary 5 (Zagier [LZ04, p. 160]). *For each integer $g \geq 1$, the polynomial $R_g(n)$ is divisible by $(n+1) \dots (n+1-2g)$.*

Proof. In short, the divisibility property comes from the fact that, in order to construct a unicellular map of genus g from a plane tree, one always needs to involve at least $2g+1$ vertices of the tree in one of the successive gluings. A more detailed proof goes as follows.

As explained in the proof of Corollary 3, one can construct all unicellular maps of genus g by first choosing an integer $r > 0$ and a sequence of genera $0 = g_0 < g_1 < \dots < g_r = g$, and then applying several times the gluing operation. We let \mathbf{m}_i be the map of genus g_i obtained after the i -th gluing, \mathbf{m}_0 being the original tree, and for $0 \leq i < r$ we let X_i be the subset of vertices of size $2(g_{i+1} - g_i) + 1$ which is distinguished in \mathbf{m}_i to apply the gluing operation. The set $\cup_{i=0}^{r-1} X_i$ naturally projects onto a subset S of vertices of the tree, namely the set of vertices which will be involved, at some step, in a gluing operation. Note that for each i the vertex resulting from the i -th gluing may be again an element of the set X_j for some $j > i$, so it is not possible to determine the size of S knowing only r and the g_i 's. However, it is easy to see that $2g+1 \leq |S| \leq 3g$.

Now, it is possible to express the number of unicellular maps of fixed genus g and size n as a (finite) sum over the possible values of $|S|$. Clearly, for fixed $s \geq 2g+1$, the number of s -element subsets of vertices of \mathbf{m}_0 is $\binom{n+1}{s}$, which is divisible by $(n+1) \dots (n+1-2g)$. Now, the contribution of such a set S to the polynomial R_g can be expressed as the sum of the quantity $\prod_i \frac{1}{2g_i}$ over all possible ways to choose numbers $0 < g_1 < \dots < g_r = g$, all possible ways to choose sets $(X_i)_{0 \leq i < r}$ such that $|X_i| = 2(g_{i+1} - g_i) + 1$ and that the projection of $\cup_{i=0}^{r-1} X_i$ on \mathbf{m}_0 is S . All the sums being finite, the contribution of the set S is some rational number (some weight), depending only on $|S|$ and g , but not on n , which concludes the proof. \square

6 Variants.

6.1 Bipartite unicellular maps

A unicellular map is *bipartite* if one can color its vertices in black and white in such a way that only vertices of different colors are linked by an edge. All bipartite unicellular maps considered in this paper will be rooted, and by convention the root vertex (the vertex containing the root half-edge) will always be colored in white.

Notation 4. We let $\beta_g(i, j)$ be the number of bipartite unicellular maps of genus g with i white vertices and j black vertices. Such a map has $i + j + 2g - 1$ edges.

It is clear that that our construction also applies to bipartite unicellular maps: the only difference is that, if we want the gluing operations Φ and Ψ to preserve the bipartition of the map, we have to paste together only vertices *of the same color*. We therefore obtain the following variant of our main identity:

Proposition 4. *The number $\beta_g(i, j)$ of bipartite unicellular maps with i white vertices and j black vertices obey the following recursion formula:*

$$2g \cdot \beta_g(i, j) = \sum_{p=0}^{g-1} \binom{i+2g-2p}{2g-2p+1} \beta_p(i+2g-2p, j) + \sum_{p=0}^{g-1} \binom{j+2g-2p}{2g-2p+1} \beta_p(i, j+2g-2p). \quad (8)$$

Corollary 6. We have $\beta_g(i, j) = S_g(i, j)\beta_0(i, j)$, where $\beta_0(i, j) = \frac{i+j-1}{ij} \binom{i+j-2}{i-1}^2$ is the number of bipartite plane trees with i white vertices and j black vertices computed in [GJ83], and S_g is the polynomial in (i, j) defined by:

$$S_g(i, j) = \frac{1}{2g} \sum_{p=0}^{g-1} \binom{i+2g-2p}{2g-2p+1} S_p(i+2g-2p, j) + \frac{1}{2g} \sum_{p=0}^{g-1} \binom{j+2g-2p}{2g-2p+1} S_p(i, j+2g-2p)$$

with $S_0 = 1$.

For example for the first genera we obtain:

$$S_0 = 1 \quad ; \quad S_1(i, j) = \frac{(i+2)(i+1)i + (j+2)(j+1)j}{12} \quad ; \quad S_2(i, j) = s_2(i, j) + s_2(j, i)$$

where $s_2(i, j) = \frac{i(i+1)(i+2)(i^5+22i^4+211i^3+2i^2j+998i^2+i^2j^3+3i^2j^2+21ij^2+2248i+7ij^3+14ij+96j^2+1920+64j+32j^3)}{5760}$.

6.2 Precubic unicellular maps

A unicellular map is *precubic* if all its vertices have degree 1 or 3. In such a map, all trisections are necessarily of type **I**: indeed, a trisection of type **II** cannot appear in a vertex of degree less than 4. Therefore, each precubic map can be obtained in exactly $2g$ different ways from a *precubic* map of genus $(g-1)$ with three distinguished *leaves*. By repeating the argument g times, we see that each precubic unicellular map can be obtained in exactly $2g \cdot 2(g-1) \dots 2 = 2^g g!$ different ways from a precubic tree (a plane tree where all vertices have degrees 1 or 3), by g successive gluings of a triple of leaves.

Now, we can easily enumerate precubic trees with n edges. First, we observe that by removing a leaf from such a tree, we find a binary tree with $n-1$ edges (and n vertices). This implies that $n = 2m+1$, where m is the number of *nodes* of the binary tree, and that the number of precubic trees with n edges which are rooted on a leaf is the Catalan number $\text{Cat}(m)$. A double-counting argument then shows that those trees whose root-vertex has degree 3 are counted by the number $\frac{3m}{m+2} \text{Cat}(m)$: indeed, the number $3m \text{Cat}(m)$ counts precubic trees which are rooted at the same time on a leaf and a vertex of degree 3, and these trees can also be obtained by distinguishing one of the $(m+2)$ leaves in a tree which is rooted on a vertex of degree 3. Finally, the number of all precubic rooted trees with n edges equals $(1 + \frac{3m}{m+2}) \text{Cat}(m) = \text{Cat}(m+1)$. We thereby obtain:

Corollary 7. The number $\xi_g(n)$ of precubic unicellular maps of genus g with $n = 2m+1$ edges is given by:

$$\xi_g(n) = \frac{1}{2^g g!} \binom{m+2}{3, 3, \dots, 3, m+2-3g} \text{Cat}(m+1) = \frac{2(2m+1)(2m)!}{12^g g! (m+2-3g)! m!}.$$

Precubic unicellular maps which have no leaves necessarily have $6g-3$ edges. These objects appeared previously in the literature ([WL72, BV02], and recently in [CMS09] under the name of *dominant schemes*). We can recover their number by setting $m = 3g-2$ in the previous formula. In that case, the bijection given here reduces to the one given in our older paper [Cha10], in which the following corollary already appeared. However, we repeat it here for completeness:

Corollary 8 ([WL72]). *The number of rooted unicellular maps of genus g with all vertices of degree 3 is:*

$$\frac{2(6g-3)!}{12^g g!(3g-2)!}.$$

Dually, this number counts rooted triangulations of genus g with only one vertex.

6.3 Labelled unicellular maps.

A *labelled unicellular map* is a pair (\mathbf{m}, l) such that \mathbf{m} is a rooted unicellular map, and l is a *labelling* of the vertices of \mathbf{m} , i.e. a mapping $l : V(\mathbf{m}) \rightarrow \mathbb{Z}$ such that if v_1, v_2 are two adjacent vertices in \mathbf{m} , then $l(v_1) - l(v_2) \in \{-1, 0, 1\}$, and such that the root-vertex has label 0. These objects were introduced in the planar case by Cori and Vauquelin [CV81] who gave a bijection relating rooted planar maps to labelled trees, that is, labelled unicellular maps of null genus². This bijection was later re-interpreted by Schaeffer [Sch99], and extended to the case of positive genus by Marcus and Schaeffer [MS01]. The Marcus-Schaeffer bijection implies the following theorem:

Proposition 5 ([MS01], see also [CMS09] for the version stated here). *Let $m_g(n)$ be the number of (all, not necessarily unicellular) rooted maps of genus g with n edges, and let $l_g(n)$ be the number of labelled unicellular maps of genus g with n edges. Then one has:*

$$(n + 2 - 2g) m_g(n) = 2 l_g(n).$$

Therefore it is interesting to see what our construction becomes on labelled unicellular maps. We let $\mathcal{L}_g^{(k)}(n)$ be the set of rooted labelled unicellular maps of genus g with n edges and k distinguished vertices *of the same label*. We also let $\mathcal{DL}_g(n)$ be the set of labelled unicellular maps carrying a distinguished trisection. We have:

Corollary 9. *The application Λ induces a bijection:*

$$\Lambda : \bigoplus_{p=0}^{g-1} \mathcal{L}_p^{(2g-2p+1)}(n) \longrightarrow \mathcal{DL}_g(n).$$

Proof. The only thing to change in our construction so that the gluing of a labelled unicellular map results in a well-defined labelled unicellular map is to restrict the gluing operation to vertices of the same label, which is exactly what we do here. \square

Note that it is not easy to compute the cardinality of $\mathcal{L}_g^{(k)}(n)$: in order to compute it from $l_g(n)$, one would need nontrivial information about the distribution of labels of vertices in a randomly labelled unicellular map of genus g , or by induction, in a randomly labelled *plane tree*. Still, Corollary 9, or even its special case already presented in [Cha10], enables one to obtain some information about the asymptotic behaviour of $m_g(n)$, in connection with continuum random trees (see [Cha10]).

²More precisely, the bijection in [CV81] relates rooted planar maps to *well labelled trees*, which are labelled trees in which the labelling function l takes only nonnegative values. The same nonnegative labellings are considered in [Sch99], and in the *well labelled* unicellular maps introduced in [MS01]. It was understood later (see [CMS09]) that the nonnegativity hypothesis can be circumvented by a conjugation argument, which is why we prefer to consider general (i.e. non necessarily well labelled) labelled unicellular maps here. That being said, Corollary 9 holds *verbatim* if one replaces everywhere "labelled" unicellular maps by "well labelled" unicellular maps.

7 More computations.

We now sketch a computation inspired by Emmanuel Guitter [Gui09], that enables one to obtain the Harer-Zagier formula from our construction. For all $n \geq 1$, we let $F_n(x) = \sum_{g \geq 0} \epsilon_g(n) x^{n+1-2g}$ be the generating function of unicellular maps with n edges, where the variable x marks the number of vertices. Then we have:

Proposition 6 ([Gui09]). *For every real sequence $(a_n)_{n \geq 0}$, the formal power series $F(x, y) = \sum_{n \geq 0} a_n y^{n+1} F_n(x)$ is a solution of the difference-differential equation:*

$$2y \cdot \frac{\partial}{\partial y} F(x, y) = F(x+1, y) - F(x-1, y) \quad (9)$$

Proof. Clearly, the series $\frac{1}{2}(F_n(x+1) - F_n(x-1))$ is the generating function of unicellular maps with n edges, in which an *odd* number of vertices have been distinguished, and are no longer counted in the exponent of x . These objects are divided into two categories: either the number of distinguished vertices is ≥ 3 , or it is equal to one. By our main theorem, objects of the first kind are in bijection with unicellular maps with n edges and a distinguished trisection; objects of the second kind are unicellular maps with n edges with a distinguished vertex. Now, by the trisection lemma and Euler's formula, in each map the number of trisections plus the number of vertices equals $n+1$. Therefore we have: $\frac{1}{2}(F_n(x+1) - F_n(x-1)) = (n+1)F_n(x)$ and the proposition follows. \square

Corollary 10 ([HZ86]). *Let $a_n = \frac{2^n n!}{(2n)!}$, and let $F(x, y) = \sum_{n \geq 0} a_n y^{n+1} F_n(x)$. Then one has:*

$$F(x, y) = \frac{1}{2} \left(\frac{1+y}{1-y} \right)^x - \frac{1}{2}. \quad (10)$$

As observed in [HZ86], the Harer-Zagier formula (3) follows from (10) by writing $\left(\frac{1+y}{1-y} \right)^x = \left(1 + \frac{2y}{1-y} \right)^x$ and expanding the x -th power with the binomial theorem.

Proof. We follow [Gui09]. First, one easily checks that the function given here is a solution of Equation (9). Moreover, a solution to Equation (9) is entirely characterized by its "planar terms", i.e. by the coefficients of $x^{n+1}y^{n+1}$ for all $n \geq 0$ (think about computing the coefficients inductively via Formula (4)). Hence the only thing to check is that $\lim_{y \rightarrow 0} F\left(\frac{u}{y}, y\right)$ is equal to $\sum_{n \geq 0} a_n \text{Cat}(n) u^{n+1} = \frac{1}{2}(\exp(2u) - 1)$, which is immediate from (10). \square

We conclude with an extension of the previous computation to bipartite unicellular maps. For these maps, the ordinary generating series is not given by a simple closed formula [Adr97], and it is convenient to work with "modified" generating series. More precisely, following Adrianov [Adr97] and Zagier [Zag95], we introduce for each integer $v \geq 0$ the series $\phi_v(x) := \sum_{k \geq 1} k^v x^{k-1}$. We consider the modified generating series of bipartite unicellular maps defined as follows:

$$B_n(x, y) = \sum_{i, j \geq 0} \beta_{\frac{n+1-i-j}{2}}(i, j) \phi_i(x) \phi_j(y).$$

Note that by Euler's formula, a unicellular map with $i + j$ vertices and genus $\frac{n+1-i-j}{2}$ has n edges, so that $B_n(x, y)$ is the generating function of bipartite unicellular maps with n edges, in which a map with i white and j black vertices is given a weight $\phi_i(x)\phi_j(y)$. By studying the effect of the deletion of vertices in the context of modified generating series, one obtains the following analogue of Proposition 6.

Proposition 7. *The formal power series $B_n(x, y)$ is a solution of the differential equation:*

$$\frac{\partial}{\partial x}((1-x^2)B_n(x, y)) + \frac{\partial}{\partial y}((1-y^2)B_n(x, y)) = (n+1)B_n(x, y). \quad (11)$$

Corollary 11 ([Adr97]). *The series $B_n(x, y)$ admits the following closed form:*

$$B_n(x, y) = n! \frac{(1-xy)^{n-1}}{((1-x)(1-y))^{n+1}}. \quad (12)$$

Proof. One easily checks that the series given here is a solution of Equation (11). Now, as in the monochromatic case, a formal power series

$$C_n(x, y) = \sum_{1 \leq i+j \leq n+1} c_{\frac{n+1-i-j}{2}}(i, j) \phi_i(x) \phi_j(y)$$

which is a solution of Equation (11) is characterized by its "planar terms", i.e. by the sequence of numbers $(c_0(i, j))_{i \geq 1, j \geq 1}$. Therefore it is enough to prove that the numbers $c_0(i, j)$ corresponding to the function $C_n(x, y) := n! \frac{(1-xy)^{n-1}}{((1-x)(1-y))^{n+1}}$ are equal to the numbers $\beta_0(i, j) = \frac{i+j-1}{ij} \binom{i+j-2}{i-1}^2$.

Now, set $X = \frac{1}{1-x}$, $Y = \frac{1}{1-y}$ so that $C_n(x, y) = n! X^2 Y^2 (X + Y - 1)^{n-1}$ is a polynomial $\tilde{C}_n(X, Y)$ in X and Y . Using the fact that around $x = 1$, one has $\phi_v(x) = \frac{v!}{(1-x)^{v+1}} + O\left(\frac{1}{(1-x)^v}\right)$, one obtains that for all i, j such that $i + j = n + 1$, the coefficient of $X^{i+1} Y^{j+1}$ in the polynomial $\tilde{C}_n(X, Y)$ is $i! j! c_0(i, j)$. Therefore we have:

$$c_0(i, j) = \frac{n!}{i! j!} \times \text{Coeff}_{X^{i-1} Y^{j-1}} \left((X + Y - 1)^{n-1} \right) = \frac{n!}{i! j!} \binom{n-1}{i-1, j-1, 0} = \beta_0(i, j).$$

□

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