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Asymptotic enumeration of constellations and related families of maps on orientable surfaces.

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We perform the asymptotic enumeration of two classes of rooted maps on orientable surfaces: m-hypermaps and m-constellations. For $m = 2$ they correspond respectively to maps with even face degrees and bipartite maps. We obtain explicit asymptotic formulas for the number of such maps with any finite set of allowed face degrees.

Our proofs combine a bijective approach, generating series techniques related to lattice walks, and elementary algebraic graph theory.

A special case of our results implies former conjectures of Z. Gao.

Keywords: Graphs on surfaces, asymptotic combinatorics, algebraic series, bijections.

1. Introduction

Maps are combinatorial objects which describe the embedding of a graph in a surface. The enumeration of maps began in the sixties with the work of Tutte, in the series of papers [24, 25, 23, 26]. By analytic techniques, involving recursive decompositions and non trivial manipulations of power series, Tutte obtained beautiful and simple enumerative formulas for several families of planar maps. His techniques were extended in the late eighties by several authors to more sophisticated families of maps or to the case of maps of higher genus. Bender and Canfield [3, 4] obtained the asymptotic number of maps on a given orientable surface. Gao [14] obtained formulas for the asymptotic number of $2k$-angulations on orientable surfaces, and conjectured a formula for more general families (namely maps where the degrees of the faces are restricted to lie in a given finite subset of $2\mathbb{N}$).

A few years later, Schaeffer [22], following the work of Cori and Vauquelin [11], gave in thesis a bijection between planar maps and certain labelled trees which enables to recover the formulas of Tutte, and explains combinatorially their remarkable simplicity. This bijection has attracted a lot of interest in probability and physics, since it also enables to study geometrical aspects of large random maps [10, 16, 18, 8, 20, 17]. Moreover, it has been generalised in two directions. First, Bouttier, Di Francesco, and Guitter [7] gave a construction that generalises Schaeffer's bijection to the large class of *Eulerian maps*, which includes for example maps with restricted face degrees, or constellations. Secondly, Marcus and Schaeffer [19] generalised Schaeffer's construction to the case of maps drawn on orientable surfaces of any genus, opening the way to a bijective derivation [9] of the results of Bender and Canfield.

The first purpose of this article is to unify the two generalisations of Schaeffer's bijection: we show that the general construction of Bouttier, Di Francesco, and Guitter stays valid in any genus, and involves the same kind of objects as developped in [19]. Our second (and main) task is then to use this bijection to perform the asymptotic enumeration of several families of maps, namely m-constellations and m-hypermaps. These maps will be defined later, but we can mention now that for $m = 2$, they correspond respectively to bipartite maps, and maps with even face degrees. In particular, a special case of our results implies the conjectures of Gao [14].

Apart of the generalized Bouttier–Di Francesco–Guitter bijection, our paper is based on a decomposition of the objects inherited from that bijection inspired from a previous work of Marcus, Schaeffer and the author [9]. In our case, this decomposition leads to the study of of certain lattice paths. Considering the generating series of these paths (which turn out to be algebraic), and inverting the decomposition, we express the generating series of m-hypermaps or constellations of genus g as a rational function of the roots of a certain characteristic polynomial. We finally perform the singularity analysis of the series, from which we deduce asymptotic formulas via transfer theorems.

2. Outline of the paper

Since the decomposition of maps presented in this paper is rather long and contains several details, we begin with a general description of the different sections which will, we hope, enlight the presentation.

- **Section 3**: we give several definitions related to maps on surfaces, and we state our two main results (Theorems 3.1 and 3.2).
- **Section 4**: we present the generalized version of the Bouttier–Di Francesco–Guitter bijection, which relates Eulerian maps of genus g with one distinguished vertex to certain objects called *g-mobiles* (Theorem 4.3). These objects are maps of genus g with one face carrying several types of vertices and edges, which are labelled by integers. The construction is similar to the planar case, only the proof of Lemma 4.1 is specific to positive genus.
- **Section 5**: the purpose of this section is to present the two main building blocks of mobiles, which are *elementary stars* and *cells* (these are certain sorts of star-graphs carrying labels on their vertices). We introduce the notion of an *m-walk*, which is a certain kind of lattice walk that describes the succession of the labels of vertices around elementary stars (these walks already appear implicitely in [7]).
- **Section 6**: we present the main decomposition of this paper. Each mobile can be decomposed into the following building blocks: a forest of *planar* mobiles, a certain number of lattice paths (whose elementary steps are precisely the cells of Section 5), and one *full-scheme*. The full-scheme is a sort of "skeleton" of the mobile, and contains also several decorations, which make the decomposition reversible (Proposition 6.6).

An important fact is that for each genus, the number of possible full-schemes is finite. (Lemma 6.5).

- **Section 7**: we study in details the *characteristic polynomial*, which is the generating polynomial of cells, with two variables (for the *size* and the *increment*). In order to study lattice walks whose steps are given by the cells, we study in details the behaviour of the roots of the characteristic polynomial; we also compute the values of several of its partial derivatives at the critical point.
- **Section 8**: using the decomposition previously stated, and a suitable resummation of the expressions of Section 7, we manage to express the generating series of g-mobiles as a rational fraction in the roots of the characteristic polynomial (Section 8.2). Then, we perform the singular analysis of the series at its radius of convergence: we show that only the principal root contributes to the singularity, thanks to a study of the cancellations of the contributions between conjugate roots (Proposition 8.4).
- **Section 9**: we compute explicitely the contribution to the generating series of mobiles, of the decorations which appear in the full-schemes. Finally, our last step is a "depointing lemma" (Lemma 9.4), which relates the asymptotic number of rooted maps with a distinguished vertex (which is what we inherit from the bijection) to the number of maps which are only rooted. We conclude the paper by several corollaries which concern special cases of our main results, for example Gao's conjecture or non degree-restricted m-constellations.

Constellations vs hypermaps: In all the paper, we treat simultaneously the cases of m-hypermaps and m-constellations (which are a special case of m-hypermaps). To this end, we define in Section 6 the *type* of a m-hypermap, which is some element of a $\mathbb{Z}/m\mathbb{Z}$ vector space of dimension $2g$; constellations correspond to m-hypermaps whose type is the null vector. In the paper, all the building blocks (elementary stars and cells), and corresponding generating series, are considered under different versions which depend on the type of the underlying hypermap; we make as explicit as possible the relations between blocks of different types (Proposition 7.5). At the very end (Section 9), we discover that the contributions of the different building blocks are well balanced, and that in very large m-hypermaps, all the possible types are equally likely. In particular, in the large size limit, m-constellations form an asymptotic proportion $1/m^{2g}$ of all m-hypermaps (Theorem 3.2).

3. Definitions and main results

Let $\mathcal{S}_g$ be the torus with g handles. A *map* on $\mathcal{S}_g$ (or map of genus g) is a proper embedding of a finite graph G in $\mathcal{S}_g$ such that the maximal connected components of $\mathcal{S}_g \setminus G$ are simply connected regions. Multiple edges and loops are allowed. The maximal simply connected components are called the *faces* of the map. The degree of a face is the number of edges incident to it, counted with multiplicity. A *corner* consists of a vertex together with an angular sector adjacent to it.

We consider maps up to homeomorphism, i.e. we identify two maps such that there exists an orientation preserving homeorphism that sends one to another. In this setting, maps become purely combinatorial objects (see [21] for a detailed discussion on this fact). In particular, there are only a finite number of maps with a given number of edges, opening a way to enumeration problems.

All the families of maps considered in this article will eventually be rooted (which means that an edge has been distinguished and oriented), pointed (when only a vertex has been distinguished), or both rooted and pointed. In every case, the notion of oriented homeomorphism involved in the definition of a map is adapted in order to keep trace of the pointed vertex or rooted edge.

The first very useful result when working with maps on surfaces is *Euler characteristic formula*, that says that if a map of genus g has f faces, v vertices, and n edges, then we have:

$$v + f = n + 2 - 2g.$$

An *Eulerian map* on $\mathcal{S}_g$ is a map on $\mathcal{S}_g$, together with a colouring of its faces in black and white, such that only faces of different colours are adjacent. By convention, the root of an Eulerian map will always be oriented with a black face on its right. This article will mainly be concerned with two special cases of Eulerian maps, namely m-hypermaps and m-constellations.

Definition. Let $m \geq 2$ be an integer. An *m-constellation* on $\mathcal{S}_g$ is a map on $\mathcal{S}_g$, together with a colouring of its faces in black and white such that:

(i) only faces of different colours are adjacent.
(ii) black faces have degree m, and white faces have a degree which is a multiple of m.
(iii) every vertex can be given a label in $\{1, \ldots, m\}$ such that around every black face, the labels of the vertices read in clockwise order are exaclty $1, \ldots, m$.

A map that satisfies conditions **(i)** and **(ii)** is called a m-hypermap.

It is a classical fact that in the planar case, conditions (i) and (ii) imply condition (iii), so that all planar m-hypermaps are in fact m-constellations; however, this is not the case in higher genus. Observe that 2-hypermaps are in bijection with maps whose all faces have even degree (in short, *even maps*). This correspondence relies on contracting every black face of the 2-hypermap to an edge of the even map. Observe also that this correspondence specializes to a bijection between 2-constellations and bipartite maps. This makes m-constellations and m-hypermaps a natural object of study. For previous enumerative studies on constellations, and for their connection to the theory of enumeration of rational functions on a surface, see [15],[6].

In the rest of the paper, $m \geq 2$ will be a fixed integer, and $D \subset \mathbb{N}_{>0}$ will be a non-empty and finite subset of the positive integers. If $m = 2$, we assume furthermore that D is not reduced to $\{1\}$. A *m-hypermap* with degree set mD is an m-hypermap in which all white faces have a degree which belongs to mD. The same definition holds for constellations. For example, a 2-constellation of degree set $2\{2\}$ is (up to contracting black faces to edges) a bipartite quadrangulation. Finally, the *size* of an m-hypermap is its number of black faces.

Our main results are the two following theorems:

Theorem 3.1. *The number $c_{g,D,m}(n)$ of rooted m-constellations of genus g, degree set mD, and size n satisfies:*

$$c_{g,D,m}(n) \sim t_g \frac{\gcd(D)}{2} \left(\frac{(m-1)^{5/2}\sqrt{2\gamma_{m,D}}}{m\beta_{m,D}^{5/2}} \right)^{g-1} n^{\frac{5(g-1)}{2}} (z_{m,D}^{(c)})^{-n}$$

when n tends to infinity along multiples of $\gcd(D)$*, and where:*

- $t_{m,D}^{(c)}$ *is the smallest positive root of:* $\sum_{k\in D} [(m-1)k-1] \binom{mk-1}{k} [t_{m,D}^{(c)}]^k = 1$
- $\beta_{m,D} = \sum_{k\in D} [(m-1)k] \binom{mk-1}{k} [t_{m,D}^{(c)}]^k$
- $\gamma_{m,D} = \sum_{k\in D} [(m-1)k][(m-1)k-1] \binom{mk-1}{k} [t_{m,D}^{(c)}]^k$
- $z_{m,D}^{(c)} = t_{m,D}^{(c)} [\beta_{m,D}]^{1-m}$

and the constant t_g is defined in [3].

Theorem 3.2. *The number $h_{g,D,m}(n)$ of rooted m-hypermaps of degree set mD and size n on a surface of genus g satisfies:*

$$h_{g,D,m}(n) \quad \sim \quad m^{2g} c_{g,M,D}(n)$$

when n tends to infinity along multiples of $\gcd(D)$.

Observe that Theorem 3.2 can be reformulated as follows: the probability that a large m-hypermap of genus g is an m-constellation tends to $1/m^{2g}$. To our knowledge, this fact had only been observed in the case of quadrangulations (which are known to be bipartite with probability $\sim 1/4^g$, see [2] and references therein). Putting Theorems 3.1 and 3.2 together gives an asymptotic formula for the number $h_{g,D,m}(n)$, which was already proved by Gao in the case where $m = 2$ and D is a singleton, and conjectured for $m = 2$ and general D in the paper [14]. All the other cases were, as far as we know, unknown.

4. The Bouttier–Di Francesco–Guitter bijection on an orientable surface.

In this section, we describe the Bouttier–Di Francesco–Guitter bijection on $\mathcal{S}_g$. This construction has been introduced in [7], as a generalisation of the Cori-Vauquelin-Schaeffer bijection, and provides a correspondence between planar maps and plane trees. Here, we unify this generalisation with the generalisation of [19], where the classical bijection of Cori, Vauquelin, and Schaeffer is extended to any genus.

All the constructions are local and are similar to the planar case. The proof that the construction is well defined (Lemma 4.1) is an adaptation of the one of [19], and is different from the one of [7], that uses the planarity. After that, everything (Lemma 4.2) is already contained in [7]. Therefore, we will not state all proofs.

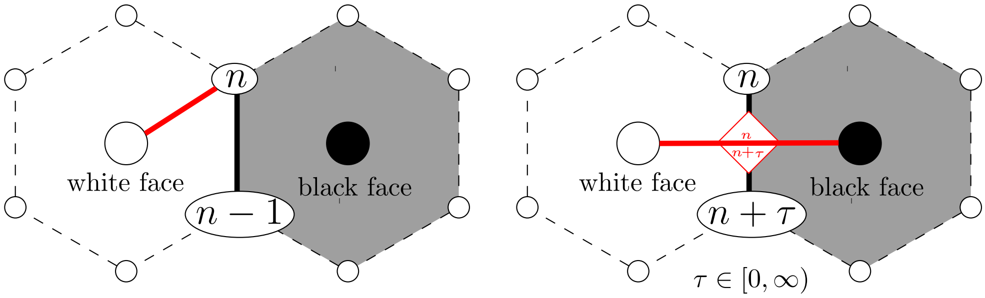

Figure 1. The Bouttier–Di Francesco–Guitter construction.

4.1. From Eulerian maps to mobiles.
Let $\mathfrak{m}$ be a rooted and pointed Eulerian map on $\mathcal{S}_g$ (i.e. $\mathfrak{m}$ has at the same time a root edge and a distinguished vertex).

The BDFG construction:

(1) orientation and labelling. First, we orient every edge of $\mathfrak{m}$ in such a way that it has a black face on its right. Then, we label each vertex v of $\mathfrak{m}$ by the minimum number of oriented edges needed to reach it from the pointed vertex. Observe that along an oriented edge, the label can either increase by 1, or decrease by any nonnegative integer.

(2) local construction. First, inside each face of $\mathfrak{m}$, we add a new vertex of the colour of the face. Then, inside each white face F of $\mathfrak{m}$, and for all edge e adjacent to F, we procede to the following construction (see Figure 1):

- if the label increases by 1 along e, we add a new edge between the unlabelled white vertex at the center of F and the extremity of e of greatest label.
- if the label decreases by $\tau \geq 0$ along e, we add an new edge between the two central vertices lying at the centers of the two faces separated by e. Moreover, we mark each side of this is edge with a *flag*, which is itself labelled by the label in $\mathfrak{m}$ of the corresponding extremity of e, as in Figure 1.

(3) erase original edges. We let $\bar{\mathfrak{m}}$ be the map obtained by erasing all the original edges of $\mathfrak{m}$ and the pointed vertex v_0 (i.e. the map consisting of all the new vertices and edges added in the construction, and all the original vertices of the map except the pointed vertex).

(4) choose a root and shift labels. We define the root of $\bar{\mathfrak{m}}$ as the edge associated to the root edge of $\mathfrak{m}$ in the above construction; we orient it such that it leaves a white unlabelled vertex. The *root label* is either the label of the only labelled vertex adjacent to the root edge (if it exists), either the label of the flag situated on the left of the root edge. We now shift all the labels in $\bar{\mathfrak{m}}$ by minus the root label, so that the new root label is 0: we let $\mathrm{Mob}(\mathfrak{m})$ be the map obtained at this step. A planar example is shown on Figure 2.

Recall that a g-tree is a map on $\mathcal{S}_g$ which has only one face. In the planar case, from Euler characteristic formula, this is equivalent to the classical graph-theoretical definition

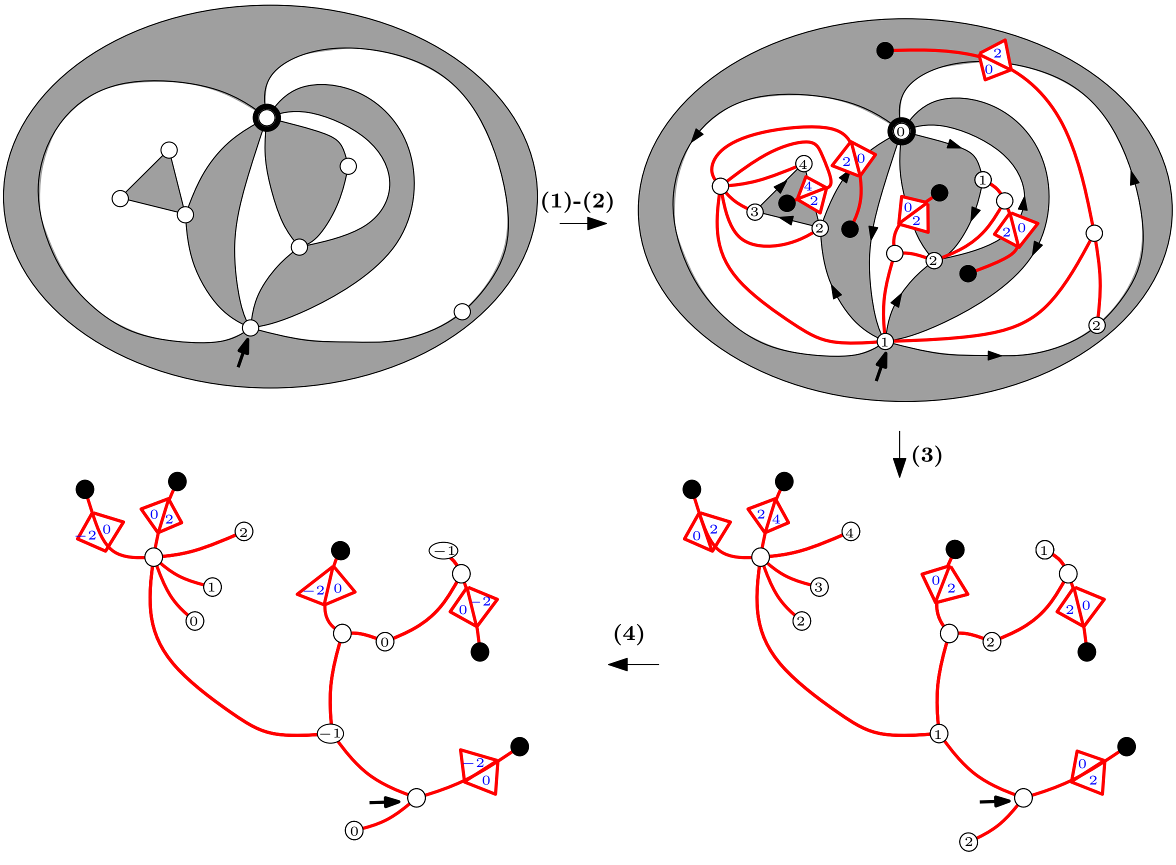

Figure 2. A rooted and pointed 3-constellation on the sphere, and its associated mobile.

of a tree. However, in positive genus, a g-tree always has cycles, and therefore will never be a tree, in the graph sense. We have:

Lemma 4.1. Mob($\mathfrak{m}$) *is a well-defined map on* $\mathcal{S}_g$*, and is moreover a* g*-tree.*

Proof. Our proof follows the arguments of [9]. We let $\mathfrak{m}'$ be the map consisting of the original map $\mathfrak{m}$ and all the new vertices and edges added in the previous construction; to avoid edge-crossings, each time a flagged edge of Mob($\mathfrak{m}$) crosses an edge of $\mathfrak{m}$, we split those two edges in their middle, and we consider the pair of flags lying in the middle of the flagged edge as a tetravalent vertex of $\mathfrak{m}'$, linked to the four ends created by the edge-splitting.

It is clear from the construction rules that each black or white unlabelled vertex is adjacent to at least one flagged edge, so that $\mathfrak{m}'$ is a well defined connected map of genus g. We now let $\widehat{\mathfrak{m}'}$ be the dual map of $\mathfrak{m}'$, and $\mathfrak{t}$ be the submap of $\widehat{\mathfrak{m}'}$ induced by the set of edges of $\mathfrak{m}'$ which are dual edges of original edges of $\mathfrak{m}$. We now examine the cycles of $\mathfrak{t}$.

By convention, we orient each edge of $\mathfrak{t}$ as follows: if the edge lies between a vertex and a flag, then we orient it in such a way that it has the flag on its left. If it lies between

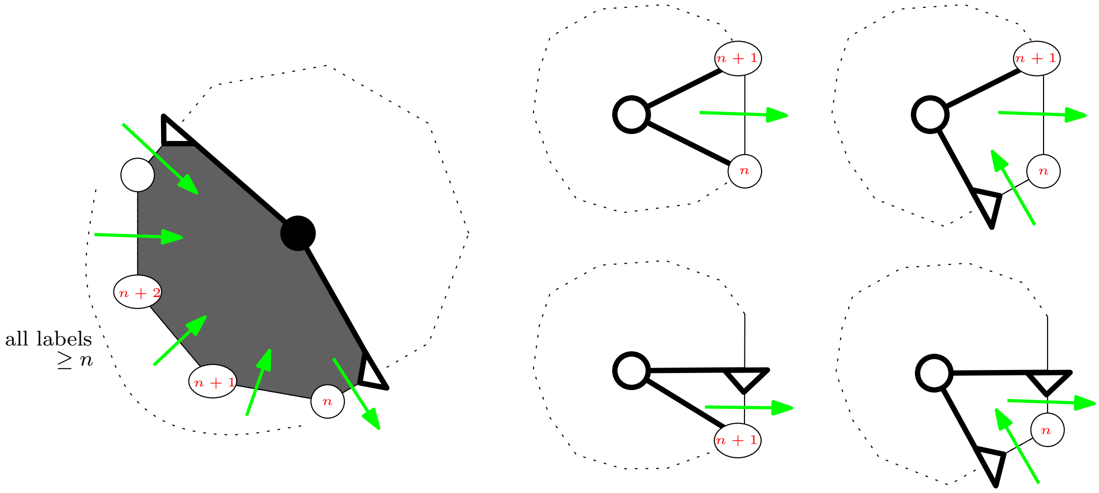

Figure 3. A typical black face, and the four types of white faces of $\mathfrak{m}'$. When a cycle of arrows crosses a face, the label at its right cannot increase. Moreover, it remains constant if and only if it turns around a single vertex.

two vertices, then we orient it in such a way that it has the vertex of greatest label on its left. Then by the construction rules (see Figure 3) each face of $\mathfrak{m}'$ carries a unique outgoing edge of $\mathfrak{t}$. Hence, if $\mathfrak{t}$ contains a cycle of edges, it is in fact an oriented cycle. Moreover, when going along an oriented cycle of edges of $\mathfrak{t}$, the label present at the right of the edge cannot increase (as seen on checking the different cases on Figure 3). Hence this label is constant along the cycle, and looking one more time at the different cases on Figure 3, this is possible only if the cycle encircles a single vertex. Such a vertex cannot be incident to any vertex with a smaller label (otherwise, by the construction rules, an edge of $\mathrm{Mob}(\mathfrak{m})$ would cut the cycle), which implies by definition of the labelling by the distance that the encycled vertex is the pointed vertex v_0.

Hence $\mathfrak{t}$ has no other cycle than the cycle encycling v_0. This means that, after removing v_0 and all the original edges of $\mathfrak{m}$, one does not create any non simply connected face, and that $\mathrm{Mob}(\mathfrak{m})$ is a well defined map of genus g (for a detailed topological discussion of this implication, see the appendix in [9]).

Finally, let b (resp. w) be the number of black (resp. white) faces of $\mathfrak{m}$, and v (resp. e) be its number of vertices (resp. edges). Then by Euler characteristic formula, one has:

$$(b+w)+v = e+2-2g$$

Now, by construction, $\mathrm{Mob}(\mathfrak{m})$ has e edges and $b+w+v-1$ vertices. Hence applying Euler characteristic formula to $\mathrm{Mob}(\mathfrak{m})$ shows that it has exactly one face, i.e. that it is a g-tree. $\square$

4.2. From mobiles to Eulerian maps

Our definition of a mobile is similar to the one given in [7]:

Definition. A g-mobile is a rooted g-tree $\mathfrak{t}$ such that:

i. $\mathfrak{t}$ has vertices of three types: unlabelled ones, which can be black or white, and labelled ones carrying integer labels.

ii. edges can either connect a labelled vertex to a white unlabelled vertex, or connect two unlabelled vertices of different color. The edges of the second type carry on each side a flag, which is itself labelled by an integer.

iii-w. when going clockwise around a white unlabelled vertex:

- a vertex labelled l is followed by a label $l-1$ (either vertex or flag).
- two successive flags of labels l and l' lying on the same edge satisfy $l' \geq l$; the second flag is followed by a label l' (either vertex or flag).

iii-b. when going clockwise around a black unlabelled vertex, two flags of labels l and l' lying on the same side of an edge satisfy $l' \leq l$; the second flag is followed by a flag labelled $\geq l'$.

iv. The root edge is oriented leaving a white unlabelled vertex. The root label (which is either the label of the labelled vertex adjacent to the root, if it exists, or the label of the flag present on its left side) is equal to 0.

One easily checks that the BDFG construction leads to a map that satisfies the conditions above. Hence, thanks to the previous lemma (Lemma 4.1), for every Eulerian map $\mathfrak{m}$, Mob($\mathfrak{m}$) is a g-mobile. We now describe the reverse construction, that associates an Eulerian map to any g-mobile. This construction takes place inside the unique face of Mob($\mathfrak{m}$). In particular, we want to insist on the fact that all the work specific to the non planar case has been done when proving that Mob($\mathfrak{m}$) is a g-tree. Until the end of this section, everything is similar to the planar case. For this reason, we refer the reader to [7] for proofs. Let $\mathfrak{t}$ be a g-mobile. The closure of $\mathfrak{t}$ is defined as follows:

Reverse construction:

(0) Translate all the labels of $\mathfrak{t}$ by the same integer in such a way that the minimum label is either a flag of label 0, or a labelled vertex of label 1.

(1) Add a vertex of label 0 inside the unique face of $\mathfrak{t}$. Connect it by an edge to all the labelled corners of $\mathfrak{t}$ of label 1, and to all the flags labelled 0.

(2) Draw an edge between each labelled corner of $\mathfrak{t}$ of label $n \geq 2$ and its succesor, which is the first labelled corner or flag with label $n-1$ encountered when going counterclockwise around $\mathfrak{t}$.

(3) Draw an edge between each flag of label n and its succesor, which is the first labelled corner or flag with label n encountered when going counterclockwise around $\mathfrak{t}$.

(4) Remove all the original edges and unlabelled vertices of $\mathfrak{t}$.

We call Map($\mathfrak{t}$) the map obtained at the end of this construction. The root of Map($\mathfrak{t}$) is either the root joining the endpoint of the root of $\mathfrak{t}$ to its succesor (if it is labelled), or the edge corresponding to the flags lying on the root edge. The fact that this construction is reciprocal to the previous one is proved in the planar case in [7], but, as we already said, every argument stay valid in higher genus. Hence we have:

Lemma 4.2 ([7]). *For every Eulerian map* $\mathfrak{m}$*, one has:* $\mathrm{Map}(\mathrm{Mob}(\mathfrak{m})) = \mathfrak{m}$ *For every* g*-mobile* $\mathfrak{t}$*, one has:* $\mathrm{Mob}(\mathrm{Map}(\mathfrak{t})) = \mathfrak{t}$

This proves:

Theorem 4.3. *The application* Mob *defines a bijection between the set of rooted and pointed Eulerian maps of genus* g *with* n *edges and the set of* g*-mobiles with* n *edges. This bijection sends a map which has* n_i *white faces of degree* i *for all* i*,* b *black faces, and* v *vertices to a mobile which has* n_i *white unlabelled vertices of degree* i *for all* i*,* b *black unlabelled vertices and* $v-1$ *labelled vertices.*

4.3. m-constellations, m-hypermaps, and mobiles

Mobiles obtained from m-hypermaps form a subset of the set of all mobiles, and satisfy additionnal properties. To keep the terminology reasonable, we make the following convention:

Convention: In the rest of the paper, the word *mobile* will refer only to mobiles which are associated to m-hypermaps of genus g by the Bouttier–Di Francesco–Guitter bijection.

Let $\mathfrak{m}$ be a rooted and pointed m-hypermap, with vertices labelled by the distance from the pointed vertex. We define the *increment* of an (oriented) edge as the label of its origin minus the label of its endpoint; since all black faces have degree m, by the triangle inequality, all increments are in $[\![-1, m-1]\!]$. More, if if a black face is adjacent to an edge e of increment $m-1$, and since the sum of the increments is null along a face, then its $m-1$ other edges must have type -1. Hence, the black unlabelled vertex of the corresponding mobile has degree 1: it is connected only to the flagged edge corresponding to e.

Now, let $\mathfrak{t}$ be a mobile. The *increment* of a flagged edge is the increment of the associated edge in the corresponding m-hypermap: it is therefore the difference of the labels of the two flags, counterclockwise around the white unlabelled vertex. All black unlabelled vertices of degree 1 are linked to a flagged edge of increment $m-1$.

Now, observe that an m-hypermap is an m-constellation if and only if the labelling of its vertices by the distance from the pointed vertex, taken modulo m, realizes the property **iii** of the definition of a constellation. Indeed, in a m-constellation, the difference modulo m between the distance labelling and any labelling realizing property **iii** is constant on a geodesic path of oriented edges from the pointed vertex to any vertex, since both increase by 1 modulo m at each step. Hence, all the edges of an m-constellation have an increment which is either -1, either $m-1$. This gives:

Lemma 4.4. *Let* $\mathfrak{m}$ *be a rooted and pointed* m*-hypermap, with vertices labelled by the distance from the pointed vertex. Then* $\mathfrak{m}$ *is an* m*-constellation if and only if one of the following two equivalent properties holds:*

- *all its edges have increment* -1 *or* $m-1$
- *all the black unlabelled vertices of its mobile have degree* 1

In particular, $\mathfrak{m}$ is an m-constellation if and only if, clockwise around any black face, the label increases by 1 exactly $m-1$ times, and decreases by $m-1$ exactly one time.

5. The building blocks of mobiles: elementary stars and cells.

5.1. Elementary stars.

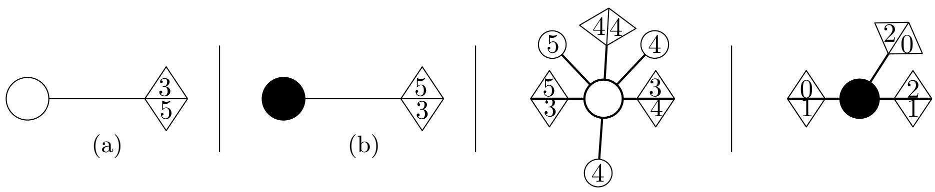

Figure 4. (a) a white split-edge of type 3; (b) a black split edge of type 3; (c) a white elementary star; (d) a black elementary star.

We now define what the building blocks of mobiles are.

Definition. (see Figure 4) A *white split-edge* is an edge that links a white unlabelled vertex to a pair formed by two flags, each one lying on one side of the edge, as in Figure 4. Each flag is labelled by integer. If those labels are l_1 and l_2, in clockwise order around the unlabelled vertex, the quantity $l_2 - l_1 + 1$ is called the *type* of the split-edge. The same definition holds for *black split-edges*, but in this case the type is defined as $l_1 - l_2 - 1$.

A *white elementary star* is a star formed by a central white unlabelled vertex, which is connected to a certain number of labelled vertices, and to a certain number of white split edges, and that satisfies the property **iii-w** of Definition 4.2. Elementary stars are considered up to translation of the labels.

The same definition holds for black elementary stars, up to replacing "white" by "black" and property **iii-w** by property **iii-b**.

The following lemma will be extremely useful:

Lemma 5.1. *Let $\mathfrak{s}$ be a white elementary star of degree km. Assume that $\mathfrak{s}$ has r split-edges, and let $\tau_1, \tau_2, \dots, \tau_r$ be their types. Then we have:*

$$\sum_{i=1}^{r} \tau_i = km.$$

Proof. We number the flags from 1 to r, in clockwise order, starting anywhere. We let l_i and l'_i be the labels carried by the i-th flag, in clockwise order, so that the corresponding type is $\tau_i = l'_i - l_i + 1$. By the property **iii-w**, the label decreases by one after each labelled vertex, so that $l'_i - l_{i+1}$ is exactly the number of labelled vertices between the i-th and $i+1$-th flags (with the convention that the $r+1$-th flag is the first one). Hence the total degree of $\mathfrak{s}$ is:

$$r+\sum_{i=1}^{r}(l'_i-l_{i+1}) = r+\sum_{i=1}^{r}(l'_i-l_i) = \sum_{i=1}^{r}\tau_i$$

which yields the result. □

Remark. If $\mathfrak{s}$ is a black elementary star which is present in a mobile $\mathfrak{t}$ such that Map($\mathfrak{t}$) is an m-hypermap, then the conclusion of the lemma also holds, with $k=1$. Indeed, if $l_1,..,l_m$ is the clockwise sequence of the distance labels around the corresponding black face of Map($\mathfrak{t}$), then $\sum_{i=1}^{m}\tau_i=\sum_{i=1}^{m}(l_{i+1}-l_i+1)=m$.

Definition. An m-walk of length l is a l-tuple of integers $(n_1,\dots,n_l)\in[\![-1,m-1]\!]^l$ such that $\sum n_i=0$. A circular m-walk of length l is an m-walk of length l considered up to circular permutation of the labels (i.e. an orbit under the action of the cyclic group $\mathbb{Z}_l$ on the indices).

We now explain how to associate an m-walk to an elementary star. Let $\mathfrak{s}$ be a white elementary star with degree multiple of m. We read clockwise the sequence of labels of vertices and split-edges around the central vertex. We interpret labelled vertices as a number -1, and split-edges of type τ as a number $\tau-1$. We obtain a sequence of integers $(n_1,\dots,n_l)$ defined up to circular permutations.

Lemma 5.2. *For each l multiple of m, the construction above defines a bijection between white elementary stars of degree l and circular m-walks of length l.*

Proof. It follows from the property **iii-w** that the walk associated to a white elementary star is indeed an m-walk. Conversely, given an m-walk of length l, and interpreting steps -1 as labelled vertices, and steps $\tau-1$ as split-edges of type τ, one reconstructs a white elementary star, which is clearly the only one from which the construction above recovers the original walk. □

Definition. We say that a split-edge is *special* if its type is not equal to m. A star is *special* if it contains at least one special split-edge, and *standard* otherwise.

5.2. Cells and chains of type 0.

Definition. (see Figure 5) A *cell of type* 0 is a standard white elementary star of degree multiple of m, which carries two distinguished labelled vertices: the *in* one and the *out* one.
The *increment* of a cell of type 0 is the difference $l_{out}-l_{in}$ of the labels of its out and in vertices. Its *size* is its number of split-edges, and its *total degree* is its degree as an elementary star (i.e. the degree of the central vertex).

A *chain of type* 0 is a finite sequence of cells type 0. Its size and increment are defined additively from the size and increment of the cells it contains. Its *in* vertex (resp *out* vertex) is the in vertex of its first cell (resp. out vertex of its last cell).

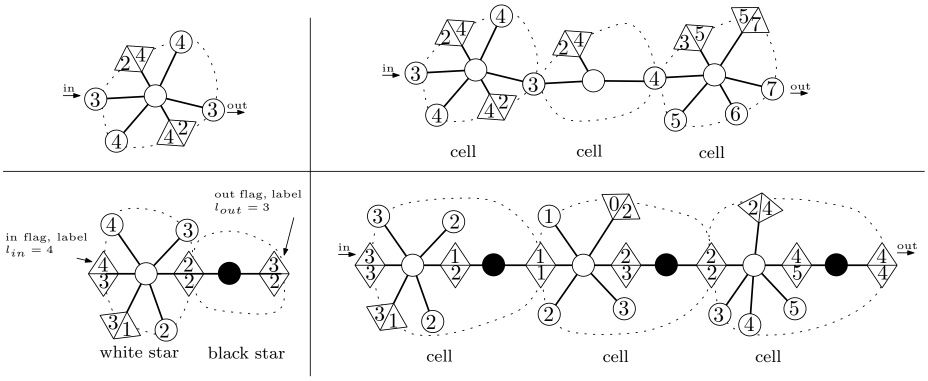

Figure 5. Four examples in the case $m = 3$. Up, a cell of type 0, and a chain of type 0; down, a cell of type 2, and a chain of type 1.

On pictures, to draw a chain of type 0, we identify the out vertex of each cell with the in vertex of the following one, as in Figure 5. From Lemma 5.2, cells of type 0 are in correspondence with m-walks which have only steps -1 and $m-1$, which implies that the total degree of a cell of type 0 equals m times its size, and that, in a chain of type 0, the total number of corners of the chain adjacent to a labelled vertex equals $(m-1)$ times the size of the chain.

5.3. Cells and chains of type $\tau \in [\![1, m-1]\!]$.

Definition. (see Figure 5) Let $\tau \in [\![1, m-1]\!]$. A *cell of type* τ is a pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ where:
- $\mathfrak{s}_1$ is a white elementary star, with exactly two special split-edges: the *in* one, of type τ, and the *out* one, of type $m - \tau$.
- $\mathfrak{s}_2$ is a black elementary star, with exactly two special split-edges: the *in* one, of type $m - \tau$, and the *out* one, of type τ.

On pictures, we identify the two split-edges of type $m - \tau$, as in Figure 5. The *in* split-edge of the cell is the in split-edge of $\mathfrak{s}_1$, and its *out* split-edge is the out split-edge of $\mathfrak{s}_2$; the corresponding labels l_{in} and l_{out} are defined with the convention of Figure 5. The *increment* of the cell is the difference l_{out} - l_{in}.

A chain of type τ is a finite sequence $\mathfrak{c}$ of cells of type τ. On pictures, we glue the flags of the out split-edge of a cell with the flags of the in split-edge of the following cell, as in Figure 5. The *increment* of the chain is the sum of the increment of the cells it contains. We let $|\mathfrak{c}|$ denote the total number of labelled vertices appearing in $\mathfrak{c}$. We also let $\langle\mathfrak{c}\rangle$ be the total number of black vertices appearing in $\mathfrak{c}$ plus its total number of split-edges of type m (equivalently, $\langle\mathfrak{c}\rangle$ is the total number of black vertices of $\mathfrak{c}$ if one links each split-edge of type m to a new univalent black vertex).

6. The full scheme of a mobile.

In this section, we explain how to reduce mobiles of genus g to a finite number of cases, indexed by minimal objects called their *full schemes*. Full schemes are a generalisation of what is called *labelled schemes* in [9] (see the remark in Section 6.5).

6.1. Schemes.

Definition. A *scheme of genus g* is a rooted map $\mathfrak{s}$ of genus g, which has only one face, and whose all vertices have degree ≥ 3. The set of schemes of genus g is denoted $\mathcal{S}_g$.

Let $\mathfrak{s}$ be a scheme of genus g, and, for all $i \geq 3$, let n_i be the number of vertices of $\mathfrak{s}$ of degree i. Then, by the hand-shaking lemma, its number of edges is $\frac{1}{2}\sum in_i$, and Euler Characteristic formula gives:

$$\sum_{i\geq 3} \frac{i-2}{2} n_i = 2g-1. \tag{6.1}$$

Hence the sequence $(n_i)_{i\geq 3}$ can only take a finite number of values. Since the number of maps with a given degree sequence is finite, this proves:

Lemma 6.1 ([9]). *The set $\mathcal{S}_g$ of all schemes of genus g is finite.*

We now need a technical discussion that will be of importance later. We assume that each scheme of genus g carries an arbitrary orientation and labelling of its edges, chosen arbitrarily but fixed once and for all. This will allow us to talk about "the i-th edge" of a scheme, or "the canonical orientation" of an edge, without more precision. Our first construction is not specific to mobiles, and applies to all maps of genus g with one face (see Figure 6):

Algorithm 1 (The scheme of g-tree $\mathfrak{t}$.). *Let $\mathfrak{t}$ be a g-tree. First, if $\mathfrak{t}$ contains a vertex of degree 1, we erase it, together with the edge it is connected to. We then repeat this step recursively until there are no vertices of degree 1 left. We are left with a map $\mathfrak{c}$, which we call* the core of $\mathfrak{t}$ *(see Figure 6, middle part). If the original root of $\mathfrak{t}$ is still present in the core, we keep it as the root of $\mathfrak{c}$. Otherwise, the root is present in some subtree of $\mathfrak{t}$ which is attached to $\mathfrak{c}$ at some vertex v: we let the root of $\mathfrak{c}$ be the first edge of $\mathfrak{c}$ encountered after that subtree when turning clockwise around v (and we orient it leaving v).*

Now, in the core, vertices of degree 2 lie on maximal paths whose end vertices are of degree at least 3 We now replace each of these paths by an edge: we obtain a map $\mathfrak{s}$, which has only vertices of degree ≥ 3. The root of $\mathfrak{s}$ is the edge corresponding to the path that was carrying the root of $\mathfrak{c}$ (with the same orientation). We say that $\mathfrak{s}$ is the scheme of $\mathfrak{t}$. *The vertices of $\mathfrak{t}$ that remain vertices of $\mathfrak{s}$ are called* the nodes of $\mathfrak{t}$.

6.2. The superchains of a mobile.

Let $\mathfrak{t}$ be a mobile whose scheme $\mathfrak{s}$ has k edges. Each edge of $\mathfrak{s}$ corresponds to a path of vertices of degree 2 of the core. For $i = 1..k$, we let $\mathfrak{p}_i$ be the path corresponding to the i-th edge of $\mathfrak{s}$, oriented by the canonical orientation of this edge (observe that each node is the extremity of several paths). A priori, $\mathfrak{p}_i$ can contain labelled vertices, black or white unlabelled vertices, and flagged or unflagged edges. We have the following important lemma:

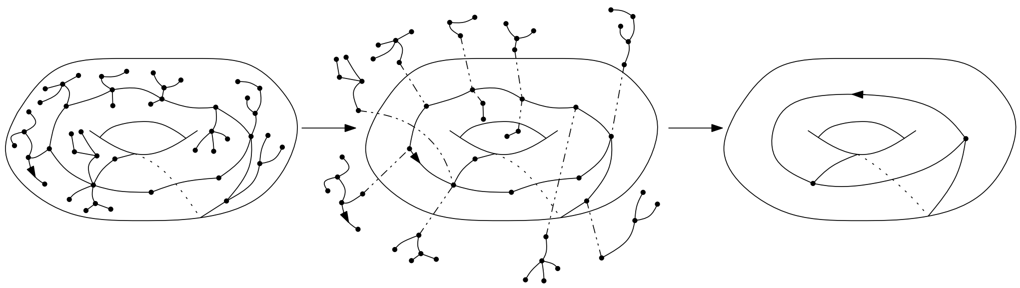

Figure 6. From a 1-tree to its scheme.

Lemma 6.2. *All the special flagged edges of* $\mathfrak{t}$ *lie on the paths* $\mathfrak{p}_i$, $i = 1..k$.

Proof. Assume that there is a special flagged edge e_0 in $\mathfrak{t} \setminus \mathfrak{c}$: e_0 belongs to a subtree $\mathfrak{t}'$ that has been detached from $\mathfrak{t}$ during the construction of its core. e_0 is connected to two unlabelled vertices, one of them, say v, being the farthest from $\mathfrak{c}$. Now, by Lemma 5.1, an unlabelled vertex (black or white) of $\mathfrak{t}$ cannot be connected to exactly one special flagged edge. Hence v is connected to another special edge e_1. Repeating recursively this argument, one constructs an infinite sequence of special edges $e_0, e_1, \ldots$. All these special edges belong to the subtree $\mathfrak{t}'$, so that the sequence cannot form a cycle: this implies that these edges are all distinct, which is impossible since a mobile has a finite number of edges. □

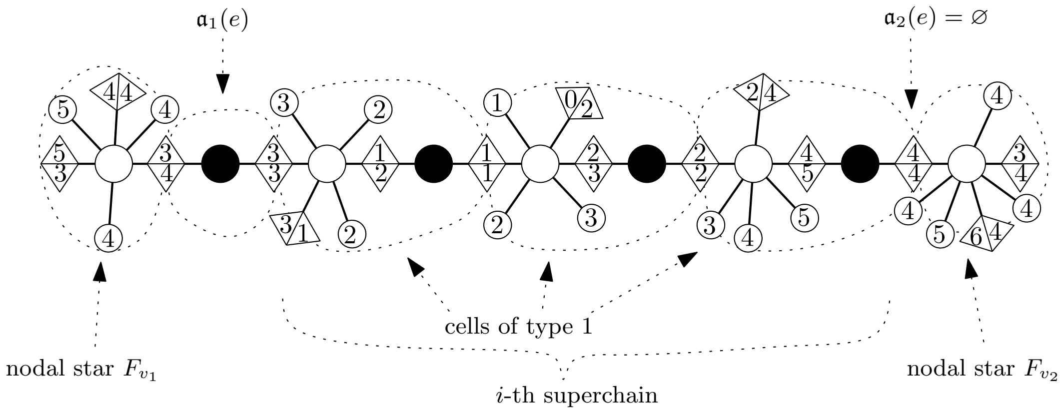

Figure 7. A typical superchain of type 1, in the case $m = 3$. It has two nodal star, only one correcting term $\mathfrak{a}_1(e)$ ($\mathfrak{a}_2(e)$ is empty), and the superchain itself is formed of three consecutive cells of type 1.

Each unlabelled vertex of $\mathfrak{p}_i$ was, in the original mobile $\mathfrak{t}$, at the center of an elementary star. We now re-draw all these elementary stars around each unlabelled vertex of $\mathfrak{p}_i$, as on Figure 7. If the extremities of $\mathfrak{p}_i$ are unlabelled vertices, we say that the corresponding stars are *nodal stars* of $\mathfrak{t}$. For the moment, we remove the nodal stars, if they exist:

we obtain a (eventually empty) sequence $\mathfrak{s}_1, \dots, \mathfrak{s}_l$ of successive stars. We now have to distinguish two cases.

case 1: $\mathfrak{p}_i$ contains no special flagged edge. In this case, $\mathfrak{p}_i$ is made of succession of edges linking white unlabelled vertices to labelled vertices (since the only remaining case, flagged edges of type m, are only linked to univalent black vertices and then cannot be part of the core). Consequently, the sequence $\mathfrak{s}_1, \dots, \mathfrak{s}_l$ is a sequence of white elementary stars, with no special flagged edges, glued together at labelled vertices, i.e. a chain of type 0 in the terminology of the preceding section. We say that $(\mathfrak{s}_1, \dots, \mathfrak{s}_l)$ is the i-th superchain of $\mathfrak{t}$.

case 2: $\mathfrak{p}_i$ contains at least one special flagged edge In this case, we will also show that our path reduces to a sequence of cells. First, from Lemma 5.1, an unlabelled vertex cannot be adjacent to exactly one special edge. Now, from Lemma 6.2, an unlabelled vertex of $\mathfrak{p}_i$ which is not one of its extremities cannot be adjacent to more than 2 special edges in $\mathfrak{t}$. Hence such a vertex is adjacent either to 0 or 2 special edges. Hence the set of special flagged edges of $\mathfrak{p}_i$ forms itself a path with the same extremities as $\mathfrak{p}_i$, i.e. is equal to $\mathfrak{p}_i$. In other terms: all the edges of $\mathfrak{p}_i$ are special flagged edges.
We now consider the sequence of stars $\mathfrak{s}_1, \dots, \mathfrak{s}_l$. If the first star of the sequence is black, we call it $\mathfrak{a}_1(i)$ and we remove it (otherwise we put formally $\mathfrak{a}_1(i) = \varnothing$). Similarly, if the last star is white, we call it $\mathfrak{a}_2(i)$ and we remove it. We now have a sequence of alternating color stars $(\mathfrak{s}'_1, \dots, \mathfrak{s}'_{l'})$ that begins with a white star and ends with a black one. From what we just said, all these stars are elementary stars with exactly two special flagged edges, glued together at these flagged edges. Since the sequence is ordered, we can talk of the ingoing and outgoing special edge of each of these stars. Now, let τ be the type of the ingoing special edge of $\mathfrak{s}'_1$. By Lemma 5.1, the type of its outgoing special edge is $m - \tau$. Now, this flagged edge is also the ingoing edge of the black star $\mathfrak{s}'_2$, and applying Lemma 5.1 again, the type of the outgoing edge of $\mathfrak{s}'_2$ is $m - (m - \tau) = \tau$. Consequently, $(\mathfrak{s}'_1, \mathfrak{s}'_2)$ is a cell of type τ, in the sense of the previous section. Applying recursively the argument, each pair $(\mathfrak{s}'_{2q-1}, \mathfrak{s}'_{2q})$ is a cell of type τ. The sequence $(\mathfrak{s}'_1, \dots, \mathfrak{s}'_l)$ is therefore a chain of type τ, which we call the i-th superchain of $\mathfrak{t}$.

In the two cases above, we have associated to the i-th edge e of $\mathfrak{s}$ a chain, which we called the i-th superchain of $\mathfrak{t}$. We now define the *type* of e as the type of this chain, and we note it $\tau(e)$.

By convention, if the i-th edge has type 0, we put $\mathfrak{a}_1(i) = \mathfrak{a}_2(i) = \varnothing$.

6.3. Typed schemes and the Kirchoff law.

Let v be a node of $\mathfrak{t}$. If v is labelled, then it is connected to no flagged edge (since flagged edges only connect unlabelled vertices). Hence all the paths $\mathfrak{p}_i$'s that are meeting at v correspond to case 1 above, or equivalently, all the edges of $\mathfrak{s}$ meeting at v are edges of type 0.

On the contrary, assume that v is unlabelled. Let e be an edge of $\mathfrak{s}$ adjacent to v, of type $\tau(e) \neq 0$. and let $\mathfrak{p}_i$ be the corresponding path of the core. We let $\tilde{\tau}(e)$ be the type of the split-edge of $\mathfrak{p}_i$ which is adjacent to v. It follows from the construction rules of the scheme that if v is *white*, then one has $\tilde{\tau}(e) = \tau(e)$ if e is incoming at v and $\tilde{\tau}(e) = m - \tau(e)$ if it is outgoing. On the contrary, if v is *black* then $\tilde{\tau}(e) = m - \tau(e)$ if e is incoming and $\tilde{\tau}(e) = \tau(e)$ if it is outgoing. Now, in both cases, Lemma 5.1 or the remark following it give: $\sum_{e \sim v} \tilde{\tau}(e) = 0 \text{ mod } m$.

In all cases, we have therefore:

Proposition 6.3 (Kirchoff law). *Let v be a vertex of $\mathfrak{s}$. We have:*

$$\sum_{e \text{ outgoing}} \tau(e) - \sum_{e \text{ ingoing}} \tau(e) = 0 \text{ mod } m \tag{6.2}$$

This leads to the following definition:

Definition. Let $\mathfrak{s}$ be a scheme of genus g. A *typing of* $\mathfrak{s}$ is an application

$$\tau : \{\text{edges of } \mathfrak{s}\} \to [\![0, m-1]\!]$$

that satisfies Equation (6.2) around each vertex.
A *typed scheme* is a pair $(\mathfrak{s}, \tau)$ formed by a scheme and one of its typings.
If $\mathfrak{s}$ is the scheme of a mobile $\mathfrak{t}$, and τ is the application that associates to each edge of $\mathfrak{s}$ the type of its corresponding superchain, we say that $(\mathfrak{s}, \tau)$ is *the typed scheme of* $\mathfrak{t}$.

For future reference, we now state the following lemma, which is a key fact in the proof of Theorem 3.2:

Lemma 6.4. *Let $\mathfrak{s}$ be a scheme of genus g. Then $\mathfrak{s}$ has exactly m^{2g} different typings.*

Proof. Observe that, if we identify $[\![0, m-1]\!]$ with $\mathbb{Z}/m\mathbb{Z}$, the set of all valid typings of $\mathfrak{s}$ is a $\mathbb{Z}/m\mathbb{Z}$ vector space. Actually, it coincides with the cycle space of $\mathfrak{s}$ in the sense of algebraic graph theory (see [27] for an introduction to this notion). Now, it is classical that the dimension of the cycle space of a connected graph equals its number of edges minus its number of vertices plus 1 (to see that, observe that the complementary edges of any spanning tree form a basis of this space). Now, since $\mathfrak{s}$ has one face, Euler characteristic formula gives:

$$\#\text{edges of } \mathfrak{s} - \#\text{vertices of } \mathfrak{s} = 2g - 1.$$

Hence the cycle space has dimension $2g$, and its cardinality is m^{2g}. □

6.4. Nodal stars and decorated schemes.

Let once again v be a node of $\mathfrak{t}$. If v is unlabelled, it is located at the center of an elementary star F_v (which, as we already said, we call a nodal star). F_v has a certain number of special split-edges, and a certain number of distinguished labelled vertices, which are connected to the paths $\mathfrak{p}_i$'s of $\mathfrak{t}$. We slightly abuse notations here, and assume the notation F_v denotes not only the elementary star itself, but the elementary star together with those distinguished vertices and the application that maps each distinguished vertex and split-edge of F_v to the corresponding half-edge of $\mathfrak{s}$.

In the case where v is labelled, we put formally $F_v = \circ$, where $\circ$ may be understood as a single labelled vertex considered up to translation (so that its label does not import).

Until the rest of the paper, if $\mathfrak{s}$ is a scheme of genus g, we note $E(\mathfrak{s})$ and $V(\mathfrak{s})$ for the sets of edges and vertices of $\mathfrak{s}$, respectively. If $|E(\mathfrak{s})| = k$, we will sometimes identify $E(\mathfrak{s})$ with $[\![1, k]\!]$.

Definition. We say that the quadruple

$$(\mathfrak{s}, \tau, F, \mathfrak{a}) = \big(\mathfrak{s}, (\tau(e))_{e\in E(\mathfrak{s})}, (F(v))_{v\in V(\mathfrak{s})}, (\mathfrak{a}_1(e), \mathfrak{a}_2(e))_{e\in E(\mathfrak{s})}\big)$$

is the *decorated scheme* of $\mathfrak{t}$.

6.5. The full scheme of a mobile.

We now present the last step of the reduction of mobiles to elementary objects. We assume that for each decorated scheme $(\mathfrak{s}, \tau, F, \mathfrak{a})$, and for each vertex v of $\mathfrak{s}$, the star F_v carries an arbitrary but fixed labelled vertex or flag, chosen once and for all, that we call the *canonical element* of v.

Now, let $\mathfrak{t}$ be a mobile, of decorated scheme $(\mathfrak{s}, \tau, F, \mathfrak{a})$. For each vertex v of $\mathfrak{s}$, we let l_v be the label in $\mathfrak{t}$ of the canonical element of v. We now normalize these labels, so that they form an integer interval of minimum 0. Precisely, we let $M = \mathrm{card}\,\{l_v,\ v \in V(\mathfrak{s})\} - 1$ and λ be the unique surjective increasing application $\{l_v,\ v \in V(\mathfrak{s})\} \to [\![0, M]\!]$.

Definition. We say that the quintuple $(\mathfrak{s}, \tau, F, \mathfrak{a}, \lambda)$ is the *full scheme* of $\mathfrak{t}$.

In few words, the full scheme of $\mathfrak{t}$ contains five informations: the combinatorial arrangement of the superchains, given by $\mathfrak{s}$; the types of the superchains, given by τ; the stars F_v that lie on the nodes of $\mathfrak{t}$; the (eventually trivial) stars $\mathfrak{a}_1(e)$ and $\mathfrak{a}_2(e)$ that ensure that each superchain of type $\neq 0$ begins with a white star, and ends with a black one; the relative order of the labels of the canonical elements, given by λ.

Remark. Our full schemes are a generalization of the *labelled schemes* of [9], which consist only of the pair $(\mathfrak{s}, \lambda)$. Here, our mobiles are more complicated, so we have to take the decoration $(F, \mathfrak{a})$ and the typing τ into account.

Recall that the number of schemes of genus g, and the number of typings of a given scheme, are finite. Moreover, since the set D of allowed face degrees is finite, there are only a finite number of elementary stars with total degree in mD. Hence $(F(v))_{v\in V(\mathfrak{s})}$ and $(\mathfrak{a}_1(e), \mathfrak{a}_2(e))_{e\in E(\mathfrak{s})})$ can only take a finite number of values, and:

Lemma 6.5. *The set $\mathcal{F}_g$ of all full schemes of genus g is finite.*

Let $\mathfrak{f} = (\mathfrak{s}, \tau, F, \mathfrak{a}, \lambda)$ be a full scheme of genus g. We say that a labelling $(l_v)_{v\in V(\mathfrak{s})}$ of its canonical elements is compatible with $\mathfrak{f}$ if normalizing it to an integer interval as we did above yields the application λ. We consider compatible labellings up to translation, or equivalently, we assume that the minimum l_v is equal to 0, so that all the compatible labellings are of the form:

$$l_v = \sum_{i=1}^{\lambda(v)} \delta_i \text{ for some } \delta \in (\mathbb{N}_{>0})^M.$$

Assume that such a labelling has been fixed. To reconstruct a mobile, we have to do the inverse of what precedes, and substitute a sequence of cells of the good type along each edge of $\mathfrak{s}$. Observe that, for each edge e, the increment $\Delta(e)$ of the superchain to be substituted to e is fixed by the choice of (l_v). Precisely, let e_+ and e_- be the extremities of e, with the convention $\lambda(e_+) \geq \lambda(e_-)$ (if $\lambda(e_+) = \lambda(e_-)$, any fixed choice will be convenient). Then, up to the sign, we have $\Delta(e) = l_{e_+} - l_{e_-} + a_{F,\mathfrak{a}}(e)$, where $a_{F,\mathfrak{a}}(e)$ is a correction term that does not depends on the l_v's, and that accounts for the fact that superchains do not necessarily begin and end at the canonical vertices. Precisely, $a_{F,\mathfrak{a}}(e)$ equals the difference of the label of the canonical element of e_+ and the label of the out vertex or flag of $\mathfrak{a}_2(e)$, from which one must substract the corresponding quantity for e_- (and it is important that these differences depend only on F and $\mathfrak{a}$). Putting things in terms of the δ_i's, we can write:

$$\Delta(e) = a_{F,\mathfrak{a}}(e) + \delta_{e_-+1} + \ldots + \delta_{e_+} = a_{F,\mathfrak{a}}(e) + \sum_j A_{e,j}\delta_j$$

where for each edge e and $j \in [\![1, M]\!]$ we put $A_{e,j} = \mathbb{1}_{e_- < j \leq e_+}$.

6.6. A non-deterministic algorithm.
We consider the following non-deterministic algorithm:

Algorithm 2. *We reconstruct a mobile by the following steps:*

(1) *we choose a full scheme* $\mathfrak{f} = (\mathfrak{s}, \tau, F, \mathfrak{a}, \lambda) \in \mathcal{F}_g$.
(2) *we choose a compatible labelling* $(l_v)_{v\in V(\mathfrak{s})}$ *of* $\mathfrak{f}$*, or equivalently, a vector* $\delta \in (\mathbb{N}_{>0})^M$.
(3) *for each edge* e *of* $\mathfrak{s}$*, we choose a chain of type* $\tau(e)$*. We then replace the edge* e *by this chain, eventually preceded by the star* $\mathfrak{a}_1(e)$ *and followed by the star* $\mathfrak{a}_2(e)$ *if they are not empty.*
(4) *on each corner adjacent to a labelled vertex, we attach a planar mobile (which can eventually be trivial).*
(5) *we distinguish an edge as the root, and we orient it leaving a white unlabelled vertex.*
(6) *we shift all the labels in order that the root label is* 0.

We have:

Proposition 6.6. *All mobiles of genus* g *can be obtained by Algorithm 2. More precisely, each mobile whose scheme has* k *edges can be obtained in exactly* $2k$ *ways by that algorithm.*

Proof. The first statement follows by the decomposition we have explained until now: we just have to re-add what we have deleted. Precisely, to reconstruct the mobile $\mathfrak{t}$

from its full scheme, one can first recover the labelling, then replace each edge by the corresponding superchain. Then, one has to re-attach the planar trees that have been detached from $\mathfrak{t}$ during the construction of its core: this can be done at step **4**. Finally, one obtains $\mathfrak{t}$ by choosing the right edge for its root, and shifting the labels to fit the convention of the definition of a mobile.

Now, let us prove the second statement. It is clear that the only way to obtain the mobile $\mathfrak{t}$ by different choices in the algorithm above is to start at the beginning with a scheme which coincides with the scheme of $\mathfrak{t}$ as an unrooted map, but may differ by the rooting. Precisely, let us call a *doubly-rooted mobile* a mobile whose scheme carries a secondary oriented root edge. Clearly, a mobile whose scheme has k edges corresponds to $2k$ doubly-rooted mobiles (since its scheme is already rooted once, it has no symmetry). Now, Algorithm 2 can be viewed as an algorithm that produces a doubly-rooted mobile: the secondary root of the scheme of the obtained mobile is given by the root of the scheme $\mathfrak{s}$ chosen at step **1** (we insist on the fact that the root of the starting scheme $\mathfrak{s}$ has no reason to be the root of the scheme of the mobile obtained at the end). Moreover, it is clear that each doubly-rooted mobile can be obtained in exaclty one way by the algorithm: the secondary root imposes the choice of the starting scheme $\mathfrak{s}$, and after that all the choices are imposed by the strucure of $\mathfrak{t}$. This concludes the proof of the proposition. $\square$

Remark. Let us consider a variant of the algorithm, where at step **1**, we choose only full schemes whose typing is identically 0. Then Proposition 6.6 is still true, up to replacing the word "mobile" by "mobile associated with an m-constellation". Indeed, a mobile is associated to an m-constellation if and only if it has no special edge, and the double-rooting argument in the proof of the proposition clearly works if we restrict ourselves to this kind of mobiles.

7. Generating series of cells and chains

Algorithm 2 and Proposition 6.6 reduce the enumeration of mobiles to the one of a few building blocks: schemes, planar mobiles, cells and chains of given type. We now compute the corresponding generating series.

Note: In what follows, m and D are fixed. To keep things lighter, the dependancy in m and D will many times be omitted in the notations.

7.1. Planar mobiles.

We let $T_\circ(z)$ be the generating series, by the number of black vertices, of planar mobiles whose root edge connects a white unlabelled vertex to a labelled vertex. Observe that $T_\circ$ is also the generating series of planar mobiles which are rooted at a corner adjacent to a labelled vertex (for example, choose the root corner as the first corner encountered clockwise after the root edge, clockwise around the labelled vertex it is connected to). Now, let $\mathfrak{t}$ be a planar mobile, whose root edge is connected to a labelled vertex, and say that the white elementary star containing the root-edge has total degree mk. This star is attached to one planar mobile on each of its $(m-1)k$ labelled vertices; each of these mobiles is naturally rooted at a labelled corner. Moreover, given this star and the sequence of those $(m-1)k$ planar mobiles, one can clearly reconstruct the mobile $\mathfrak{t}$. Finally, by Lemma 5.2, the number of white elementary stars with total degree mk and a distinguished edge connected to a labelled vertex is equal to the number of m-walks of length mk that begin with a step -1, and have $(m-1)k$ steps -1 and k steps $m-1$ in total, which is $\binom{mk-1}{k}$. This gives the equation:

$$T_\circ(z) = 1 + \sum_{k\in D} \binom{mk-1}{k} z^k T_\circ(z)^{(m-1)k} \tag{7.1}$$

Observe that the hypotheses made on D ensure that this equation has degree at least 2 in $T_\circ$. Moreover, $T_\circ$ has a positive radius of convergence $z^{(c)}_{m,D}$, and letting $T_c = T_\circ(z^{(c)}_{m,D})$, one has:

$$T_c = (m-1)\sum_{k\in D} k\binom{mk-1}{k}[z^{(c)}_{m,D}{T_c}^{m-1}]^k \tag{7.2}$$

Subtracting Equation (7.1) to Equation (7.2) shows that $z^{(c)}_{m,D}{T_c}^{m-1} = t^{(c)}_{m,D}$, and $T_c = \beta_{m,D}$, where $t^{(c)}_{m,D}$ and $\beta_{m,D}$ are defined in the statement of Theorem 3.1.

Writing down the multivariate Taylor expansion of Equation (7.1) near $z = z^{(c)}_{m,D}$ easily leads to the following lemma:

Lemma 7.1. *When z tends to $z^{(c)}_{m,D}$, the following Puiseux expansion holds:*

$$1 - \frac{T_\circ(z)}{T_c} = \sqrt{\frac{2\beta_{m,D}}{(m-1)\gamma_{m,D}}}\sqrt{1-\frac{z}{z^{(c)}_{m,D}}} + O\left(z^{(c)}_{m,D} - z\right)$$

7.2. The characteristic polynomial of type 0.

Let $\mathcal{F}^{\circ\circ}_{m,D}$ be the set of all cells of type 0 whose total degree belongs to mD. For $F \in \mathcal{F}^{\circ\circ}_{m,D}$, we denote respectively $|F|$ and $\mathrm{i}(F)$ the size and the increment of F. The *characteristic polynomial* of type 0 is the polynomial $1 - P_{m,D}(X,t)$, where the generating Laurent polynomial $P_{m,D}$ is defined by:

$$P_{m,D}(X,t) = \sum_{F\in\mathcal{F}^{\circ\circ}_{m,D}} t^{|F|}X^{i(F)}$$

For example, in the case $m = 2$, $D = \{2\}$, Figure 8 shows that the characteristic polynomial is $1 - t^2(X^{-1} + 1 + X)$.

For every $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we let $a_{n,i}$ be the number of chains of type 0 of total size n and increment i. Note that for every n, $a_{n,i} = 0$ except for a finite number of values of i. Hence, if $\mathbb{C}[X,X^{-1}][[t]]$ denotes the ring of formal power series in t with coefficients that are Laurent polynomials in X, the generating function $S_{m,D}(X,t) = \sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty} a_{n,i}t^nX^i$ of chains of type 0 by the size and the increment is a well defined element of $\mathbb{C}[X,X^{-1}][[t]]$. Since, by definition, a chain of type 0 is a sequence of cells of type 0, and since the size and the increment are additive parameters, we have by classical

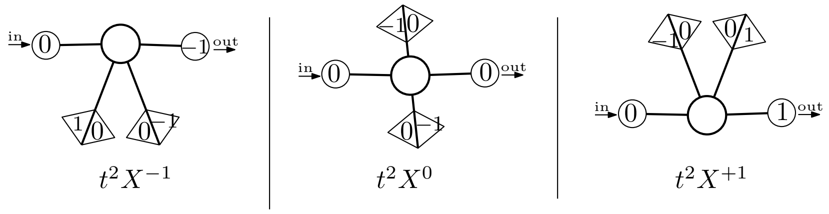

Figure 8. The three cells of type 0 and total degree 4.

symbolic combinatorics:

$$S_{m,D}(X,t) = \frac{1}{1 - P_{m,D}(X,t)}.$$

This is the reason why we will spend some time on the study of the polynomial $1 - P_{m,D}(X,t)$ (which is called the *kernel* in the standard terminology of lattice walks, see [1, 5]).

Observe that, in the m-walk reformulation, a cell of type 0 is a circular m-walk with two distinguished steps -1, or equivalently, a m-walk beginning with a step -1, with another step -1 distinguished. Hence the number of cells of type 0 and total degree mk equals $[(m-1)k-1]\binom{mk-1}{k}t^k$, so that $P(1,t) = \sum_{k\in D}[(m-1)k-1]\binom{mk-1}{k}t^k$, and $P(1, t^{(c)}_{m,D}) = 1$. Consequently, $t^{(c)}_{m,D}$ is the radius of convergence of the series $S_{m,D}(1,t)$. We now study the partial derivatives at the critical point. We have:

Lemma 7.2.

$$\frac{t\partial P_{m,D}}{\partial t}(1, t^{(c)}_{m,D}) = \frac{\gamma_{m,D}}{m-1} \tag{7.3}$$

$$\frac{\partial P_{m,D}}{\partial X}(1, t^{(c)}_{m,D}) = 0 \tag{7.4}$$

$$\frac{\partial^2 P_{m,D}}{\partial X^2}(1, t^{(c)}_{m,D}) = \frac{m}{6}\gamma_{m,D} \tag{7.5}$$

Proof. The first equation comes immediately from the definition of $\gamma_{m,D}$ and the fact that there are $[(m-1)k-1]\binom{mk-1}{k}$ distinct cells of type 0 and size k.

For the second equation, observe that since the operation consisting in inverting the in and out vertices of a cell is an involution of $\mathcal{F}^{\circ\circ}_{m,D}$, then for every t on has: $P_{m,D}(X,t) = P_{m,D}(X^{-1},t)$, which implies the second claim after derivating.

We now prove the third equation. First, recall that in the m-walk reformulation, $P_{m,D}$ is the generating function of linear m-walks of length mk, beginning with a step -1, and where a position preceding a step -1 is distinguished. Since the first derivative vanishes (Equation (7.4)), we have:

$$\begin{aligned} \frac{\partial^2 P_{m,D}}{\partial X^2}(1,t) &= \sum_{F\in\mathcal{F}^{\circ\circ}_{m,D}} i(F)(i(F)-1)t^{|F|} \\ &= \sum_{F\in\mathcal{F}^{\circ\circ}_{m,D}} i(F)^2 t^{|F|} = \sum_{k\in D} p_k t^k \end{aligned}$$

where $p_k(t) = \sum_{F\in\mathcal{F}^{\circ\circ}_{m,\{k\}}} i(F)^2$. We now fix $k \in D$, and we let $\mathcal{W}^{\circ}_{m,k}$ be the set of m-walks of length mk beginning with a step -1. We let $u = (m-1)k$ be the number of step -1 of such a walk, and for each $w \in \mathcal{W}^{\circ}_{m,k}$, we let $x_0(w), x_1(w), \ldots, x_{u-1}(w)$ be the ordinates of the points preceding a step -1 in w (so that $x_0(w) = 0$). Choosing first the m-walk, and then distinguishing a step -1, we can write:

$$p_k \quad = \sum_{w\in\mathcal{W}^{\circ}_{m,k}} (x_1(w)^2 + x_2(w)^2 + \ldots x_{u-1}(w)^2) \tag{7.6}$$

We now introduce the *risings* as the quantities $\lambda_i(w) = x_i(w) - x_{i-1}(w)$, for $i \in [\![1,u]\!]$. Then we have the following facts:

- By symmetry, the two following quantities are independant of j:

$$V_k = \sum_{w\in\mathcal{W}^{\circ}_{m,k}} \lambda_j(w)^2 t^k \quad (\text{for } j = 1..u-1)$$

$$W_k = \sum_{w\in\mathcal{W}^{\circ}_{m,k}} \lambda_1(w)\lambda_j(w) t^{k(F)} \quad (\text{for } j = 2..u-1)$$

- Since we have for all w: $\delta_1(\delta_1 + \cdots + \delta_u) = 0$ then it is still true after sumation and:

$$V_k + [u-1]W_k = 0$$

Putting the last fact together with Equation (7.6), one gets after replacing $x_i(w)$ by $\delta_1(w) + \cdots + \delta_i(w)$ and expanding:

$$\begin{aligned} p_k = &= \frac{u(u-1)}{2}V_k + \frac{u(u-1)(u-2)}{3}W_k \\ &= \frac{u(u+1)}{6}V_k \end{aligned} \tag{7.7}$$

Now, for any integer i, the number of rooted polygons of size k such that $\delta_1 = (m-1)i-1$ is easily seen to be equal to $\binom{mk-2-i}{k-i}$, hence:

$$\begin{aligned} V_k &= \sum_i [(m-1)i-1]^2 \binom{mk-2-i}{k-i} \\ &= [Y^{k-i}] \sum_i [(m-1)i-1]^2 (1+Y)^{mk-2-i} \end{aligned}$$

Expressing the last sum as an explicit rational fraction in Y, one obtains the exact value of V_k, and putting it together with Equation (7.7) leads to:

$$p_k = \frac{mk(m-1)[(m-1)k-1]}{6}\binom{mk-1}{k}$$

Hence

$$\frac{\partial^2 P_{m,D}}{\partial X^2}(1,t) = \sum_{k\in D} \frac{mk(m-1)[(m-1)k-1]}{6}\binom{mk-1}{k} t^k$$

which together with the definition of $\gamma_{m,D}$ concludes the proof of the lemma. □

7.3. The roots of the characteristic polynomial.

In this section, we study the roots of $1 - P_{m,D}$. Some of the arguments are general for lattice walks and already contained in [1],[5].

We first observe that for a cell of type 0 and size k, the maximal possible increment is $(m-1)k-1$, which corresponds to the case where the associated m-walk begins with all its steps (-1), the last one being distinguished. Similarly, the minimal possible increment is $1-(m-1)k$. Moreover, by definition, the maximal exponent of t in $P_{m,D}(X,t)$ is $\max(D)$. Therefore if we define

$$r := (m-1)\max(D) - 1$$

then in $P_{m,D}(X,t)$ the maximal power of X is r, and its minimal power is $-r$. The (true) polynomial $X^r(1-P_{m,D})$ therefore has degree $2r$ in X, and it has $2r$ roots, counted with multiplicity. Since $0 \notin D$ we have $P_{m,D}(X,0) = 0$, so that for $t = 0$ exactly r of these roots are finite and equal 0. We let $\alpha_1(t), \dots, \alpha_r(t)$ be these roots. Since exchanging the in and out vertices of a cell is an involution, $P_{m,D}$ is symmetrical under the exchange $X \leftrightarrow X^{-1}$, and the r other roots are $\alpha_1{}^{-}1(t), \dots, \alpha_r^{-1}(t)$, and they are infinite at $t = 0$.

The $\alpha_i(t)$ are formal Puiseux series in t. To understand their behaviour when t tends to 0, we consider the Newton polygon of the polynomial $X^r(1-P_{m,D}(X,t))$, i.e. the convex hull of the points (i,j) of the plane such that the monomial t^iX^j appears with a non-zero coefficient (see [13, p.498]). From the previous discussion on the maximal increments of cells of given size, the Newton polygon is the convex hull of the following set of points:

$$\big\{(0,r)\big\} \bigcup \big\{(k, r+(m-1)k-1), k \in D\big\} \bigcup \big\{(k, r+1-(m-1)k), k \in D\big\}.$$

Now, since r equals $(m-1)\max(D)-1$, it is easily seen (Figure 9) that the lower segment

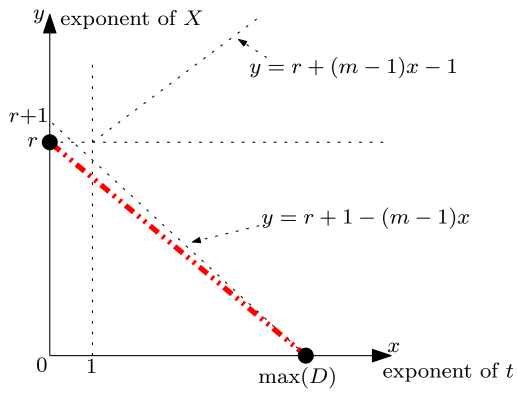

Figure 9. The Newton polygon of the polynomial $X^r(1-P_{m,D}(X,t))$.

of the Newton polygon has slope $-\frac{r}{\max(D)}$. Therefore the r branches $\alpha_1(t), \dots, \alpha_r(t)$ have a Puiseux expansion at $t = 0$ whose first term is :

$$\alpha_i(t) = c_0(\xi^i t^{1/r})^{\max(D)} + \dots$$

for a non-zero constant c_0 and a primitive r-th root of unity ξ. Moreover, the integers r and $\max(D)$ are coprime, which implies the following important fact: the r branches $\alpha_1(t), \dots, \alpha_r(t)$ are r *distinct* formal Puiseux series. We have moreover:

Lemma 7.3. *Up to renumbering the roots, we have:*

(i) $\alpha_1 \in \mathbb{R}$ *for* $t \in [0, t^{(c)}_{m,D}]$, *and* $\alpha_1(t)$ *is an increasing function on this interval. Moreover,* $\alpha_1 \longrightarrow 1$ *when* $t \longrightarrow t^{(c)}_{m,D}$.

(ii) for all $i \neq 1$, *and for all* $t \in [0, t^{(c)}_{m,D}]$, $|\alpha_i(t)| < |\alpha_1(t)|$. *There exists* $\epsilon > 0$ *such that for all* $i \neq 1$ *and for all* $t \in [0, t_c]$, $|\alpha_i(t)| < 1 - \epsilon$.

In the rest of the paper, we will keep the renumbering of the roots given by the lemma. The root $\alpha_1(t)$ is called the *principal branch*.

Proof. We already observed that 1 is a root of $P_{m,D}(X, t_c)$, and by Lemma 7.2, it is of multiplicity exactly two.

Now, for every $t \in (0, t_c)$ one knows by positivity of the coefficients of $P_{m,D}$ that $P_{m,D}(1,t) \leq P_{m,D}(1, t_c) = 1$, and $P_{m,D}(0,t) = \infty$. Moreover $X \mapsto P_{m,D}(X,t)$ is a decreasing function on $[0,1]$ (since for all i, $X^i + X^{-i}$ is) so that there exists a unique $\alpha = \alpha(t) \in [0,1]$ such that $P_{m,D}(\alpha) = 1$. Now, since $P_{m,D}$ has positive coefficients, $\alpha(t)$ is an increasing function of t. This proves claim (i).

Now, for every $\lambda \in \mathbb{C}$, one has $|P_{m,D}(\lambda)| \leq P_{m,D}(|\lambda|)$, with equality if and only if $\lambda > 0$. Hence if $|\lambda| \leq \alpha_1(t)$ one has $P_{m,D}(\lambda, t) \leq 1$, with equality if and only if $\lambda = \alpha_1$. This, together with a compacity argument, implies claim (ii). □

Lemma 7.2 then gives:

Lemma 7.4. *The following Puiseux expansion holds near* $t = t_c$*:*

$$\alpha_1(t) = 1 - \sqrt{\frac{12}{m(m-1)}}\left(1 - \frac{t}{t^{(c)}_{m,D}}\right)^{\frac{1}{2}} + o\left(\left(1 - \frac{t}{t^{(c)}_{m,D}}\right)^{\frac{1}{2}}\right)$$

Let us now define, for $i \in [\![1, r]\!]$, the following Puiseux series in t:

$$C_i = \frac{1}{-t^{\max(D)}\alpha_i \prod_j (1 - \frac{1}{\alpha_i \alpha_j}) \prod_{j \neq i} (\alpha_i - \alpha_j)} \tag{7.8}$$

Then the following partial fraction expansion holds:

$$S_{m,D}(X,t) = \frac{1}{-t^{\max(D)} \prod_i (X - \alpha_i)(1 - {\alpha_i}^{-1} X^{-1})} \tag{7.9}$$

$$= \sum_i \frac{C_i \alpha_i}{X - \alpha_i} + \sum_i \frac{C_i}{1 - X\alpha_i} \tag{7.10}$$

For all $n \in \mathbb{Z}$ we let $M_n(t) = \sum_{k=0}^{\infty} a_{k,n} t^k$ be the generating series of chains of type 0 of total increment n, by the size. Now, it easy to extract the coefficient of X^n in

Equation (7.10), via the following manipulations [1], which are valid operations in the ring of formal Puiseux series in t whose coefficients are Laurent polynomials in X :

$$\begin{aligned} S_{m,D}(X,t) &= \sum_i \frac{X^{-1}C_i\alpha_i}{1-X^{-1}\alpha_i} + \sum_i \frac{C_i}{1-X\alpha_i} \\ &= \sum_i \sum_{n=0}^{\infty} C_i {\alpha_i}^{n+1} X^{-n-1} - \sum_i \sum_{n=0}^{\infty} C_i {\alpha_i}^{n} X^{n} \end{aligned}$$

so that one obtains the generating function of chains of increment $n \in \mathbb{Z}$:

$$M_n(t) \quad = \quad \text{"}[X^n]S_{m,D}(X,t)\text{"} = \sum_{i=1}^{r} C_i(t)\alpha_i(t)^{|n|} \tag{7.11}$$

Observe that in the series $M_0(t)$, the empty walk of length 0 is counted.

7.4. Chains of all types.

We will see now that the generating series of chains of type 0, and of type $\tau \neq 0$ are closely related. To put this relation in a more fancy form, we consider not only chains, but chains where a planar mobile has been attached to each labelled vertex. For all $\tau \in [\![0, m-1]\!]$, we let $H_n^{\tau}(z)$ be the generating series of chains of type τ, that carry on each labelled corner a planar mobile (which can eventually be trivial). The variable z counts the total number of flagged edges.

In the case $\tau = 0$, this series is easily related to M_n: since a chain of type 0 and size k has $(m-1)k$ labelled vertices, and k flagged edges, H_n is obtained from M_n by the substitution $z \leftarrow zT_\circ(z)^{m-1}$.

Definition. In the rest of the paper, we note $t(z) := zT_\circ(z)^{m-1}$

We have then:

$$H_n^0(z) = M_n\left(t(z)\right) = \sum_{i=1}^{r} C_i(t(z))\alpha_i(t(z))^{|n|}$$

We now examine the case $\tau \in [\![1, m-1]\!]$. For such τ, we let $P_{m,D}^{\tau}(X,t,u)$ be the generating polynomial of elementary cells of type τ, where X, t, u count respectively the increment, the number of black vertices, and the number of labelled vertices. We also let $r_k^{(\tau)}(X)$ be the generating series of white elementary stars of total degree mk, with exactly two special split-edges, one of type τ and one of type $m-\tau$. Here, the variable X counts the increment between the two special edges. Since such a star has exactly $(m-1)k-1$ labelled vertices, $k-1$ black vertices, and since the generating series of black stars of degree m with two special edges is $1+X+\cdots+X^{m-2}$, one has, recalling that a cell of type τ is the juxtaposition of a white and a black star:

$$P_{m,D}^{(\tau)}(X,t,u) = (1+X+\cdots+X^{m-2}) \sum_{k\in D} t^k u^{(m-1)k-1} r_k^{(\tau)}(X). \tag{7.12}$$

[1] The author knows this technique from Mireille Bousquet-Mélou.

Now, by Lemma 5.2, $r_k^{(\tau)}(X)$ is also the generating series of walks of length mk, with

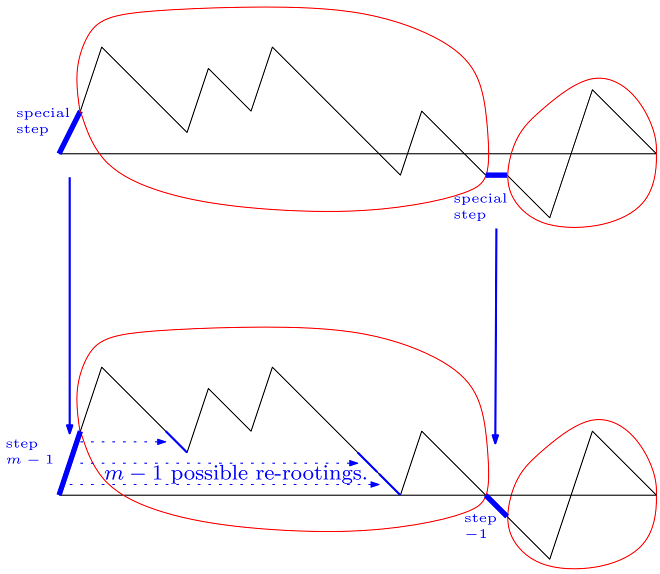

Figure 10. Walks with two special steps are in correspondence with walks with two distinguished steps, of increments $m-1$ and -1 (vertical arrows). These walks can be re-rooted in $m-1$ different ways to obtain walks with two distinguished steps -1 (horizontal arrows). In the re-rooting operation, the increment between the two steps is modified by a quantity among $0, \dots, m-2$, inducing a factor $1 + X + \cdots + X^{m-2}$ in the generating series.

$k-1$ steps $m-1$, $(m-1)k-1$ steps -1, beginning with a step $\tau-1$ and ending by a step $m-\tau-1$. These walks are in bijection with walks with of length km with only steps -1 and $m-1$, beginning with a step $m-1$, and with a distinguished step -1: to see that, exchange the steps τ, $m-2-\tau$ by two steps $-1, m-1$ [2]. Since in that walk the only decreasing steps are steps -1, the distinguished step $m-1$ lies in front of exactly $m-1$ steps -1. Hence (see Figure 10) $(1 + X + \cdots + X^{m-2}) r_k^{(\tau)}(X)$ is the generating series of walks with two distinguished steps -1, where X counts the increment between them.

Observe that these two distinguished steps are not necessarily distinct. If they are equal, we have a circular walk with one marked step -1: there are $\binom{mk-1}{k}$ of those. If they are not equal, the object considered is, up to the correspondence of Lemma 5.2, a cell of type 0. Hence we have:

$$(1 + X + \cdots + X^{m-2}) r_k^{(\tau)}(X) = \binom{mk-1}{k} + [t^k] P_{m,D}(X, t)$$

[2] In particular, $r_k^{(\tau)}(X)$ does not depend on τ.

This gives with Equation (7.12):

$$T_\circ(z)P^\tau_{m,D}(X,z,T_\circ(z)) = \left(\sum_{k\in D}\binom{mk-1}{k}t(z)^k + P_{m,D}(X,t(z))\right)$$

And using Equation (7.1) gives:

$$\frac{T_\circ(z)}{1-P_{m,D}(X,t(z))} = \frac{1}{1-P^\tau_{m,D}(X,z,T_\circ(z))}$$

Now, observe that the coefficient of X^n in the right-hand side is precisely the series $H^\tau_n(z)$. On the other hand, the coefficient of X^n in the left-hand side equals $T_\circ M_n(t(z))$. This gives the following proposition, which is the key that relates the enumeration of m-hypermaps and m-constellations:

Proposition 7.5. *For all $\tau \in [\![1, m-1]\!]$, and for all $n \in \mathbb{Z}$, we have:*

$$H^\tau_n(z) = T_\circ(z)H^0_n(z) \tag{7.13}$$

8. Generating series of mobiles

8.1. Translating Proposition 6.6 into generating series.

The previous section gave us all the building blocks to translate Proposition 6.6 in terms of generating series.

Let $(\mathfrak{s}, \tau, F, \mathfrak{a}, \lambda)$ be a full scheme of genus g. We are going to use Algorithm 2, and substitute each edge of $\mathfrak{s}$ with a chain. We first choose a compatible labelling $(l_v)_{v\in V(\mathfrak{s})}$ of that scheme. We need a little discussion on a special case. Imagine that the labelling imposes to substitute an edge e of type 0 to a chain of type 0 of increment $\Delta(e) = 0$. Then, if one of the extremities of e is associated with a non trivial nodal star, it is possible to substitute e to an empty chain; otherwise, if the two extremities are associated with the trivial nodal star $\circ$, the chain of length 0 is excluded: this would identify the two corresponding vertices of the scheme. Hence, if e is an edge of $\mathfrak{s}$, of extremities v_1 and v_2, we set:

$$r_{\mathfrak{f},(l_v)}(e) = \begin{cases} 1 \text{ if } \tau(e) = 0 \text{ and } \Delta(e) = 0 \text{ and } F_{v_1} = F_{v_2} = \circ \\ 0 \text{ otherwise.}\end{cases}$$

Then the edge e can be replaced by the empty walk if and only if $r_{\mathfrak{f},(l_v)}(e) \neq 1$. Observe that, as the notation suggests, $r_{\mathfrak{f},(l_v)}(e)$ not only depends on the full scheme $\mathfrak{f}$, but also on the compatible labelling (l_v).

We let $|\mathfrak{a}| = |\mathfrak{a}_1| + |\mathfrak{a}_2|$, $\langle\mathfrak{a}\rangle = \langle\mathfrak{a}_1\rangle + \langle\mathfrak{a}_2\rangle$, and similarly $|F| = \sum_v |F_v|$ and $\langle F\rangle = \sum_v\langle F_v\rangle$. Hence the series:

$$R_{\mathfrak{s},\tau,F,\mathfrak{a},\lambda}(z) := z^{\langle\mathfrak{a}\rangle+\langle\mathfrak{c}\rangle}t(z)^{|\mathfrak{a}|+|\mathfrak{c}|}\sum_{\text{labellings}}\prod_{e\in E(\mathfrak{s})}\left(H^{\tau(e)}_{\Delta(e)}(z) - r_{\mathfrak{f},(l_v)}(e)\right)$$

is the generating series of objects generated by the first four steps of Algorithm 2. Observe the first and second factor, that account respectively for the fact that black vertices appearing in the full scheme must be counted, and that planar mobiles must be attached also on the labelled vertices of the full scheme.

We now let $R_g(z)$ be the generating series of all mobiles of genus g, by the number of black vertices. Again, dependency in m and D are omitted in the notation. Since a mobile with k black vertices has in total mk edges, step 5 in Algorithm 2 corresponds to an operator $m\frac{zd}{dz}$ on the generating series. Hence, in terms of generating series, Proposition 6.6 admits the following reformulation:

Corollary 8.1.

$$R_g(z) = m\frac{zd}{dz} \sum_{(\mathfrak{s},\tau,F,\mathfrak{a},\lambda)\in\mathcal{F}_g} \frac{1}{2|E(\mathfrak{s})|} R_{\mathfrak{s},\tau,F,\mathfrak{a},\lambda}(z) \tag{8.1}$$

Remark. It follows from Remark 6.6 that the generating series of mobiles corresponding to m-constellations of degree set mD can be written:

$$R_g^{cons}(z) = m\frac{zd}{dz} \sum_{(\mathfrak{s},\vec{0},F,\mathfrak{a},\lambda)\in\mathcal{F}_g} \frac{1}{2|E(\mathfrak{s})|} R_{\mathfrak{s},\vec{0},F,\mathfrak{a},\lambda}(z) \tag{8.2}$$

where the sum is restricted to the full schemes $(\mathfrak{s},\tau,F,\mathfrak{a},\lambda)\in\mathcal{F}_g$ such that τ associates 0 to all edges.

8.2. An exact computation.

We fix a full scheme $\mathfrak{f} = (\mathfrak{s},\tau,F,\mathfrak{a},\lambda)$. We let E_1 be the set of edges of $\mathfrak{s}$ such that $\lambda(e_+) = \lambda(e_-)$, and E_2 be its complementary. Observe that for $e \in E_1$, the quantity $r_{\mathfrak{f},(l_v)}(e)$ does not depend on the labelling : for such an edge, we will therefore note $r_{\mathfrak{f},(l_v)}(e) = r_{\mathfrak{f}}(e)$.

To lighten notations, we note $T_\circ$, C_i, α_i for $T_\circ(z)$, $C_i(t(z))$ and $\alpha_i(t(z))$, respectively. We also note $z^{\mathfrak{f}} := z^{\langle\mathfrak{a}\rangle+\langle\mathfrak{c}\rangle}t(z)^{|\mathfrak{a}|+|\mathfrak{c}|}$. Then we have from Equation (7.13)

$$\begin{aligned} R_{\mathfrak{f}} &= z^{\mathfrak{f}} \sum_{\delta_1,..\delta_M>0} \prod_{e\in E_1} \left(\sum_{i=1}^{r} C_i - r_{\mathfrak{f}}(e)\right) \prod_{e\in E_2} \left({T_\circ}^{\mathbb{1}_{\tau(e)\neq 0}} \sum_{i=1}^{r} C_i \alpha_i^{|\Delta(e)|} - r_{\mathfrak{f},(l_v)}(e)\right) \qquad (8.3) \\ &= z^{\mathfrak{f}} {T_\circ}^{n_{\neq}} \prod_{e\in E_1} \left(\sum_{i=1}^{r} C_i - r_{\mathfrak{f}}(e)\right) \sum_{\delta_1,..\delta_M>0} \prod_{e\in E_2} \left(\sum_{i=1}^{r} C_i \alpha_i^{|a_{F,\mathfrak{a}}(e)+\sum_{j=1}^M A_{e,j}\delta_j|} - r_{\mathfrak{f},(l_v)}(e)\right) \end{aligned}$$

where $n_{\neq}$ is the number of edges of $\mathfrak{s}$ of type $\neq 0$. Now, observe that when the $\delta_i's$ are large enough, all the quantities $a_{F,\mathfrak{a}}(e)+\sum_{j=1}^M A_{e,j}\delta_j$ are positive, so that we can remove the absolute values in the expression above. Similarly, when the δ_i's are large enough, all the quantities $r_{\mathfrak{f},(l_v)}(e)$, for $e \in E_2$, are equal to 0. Therefore, if we define the following rational fraction of the α_i's:

$$\mathfrak{p}(\alpha_1,\ldots,\alpha_r) := \sum_{\delta_1,..\delta_M<K} \prod_{e\in E_2} \left(\sum_{i=1}^{r} C_i \alpha_i^{|a_{F,\mathfrak{a}}(e)+\sum_{j=1}^M A_{e,j}\delta_j|} - r_{\mathfrak{f},(l_v)}(e)\right)$$

where K is a large enough integer, we can write :

$$\sum_{\delta_1,..\delta_M>0}\prod_{e\in E_2}\left(\sum_{i=1}^{r}C_i\alpha_i^{|a_{F,\mathfrak{a}}(e)+\sum_{j=1}^{M}A_{e,j}\delta_j|}-r_{\mathfrak{f},(l_v)}(e)\right)$$
$$=\quad \mathfrak{p}(\alpha_1,\ldots,\alpha_r)+\sum_{\delta_1,..\delta_M\geq K}\prod_{e\in E_2}\left(\sum_{i=1}^{r}C_i\alpha_i^{a_{F,\mathfrak{a}}(e)+\sum_{j=1}^{M}A_{e,j}\delta_j}\right)$$

Then we have, by expanding the product:

$$\begin{aligned}\sum_{\delta_1,..\delta_M\geq K}\prod_{e\in E_2}\sum_{i=1}^{r}C_i\alpha_i^{a_{F,\mathfrak{a}}(e)+\sum_{j=1}^{M}A_{e,j}\delta_j} &= \sum_{\delta_1,..\delta_M\geq K}\ \sum_{i\in[\![1,r]\!]^{E_2}}\prod_{e\in E_2}C_{i_e}\alpha_{i_e}^{a_{F,\mathfrak{a}}(e)+\sum_{j=1}^{M}A_{e,j}\delta_j}\\ &= \sum_{i\in[\![1,r]\!]^{E_2}}\prod_{e\in E_2}C_{i_e}\alpha_{i_e}^{a_{F,\mathfrak{a}}(e)}\prod_{j=1}^{M}\frac{\left(\prod_e\alpha_{i_e}^{A_{e,j}}\right)^K}{1-\prod_e\alpha_{i_e}^{A_{e,j}}}\end{aligned}$$

where the last equality follows from a geometric summation on each variable δ_j. Observe that it remains only sums and products over finite sets. This gives the statement:

Proposition 8.2. *The series $R_{\mathfrak{f}}(z)$ is an algebraic series of z, given by the following expression:*

$$\begin{aligned}R_{\mathfrak{f}}(z)=z^{\mathfrak{f}}{T_\circ}^{n_{\neq}}\prod_{e\in E_1}\left(\sum_{i=1}^{r}C_i-r_{\mathfrak{f}}(e)\right)\times&\\ \times\left(\mathfrak{p}(\alpha_1,\ldots,\alpha_r)+\sum_{i\in[\![1,r]\!]^{E_2}}\prod_{e\in E_2}C_{i_e}\alpha_{i_e}^{a_{F,\mathfrak{a}}(e)}\prod_{j=1}^{M}\frac{\left(\prod_e\alpha_{i_e}^{A_{e,j}}\right)^K}{1-\prod_e\alpha_{i_e}^{A_{e,j}}}\right)&\end{aligned}\tag{8.4}$$

One should not worry to much about the form of the last equation. In the asymptotic regime, many terms will disappear, and it will look much nicer.

8.3. The singular behaviour of $R_{\mathfrak{f}}$.

Lemma 8.3. *The radius of convergence of $R_{\mathfrak{f}}(z)$ is at least $z_{m,D}^{(c)}$.*

Proof. Let us consider the family of all objects obtained by replacing each edge e of the scheme $\mathfrak{s}$ by a chain of type $\tau(e)$, without any constraint on the increment of the chains. These objects are not all valid mobiles (most of them are not) but clearly, this family contains all the mobiles counted by the series $R_{\mathfrak{f}}(z)$. Now, if $\mathfrak{s}$ has n_0 edges of type 0 and n_1 edges of type $\neq 0$, the generating series of these objects is:

$$z^{\mathfrak{f}}\left(\frac{1}{1-P_{m,D}(1,t(z))}\right)^{n_0}\left(\frac{T_\circ(z)}{1-P_{m,D}(1,t(z))}\right)^{n_1}$$

so that:

$$R_{\mathfrak{f}}(z)\preccurlyeq z^{\mathfrak{f}}T_\circ(z)^{n_1}\left(\frac{1}{1-P_{m,D}(1,t(z))}\right)^{n_0+n_1}$$

where $\sum f_n z^n \preccurlyeq \sum g_n z^n$ means that $f_n \leq g_n$ for all n. Since all the coefficients of these two series are nonnegative, this implies that the radius of convergence of $R_{\vec{\mathfrak{d}}}$ is at least $z^{(c)}_{m,D}$ (recall that $P_{m,D}(1, t^{(c)}_{m,D}) = 1$ and that $P_{m,D}$ has positive coefficients, so that $z^{(c)}_{m,D}$ is indeed the radius of convergence of the right hand side). $\square$

We now study the behaviour of $R_{\mathfrak{f}}(z)$ near $z = z^{(c)}_{m,D}$. Several things happen that create a singularity: First, $z^{(c)}_{m,D}$ is the radius of convergence of $T_\circ$ and $t(z)$. Second, we saw that at $t = t^{(c)}_{m,D}$, at least $\alpha_1(t)$ ceases to be analytic: we are thus in a regime of *composition of singularities*. Third, at $t = t^{(c)}_{m,D}$, $\alpha_1(t^{(c)}_{m,D}) = 1$ so that denominators in Equation (8.4) can vanish. These three factors are easy to control. There is a last one, however, that could happen. Indeed, if $P_{m,D}(X, t^{(c)}_{m,D})$ has other multiple roots than 1, the corresponding series C_i diverge. However, if ever this happens the corresponding divergences will cancel between multiple roots, and *everything works as if* 1 *was the only multiple root.* Precisely, we have:

Proposition 8.4. *The only dominating term in Expression (8.4) is the one corresponding to $i_e = 1$ for all e, and when z tends to $z^{(c)}_{m,D}$ we have:*

$$R_{\mathfrak{f}}(z) \quad = \quad c_{\mathfrak{s},\lambda} z^{\mathfrak{f}} {T_c}^{n_{\neq}} \frac{C_1(t(z))^{|E(\mathfrak{s})|}}{[1-\alpha_1(t(z))]^M} [1 + o(1)] \tag{8.5}$$

where the constant $c_{\mathfrak{s},\lambda} = \dfrac{1}{\prod_{j=1}^{M} \sum_{e\in E} A_{e,j}}$ *depends only on* $\mathfrak{s}$ *and* λ.

Proof. First, the expansion of $C_1(t)$ near $t = t^{(c)}_{m,D}$ gives : $C_1(t) \sim \dfrac{1-\alpha_1}{2(1-P(1,t))}$ which implies that $C_1(t) = \Theta\Big((t^{(c)}_{m,D} - 1)^{-1/2}\Big)$.

We now consider the contribution of the roots $\alpha_i(t)$ for $i \neq 1$. The definition of $C_i(t)$ shows that $C_i(t)$ diverges at $t = t^{(c)}_{m,D}$ if and only if $\alpha_i(t^{(c)}_{m,D})$ is a multiple root of $P(X, t^{(c)}_{m,D})$. We let $1 = \rho_1, \rho_2, \ldots, \rho_l$ be the roots of $P(X, t^{(c)}_{m,D})$ of modulus less than or equal to 1, without multiplicity, and for $j \leq l$ we note $I_j = \{i \in [\![1, r]\!],\ \alpha_i(t^{(c)}_{m,D}) = \rho_j\}$. In particular, $I_1 = \{1\}$, and $[\![1, r]\!] = \uplus_{j=1}^{l} I_j$.

We now fix $j \geq 2$. We let t in a pointed neighborhood of $t^{(c)}_{m,D}$ on which the $\alpha_i(t)$ are all distinct. We have, from the partial fraction expansion of $S(X, t)$:

$$C_i(t)\alpha_i(t) = \mathrm{Res}_{X=\alpha_i(t)} S(X, t) = \frac{1}{2\pi i} \oint_{T_i} S(X, t) dX$$

where we consider $S(X, t)$ as an analytic function of the variable X, t being fixed, and where T_i is a contour encircling $\alpha_i(t)$ and containing no other root (T_i may, and will, depend on t). Now we let $\epsilon > 0$ be such that for t close enough to $t^{(c)}_{m,D}$, the roots $\alpha_i(t)$, $i \in I_j$ are all contained in the interior of the circle C of center ρ_j and radius ϵ, and such that this circle encloses no other root $\alpha_{i'}(t)$ for $i' \notin I_j$. We then consider a tessellation of the circle C such that each face contains exactly one $\alpha_i(t)$, and we let T_i be the border of the face encircling $\alpha_i(t)$. Since the contributions of the contour integrals on each interior edge of the tessellation cancels, we obtain:

$$\sum_{i\in I_j} C_i(t)\alpha_i(t) = \frac{1}{2\pi i}\oint_C S(X,t)dX$$

Now, when t tends to $t^{(c)}_{m,D}$, $S(X,t)$ converges to $S(X,t^{(c)}_{m,D})$ *uniformly* in $X\in C$. Consequently, $\sum_{i\in I_j} C_i(t)\alpha_i(t)$ tends to $\frac{1}{2\pi i}\oint_C S(X,t^{(c)}_{m,D})dX$, which is *finite*. Hence, even if the $C_i(t)$ can diverge, we have shown that $\sum_{i\in I_j} C_i(t)\alpha_i(t) = O(1)$ at $t=t^{(c)}_{m,D}$. Similarly, by considering the residue of $X^{q-1}S(X,t)$, we obtain that for all $q\geq 0$ one has:

$$\sum_{i\in I_j} C_i(t)\alpha_i(t)^q = O(1) \text{ at } t=t^{(c)}_{m,D}. \tag{8.6}$$

We now show that this equation implies important simplifications at the critical point in Equation (8.4). First, we arrange the summation on i, in order to group together the indices whose corresponding roots meet at the critical point:

$$\sum_{i\in[\![1,r]\!]^{E_2}} \prod_{e\in E_2} C_{i_e}\alpha_{i_e}^{a_{F,\mathfrak{a}}(e)} \prod_{j=1}^{M} \frac{\left(\prod_e \alpha_{i_e}^{A_{e,j}}\right)^K}{1-\prod_e \alpha_{i_e}^{A_{e,j}}} \tag{8.7}$$

$$= \sum_{w\in[\![1,l]\!]^{E_2}} \sum_{i_1\in I_{w_1}} \cdots \sum_{i_{k'}\in I_{w_{k'}}} \prod_{e\in E_2} C_{i_e}\alpha_{i_e}^{a_{F,\mathfrak{a}}(e)} \prod_{j=1}^{M} \frac{\left(\prod_e \alpha_{i_e}^{A_{e,j}}\right)^K}{1-\prod_e \alpha_{i_e}^{A_{e,j}}} \tag{8.8}$$

where we have identified E_2 with the interval $[\![1,k']\!]$. For $w\in[\![1,l]\!]^{E_2}$, we let $|w|_1$ be its number of coordinates equal to 1, and we let $k_w=\{j,\ \forall e\ i_e=1 \text{ or } A_{e,j}=0\}$ be the number of factors such that the denominator vanishes in the previous equation. Then the previous sum rewrites:

$$\sum_{w\in[\![1,l]\!]^{E_2}} \frac{C_1(t)^{|w|_1}}{(1-\alpha_1(t))^{k_w}} \sum_{i_1\in I_{w_1}} \cdots \sum_{i_{k'}\in I_{w_{k'}}} \left(\prod_{e,w_e\neq 1} C_{i_e}\right) f_w(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_{k'}})$$

where f_w is a function of k' variables which is *analytic* at the point $(\rho_{w_1},\rho_{w_2},\ldots,\rho_{w_{k'}})$. We now consider the multivariate Taylor expansion of f_w at this point, up to a certain order. This expansion is a linear combination of monomials of the form $\alpha_{i_1}^{l_1}\alpha_{i_2}^{l_2}\ldots\alpha_{i_{k'}}^{l_{k'}}$. Now, Equation (8.6) implies that the quantity

$$\sum_{i_1\in I_{w_1}} \cdots \sum_{i_{k'}\in I_{w_{k'}}} \left(\prod_{e,w_e\neq 1} C_{i_e}\right) \alpha_{i_1}^{l_1}\alpha_{i_2}^{l_2}\ldots\alpha_{i_{k'}}^{l_{k'}}$$

is finite at the critical point. Hence, if we choose the order of the Taylor expansion large enough to be sure that the rest multiplied by $\prod_{e,w_e\neq 1} C_{i_e}$ is finite, we see that the quantity $\sum_{i_1\in I_{w_1}}\cdots\sum_{i_k\in I_{w_{k'}}}\left(\prod_{e,w_e\neq 1} C_{i_e}\right) f_w(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_{k'}})$ tends to a finite value at $t=t^{(c)}_{m,D}$.

This shows that in Equation (8.8), the term corresponding to $w = (1, \dots, 1)$ dominates strictly all the others. In particular we have:

$$\sum_{i\in[\![1,r]\!]^{E_2}} \prod_{e\in E_2} C_{i_e} \alpha_{i_e}^{a_{F,\mathfrak{a}}(e)} \prod_{j=1}^{M} \frac{\left(\prod_e \alpha_{i_e}^{A_{e,j}}\right)^K}{1-\prod_e \alpha_{i_e}^{A_{e,j}}} \sim C_1(t(z))^{|E_2|} \prod_{j=1}^{M} \frac{1}{1-\alpha_1(t(z))^{\sum_e A_{e,j}}} \qquad (8.9)$$

A similar reasoning shows, on the one hand, that no unexpected singularity occurs in $\mathfrak{p}(\alpha_1, \dots, \alpha_r)$, so that this quantity is negligible with respect to the one we have just examined, and on the other hand that the quantity $\sum_i C_i(t)$ is equivalent to $C_1(t)$. Hence we have analysed the behaviour of all quantities appearing in Equation (8.4), and the statement is proved. □

The expansion of $S_{m,D}(1,t)$ near $t = t_{m,D}^{(c)}$ gives :

$$C_1(t) = \frac{1-\alpha_1(t)}{2(1-P_{m,D}(1,t))}[1+o(1)]$$

Since Lemma 7.4 gives the singular expansion of $\alpha_1(t)$, and since the expansion of $P_{m,D}(1,t)$ follows from Lemma 7.2, we obtain:

Lemma 8.5. *When t tends to $t_{m,D}^{(c)}$, the following Puiseux expansion holds:*

$$C_1(t) = \sqrt{\frac{3(m-1)}{m}}\gamma_{m,D}^{-1}\left(1-\frac{t}{t_{m,D}^{(c)}}\right)^{-1/2} + o\left(\left(1-\frac{t}{t_{m,D}^{(c)}}\right)^{-1/2}\right) \qquad (8.10)$$

Setting $t = t(z)$, the last proposition and Lemmas 7.4, 8.5, 7.1 finally give:

Lemma 8.6. *When z tends to $z_{m,D}^{(c)}$, the following Puiseux expansion holds:*

$$\begin{aligned} R_{\mathfrak{f}}(z) &= c_{\mathfrak{s},\lambda}(z_{m,D}^{(c)})^{\mathfrak{f}}(T_c)^{n_{\neq}}(m-1)^{\frac{k+m}{4}} m^{\frac{M-k}{2}} \\ &\quad \times \gamma_{m,D}^{\frac{M-3k}{4}} \beta_{m,D}^{-\frac{k+M}{4}} 3^{\frac{k-M}{2}} 2^{\frac{-k-5M}{4}} \left(1-\frac{z}{z_{m,D}^{(c)}}\right)^{-\frac{k+M}{4}} [1+o(1)] \qquad (8.11) \end{aligned}$$

where k is the number of edges of $\mathfrak{s}$.

8.4. The dominant pairs.

From the last lemma, the singular behaviour of the sum (8.1) is dominated by the full schemes $\mathfrak{f}$ for which the quantity $k+M$ is maximal. First, to maximize the quantity $k+M$, we can assume that λ is injective, i.e. that $M = |V(\mathfrak{s})|-1$, so that the dominant terms will be given by schemes such that the quantity $|E(\mathfrak{s})|+|V(\mathfrak{s})|-1$ is maximal. Now, if a scheme $\mathfrak{s}$ of genus g has n_i vertices of degree i for all $i \geq 3$ we have:

$$|E(\mathfrak{s})| + |V(\mathfrak{s})| = \sum_{i\geq 3} \frac{i+2}{2} n_i.$$

Maximizing this quantity with the constraint of Equation (6.1) imposes that $\sum_i n_i$ is maximal, and since $\sum(i-2)n_i$ is fixed, this is realized if and only if $n_3 \neq 0$ and $n_i = 0$ for $i \neq 3$, i.e. if $\mathfrak{s}$ has only vertices of degree 3. From Euler characteristic formula, such a scheme has $6g-3$ edges and $4g-2$ vertices. This leads to:

Definition. A *dominant pair* of genus g is a pair $(\mathfrak{s}, \lambda)$, where $\mathfrak{s}$ is a rooted scheme of genus g with $6g-3$ edges and $4g-2$ vertices of degree 3, and λ is bijection: $V(\mathfrak{s}) \to [\![0, 4g-3]\!]$.
The set of all dominant pairs of genus g is denoted $\mathcal{P}_g$.

Hence, only dominant pairs appear at the first order in the sum (8.1) (this was already the case in [9]).

9. The multiplicative contribution of the nodal stars.

Observe that Equation (8.11) has a remarkable multiplicative form: the contribution of the pair $(\mathfrak{s}, \lambda)$ is clearly separated from the one of $(\tau, F, \mathfrak{a})$. In this section, we will perform a summation on $(F, \mathfrak{a})$. Since we are only interested in the asymptotics, we consider only the case of dominant pairs.

9.1. Four types of nodes

We fix a triple $(\mathfrak{s}, \lambda, \tau)$ such that $(\mathfrak{s}, \tau)$ is a typed scheme and $(\mathfrak{s}, \lambda) \in \mathcal{P}_g$.

We say that an edge $e \in E(\mathfrak{s})$ is *special* if $\tau(e) \neq 0$. Let $v \in V(\mathfrak{s})$ be a vertex of $\mathfrak{s}$ adjacent to l special edges, and let $\tau_1, ..\tau_l$ be their types. We let $\tilde{\tau}_i = \tau_i$ if the corresponding edge is incoming at v, and $\tilde{\tau}_i = m - \tau_i$ if it is outgoing. Hence, from the discussion of subsection 6.3, in any full scheme of the form $(\mathfrak{s}, \tau, F, \mathfrak{a}, \lambda)$, $\tilde{\tau}_i$ is the type of the corresponding split-edge of F_v if F_v is a white elementary star; if F_v is a black elementary star, the corresponding type will be $m - \tilde{\tau}_i$. We have:

Lemma 9.1. *The vertices of $\mathfrak{s}$ can be of four types:*

1. *vertices such that none of the three adjacent edges are special.*
2. *vertices such that exactly two adjacent edges are specials. In this case, one has: $\tilde{\tau}_1 + \tilde{\tau}_2 = m$*
3.1. *vertices such that exactly three edges are specials, and such such that: $\tilde{\tau}_1 + \tilde{\tau}_2 + \tilde{\tau}_3 = m$.*
3.2. *vertices such that exactly three edges are specials, and such that: $\tilde{\tau}_1 + \tilde{\tau}_2 + \tilde{\tau}_3 = 2m$*

Proof. The lemma is a straightforward consequence of the Kirchoff law (Proposition 6.3), and the fact that the $\tilde{\tau}_i$'s are elements of $[\![1, m-1]\!]$. □

Observe that, in a full scheme, vertices of type **3.2** can correspond either to black or white elementary stars, whereas all the other correspond to white elementary stars only. We denote by v_1 (resp. v_2, $v_3^{(1)}$, $v_3^{(2)}$) the number of vertices of type **1** (resp. **2**, **3.1**, **3.2**). Then we have:

Lemma 9.2.

$$v_3^{(1)} = v_3^{(2)}$$

Proof. Recall that $n_{\neq}$ is the number of edges of type $\neq 0$. Counting half-edges implies:

$$2n_{\neq} = 3v_3^{(1)} + 3v_3^{(2)} + 2v_2$$

Now, we compute the total sum, over all edges of type $\neq 0$, of the quantity $\tau+(m-\tau)$. It is of course equal to $mn_{\neq}$, but also to the total sum of the types of the special half-edges leaving all the vertices, i.e.:

$$mv_3^{(1)} + 2mv_3^{(2)} + mv_2.$$

So we have :

$$\begin{cases} 2n_{\neq} &= 3v_3^{(1)} + 3v_3^{(2)} + 2v_2 \\ mn_{\neq} &= mv_3^{(1)} + 2mv_3^{(2)} + mv_2 \end{cases}$$

and eliminating $n_{\neq}$ implies the lemma. □

We let $D_{\mathfrak{s},\lambda,\tau}$ be the set of all pairs $(F,\mathfrak{a})$ such that $(\mathfrak{s},\tau,F,\mathfrak{a},\lambda) \in \mathcal{F}_g$. We say that such a pair is a *decoration* of $\mathfrak{s},\tau,\lambda$. We let

$$R_{\mathfrak{s},\tau,\lambda}(z) = \sum_{(F,\mathfrak{a})\in D_{\mathfrak{s},\lambda,\tau}} R_{\mathfrak{s},\tau,F,\mathfrak{a},\lambda}(z).$$

Due to the nature of Equation (8.11), we need to compute the sum:

$$\sum_{(F,\mathfrak{a})\in D_{\mathfrak{s},\lambda,\tau}} {z_{m,D}^{(c)}}^{(\mathfrak{s},\tau,F,\mathfrak{a},\lambda)}. \tag{9.1}$$

Each vertex of $\mathfrak{s}$ will contribute a certain multiplicative factor to this quantity.

9.1.1. vertices of type 1. A vertex v of type 1 is ajacent to three edges of type 0. Hence the star F_v can be either a single vertex $\circ$, or a white elementary star with three distinguished labelled vertices. The corresponding multiplicative factor is therefore:

$$1 + \sum_{k\in D} \frac{[(m-1)k][(m-1)k-1]}{2} \binom{mk-1}{k} {t_{m,D}^{(c)}}^{(m-1)k} = \frac{\gamma_{m,D}}{2}.$$

Moreover, in this case, the half-edges ajacent to e are all of type 0, so they do not carry any correcting star of $\mathfrak{a}$.

9.1.2. vertices of type 2. First, a vertex of type two cannot be decorated by a black star, since it is linked to an edge of type 0. Then, a vertex of type 2 corresponds to a white elementary star with exactly two special edges, which is rooted at a labelled vertex. There are $\frac{k[(m-1)k-1]}{2}\binom{mk-1}{k}$ of those. Moreover, each time a special half-edge is outgoing at v, we need to add a correction black star in $\mathfrak{a}$ for the corresponding superchain to begin with a white star. Observe that the number of black stars with two marked special edges is $(m-1)$, so that each black star added in $\mathfrak{a}$ contributes a factor $(m-1)z_{m,D}^{(c)}$ at the critical point. Hence the multiplicative contribution of a vertex of type **2** is:

$$
\begin{aligned}
& [z_{m,D}^{(c)}(m-1)]^{out(v)} \sum_{k\in D} \frac{k[(m-1)k-1]}{2}\binom{mk-1}{k} {z_{m,D}^{(c)}}^{k} {T_c}^{(m-1)k-1} \\
= \quad & \frac{[z_{m,D}^{(c)}(m-1)]^{out(v)}}{T_c}\frac{\gamma_{m,D}}{2}
\end{aligned}
$$

where $out(v)$ denotes the number of outgoing special half-edges at v.

9.1.3. vertices of type 3.1 Such a vertex can correspond only to a white star. In the m-walk reformulation, this star is a walk of length $mk \in mD$, with $(m-1)k-2$ steps -1, $k-1$ steps $m-1$, that begins with a special step, and with two other special steps. For a given k, the number of such walks is $\binom{mk-1}{(m-1)k-2,k-1,2} = \frac{k[(m-1)k-1]}{2}\binom{mk-1}{k}$. Moreover, as before, for each outgoing edge, we have to add a black polygon in the sequence $\mathfrak{a}$, so that the multiplicative contribution of a vertex of type **3.1** is finally:

$$
\begin{aligned}
& [z_{m,D}^{(c)}(m-1)]^{out(v)} \sum_{k\in D} \frac{k[(m-1)k-1]}{2}\binom{mk-1}{k} {z_{m,D}^{(c)}}^{k-1} {T_c}^{(m-1)k-2} \\
= \quad & \frac{[z_{m,D}^{(c)}(m-1)]^{out(v)}}{(m-1)z_{m,D}^{(c)}{T_c}^2}\frac{\gamma_{m,D}}{2} = [z_{m,D}^{(c)}(m-1)]^{out(v)-1}\frac{\gamma_{m,D}}{2{T_c}^2}
\end{aligned}
$$

9.1.4. vertices of type 3.2 Such a vertex can correspond to a white or black star.

If it is decorated by a white star, it corresponds to a walk of length $mk \in mD$, with $(m-1)k-1$ steps -1, $k-2$ steps $m-1$, beginning with a special step, and with two other special steps. The number of such walks being $\binom{mk-1}{(m-1)k-1,k-2,2} = \frac{k[k-1]}{2}\binom{mk-1}{k}$, the corresponding contribution is:

$$
\begin{aligned}
& [z_{m,D}^{(c)}(m-1)]^{out(v)} \sum_{k\in D} \frac{k[k-1]}{2}\binom{mk-1}{k} {z_{m,D}^{(c)}}^{k-2} {T_c}^{(m-1)k-1} \\
= \quad & [z_{m,D}^{(c)}(m-1)]^{out(v)} \left(\frac{1}{{z_{m,D}^{(c)}}^2 T_c}\frac{\gamma_{m,D}-(m-2)\beta_{m,D}}{2(m-1)^2}\right)
\end{aligned}
$$

In the other case, v is decorated by a black star with three marked special edges: there are $\frac{(m-1)(m-2)}{2}$ of those, so that the contribution of the black star is $\frac{(m-1)(m-2)}{2}z_{m,D}^{(c)}$. Now, for each *ingoing* special edge of v, we need to add a white elementary star with two special split-edges: the multiplicative contribution for adding such a star is $\sum_{k\in D}[(m-1)k-1]\binom{mk-1}{k}{z_{m,D}^{(c)}}^{k-1}{T_c}^{(m-1)k} = \frac{1}{(m-1)z_{m,D}^{(c)}}$. The multiplicative factor for the second case is therefore:

$$
\frac{(m-1)(m-2)}{2}z_{m,D}^{(c)}\left[\frac{1}{(m-1)z_{m,D}^{(c)}}\right]^{3-out(v)}
$$

Putting the two cases together, the multiplicative contribution of a vertex of the type

3.2 is:

$$
\begin{aligned}
& [z^{(c)}_{m,D}(m-1)]^{out(v)} \left(\frac{1}{{z^{(c)}_{m,D}}^2 T_c} \frac{\gamma_{m,D}-(m-2)\beta_{m,D}}{2(m-1)^2} + \frac{m-2}{2(m-1)^2 {z^{(c)}_{m,D}}^2} \right) \\
= \quad & [z^{(c)}_{m,D}(m-1)]^{out(v)-2} \frac{\gamma_{m,D}}{2T_c}
\end{aligned}
$$

where we used that $T_c = \beta_{m,D}$.

9.2. Final asymptotics

Putting the four cases together, it finally comes that:

$$
\begin{aligned}
& \sum_{(F,\mathfrak{a}) \in D_{\mathfrak{s},\lambda,\tau}} {z^{(c)}_{m,D}}^{(\mathfrak{s},\tau,F,\mathfrak{a},\lambda)} \\
= \quad & \prod_{v:type\ 1} \frac{\gamma_{m,D}}{2} \prod_{v:type\ 2} \frac{[z^{(c)}_{m,D}(m-1)]^{out(v)-1}}{T_c} \frac{\gamma_{m,D}}{2} \\
& \prod_{v:type\ 3.1} [z^{(c)}_{m,D}(m-1)]^{out(v)-1} \frac{\gamma_{m,D}}{2{T_c}^2} \prod_{v:type\ 3.2} [z^{(c)}_{m,D}(m-1)]^{out(v)-2} \frac{\gamma_{m,D}}{2T_c} \\
= \quad & \left(\frac{\gamma_{m,D}}{2}\right)^{|V(\mathfrak{s})|} [z^{(c)}_{m,D}(m-1)]^{out(\mathfrak{s})-v_2-v_3^{(1)}-2v_3^{(2)}} {T_c}^{-v_2-2v_3^{(1)}-v_3^{(2)}}
\end{aligned}
$$

where $out(\mathfrak{s}) = \sum_{v\ type\ 2;3.1;3.2} out(v)$ is the total number of special half-edges that are outgoing. Observe that $out(\mathfrak{s})$ is also the total number of special edges (since each edge has exactly one outgoing half-edge), i.e. $out(\mathfrak{s}) = n_{\neq}$. Moreover, since $v_3^{(1)} = v_3^{(2)}$, we have: $v_2 + v_3^{(1)} + 2v_3^{(2)} = v_2 + \frac{3}{2}v_3 = n_{\neq}$.

Hence the multiplicative factor corresponding to all decorations of $\mathfrak{s}, \tau, \lambda$ is:

$$
\sum_{(F,\mathfrak{a}) \in D_{\mathfrak{s},\lambda,\tau}} {z^{(c)}_{m,D}}^{(\mathfrak{s},\tau,\mathfrak{F},\mathfrak{a},\lambda)} = \left(\frac{1}{T_c}\right)^{n_{\neq}} \left(\frac{\gamma_{m,D}}{2}\right)^{|V(\mathfrak{s})|}.
$$

That is where something great happens: the factor $\left(\frac{1}{T_c}\right)^{n_{\neq}}$ simplifies with ${T_c}^{n_{\neq}}$ in Equation (8.11). Hence, the first term in the singular expansion of $R_{\mathfrak{s},\lambda,\tau}$ does not depend on the typing ! This is, with Lemma 6.4, the main argument leading to Theorem 3.2. Precisely, summing Equation (8.11) over all the decorations gives:

$$
\begin{aligned}
R_{\mathfrak{s},\lambda,\tau}(z) = c_{\mathfrak{s},\lambda} & \left(\frac{\gamma_{m,D}}{2}\right)^{|V(\mathfrak{s})|} \\
& \times (m-1)^{\frac{k+m}{4}} m^{\frac{M-k}{2}} \gamma_{m,D}^{\frac{M-3k}{4}} \beta_{m,D}^{-\frac{k+M}{4}} 3^{\frac{k-M}{2}} 2^{\frac{-k-5M}{4}} \left(1 - \frac{z}{z^{(c)}_{m,D}}\right)^{-\frac{k+M}{4}} [1+o(1)]
\end{aligned}
$$

where $k = 6g-3$ and $|V(\mathfrak{s})| = M+1 = 4g-2$. This gives our main estimate:

Proposition 9.3. *When z tends to $z^{(c)}_{m,D}$, the following Puiseux expansion holds:*

$$R_{\mathfrak{s},\lambda,\tau}(z) = c_{\mathfrak{s},\lambda}(m-1)^{\frac{5g-3}{2}} m^{-g} \gamma_{m,D}^{\frac{g-1}{2}} \beta_{m,D}^{\frac{3-5g}{2}} 3^g 2^{\frac{13-21g}{2}} \left(1 - \frac{z}{z^{(c)}_{m,D}}\right)^{-\frac{k+M}{4}} [1+o(1)] \quad (9.2)$$

Let $d = \gcd(D)$. Then $T_\circ(z)$ is actually a series in z^d. It has therefore at least d dominant singularities, which are the $z^{(c)}_{m,D}\xi^k$ for a primitive d-th root of unity ξ. Now, the positivity of the coefficients in Equation (7.1) easily shows that these are the only singularities of $T_\circ(z)$, and hence of $t(z)$. Hence, due to the compositional nature of the series $R_{\mathfrak{f}}(z)$ (up to the prefactor $z^{\mathfrak{f}}$, $R_{\mathfrak{f}}(z)$ is in fact a power series with positive coefficients in $t(z)$), this implies that the $z^{(c)}_{m,D}\xi^k$ are the d only dominant roots of $R_{\mathfrak{f}}(z)$ for all $\mathfrak{f}$, so that they are also the d only dominant roots of $R_{\mathfrak{s},\tau,\lambda}(z)$.

Now, $R_{\mathfrak{s},\tau,\lambda}(z)$ being an algebraic series, it is amenable to singularity analysis, in the classical sense of [12]. Hence Equation (9.2) and the classical transfer theorems of [12] imply that the coefficient of z^n in $R_{\mathfrak{s},\tau,\lambda}(z)$ satisfies:

$$[z^n]R_{\mathfrak{s},\lambda,\tau}(z) \sim \frac{dc_{\mathfrak{s},\lambda}}{\Gamma\left(\frac{5g-3}{2}\right)}(m-1)^{\frac{5g-3}{2}} m^{-g} \gamma_{m,D}^{\frac{g-1}{2}} \beta_{m,D}^{\frac{3-5g}{2}} 3^g 2^{\frac{13-21g}{2}} \cdot n^{\frac{5g-5}{2}} {z^{(c)}_{m,D}}^{-n}$$

when n goes to infinity along multiples of d. Using Corollary 8.1 and Theorem 4.3, we obtain that the number $h^\bullet_{g,m,D}(n)$ of *rooted and pointed* m-hypermaps of degree set mD with n black faces satisfies, when n tends to infinity along multiples of d:

$$h^\bullet_{g,m,D}(n) \sim \frac{dc_g}{\Gamma\left(\frac{5g-3}{2}\right)}(m-1)^{\frac{5g-3}{2}} m^{1-g} \gamma_{m,D}^{\frac{g-1}{2}} \beta_{m,D}^{\frac{3-5g}{2}} 3^g 2^{\frac{11-21g}{2}} \cdot n^{\frac{5g-3}{2}} {z^{(c)}_{m,D}}^{-n}$$

where $c_g = \frac{m^{2g}}{6g-3}\sum_{(s,\lambda)\in\mathcal{P}_g} c_{\mathfrak{s},\lambda}$; observe the factor m^{2g}, that comes from Lemma 6.4.

Moreover, it follows from the remark after Corollary 8.1 that the number $c^\bullet_{g,m,D}(n)$ of *rooted and pointed* m-constellations of degree set mD with n black faces satisfies, when n tends to infinity along multiples of d:

$$m^{2g} c^\bullet_{g,m,D}(n) \sim h^\bullet_{g,m,D}(n).$$

9.3. A "de-pointing lemma".

The last thing that remains to do to prove Theorems 3.1 and 3.2 is to relate maps which are both rooted and pointed to maps which are only rooted. First, observe that each rooted map with v vertices corresponds to exactly v distinct rooted and pointed maps. Moreover, the vertices of an m-hypermap correspond, except for the pointed vertex, to the labelled vertices of its mobile. Therefore counting rooted m-hypermaps is equivalent to counting mobiles with a *weight* inverse of their number of labelled vertices plus 1.

Now, let $\mathfrak{t}_n$ be a mobile corresponding to an m-hypermap of degree set mD and size n, chosen uniformly at random. We note Y_n its number of labelled vertices, so that the ratio between the numbers of rooted and pointed, and rooted only m-hypermaps equals $\mathbb{E}\left[\frac{1}{Y_n+1}\right]$.

We restrict to the case of mobiles of associated full scheme $\mathfrak{f} \in \mathcal{F}_g$. Recall that, up to a multiplicative factor, the series $R_{\mathfrak{f}}(z)$ is actually a series in $t(z) = zT_\circ(z)^{m-1}$:

$$R_{\mathfrak{f}}(z) = z^{\mathfrak{f}} {T_\circ}^{n_{\neq}} H(t(z))$$

for a certain series $H(t)$ given by equation (8.4). Now, let $T(z,u)$ be the generating series of planar mobiles, where z counts black vertices and u counts labelled vertices. Then the series :

$$K(z,u) := (z,u)^{\mathfrak{f}} T(z,u)^{n_{\neq}} H(zT(z,u)^{m-1})$$

is the generating series counting the same objects as $R_{\mathfrak{f}}$, by the number of black vertices and labelled vertices, where we noted $(z,u)^{\mathfrak{f}}$ the bivariate generating polynomial of the decorations of the scheme. Now, we have:

$$\mathbb{E}\left[Y_n\right] = \frac{[z^n]K'_u(z,1)}{[z^n]K(z,1)}$$

Moreover, the series $T(z,u)$ is given by the following equation:

$$T(z,u) = u + \sum_{k\in D} \binom{mk-1}{k} z^k T(z,u)^{(m-1)k}$$

A little computation shows that when z tends to $z^{(c)}_{m,D}$, the following expansion holds:

$$\frac{T'_u(z,1)}{zT'_z(z,1)} = \frac{(m-1)}{\beta_{m,D}}(1+o(1)),$$

which, since $t(z) = zT(z,1)^{m-1}$, implies the equation:

$$\left.\frac{d}{du}\right|_{u=1} H(zT(z,u)^{(m-1)}) = \frac{(m-1)}{\beta_{m,D}} \cdot z\frac{d}{dz}H(t(z))(1+o(1))$$

Moreover, we have seen that H has an expansion of the form:

$$H(t) = (cte)(t^{(c)}_{m,D} - t)^{-K}(1+o(1))$$

for a certain $K > 0$. Recall that in what precedes, we have already examined the singularity type of all the series under consideration. Therefore we know that all these series are algebraic series amenable to singularity analysis, in the sense of [12], and we can write down the following computations:

$$\begin{aligned}\frac{[z^n]K'_u(z,1)}{[z^n]K(z,1)} &\sim \frac{[z^n]\left.\frac{d}{du}\right|_{u=1} H(zT(z,u)^{(m-1)})}{[z^n]H(t(z))}\\ &\sim \frac{m-1}{\beta_{m,D}} \cdot \frac{[z^n]z\frac{d}{dz}H(t(z))}{[z^n]H(t(z))}\\ &\sim \frac{(m-1)n}{\beta_{m,D}}\end{aligned}$$

Therefore we have shown the convergence: $\mathbb{E}\left[\frac{Y_n}{n}\right] \to \frac{m-1}{\beta_{m,D}}$. The same computation (with this time a second order derivative) shows that the second moment satisfies: $\mathbb{E}\left[\left(\frac{Y_n}{n}\right)^2\right] \to \left(\frac{m-1}{\beta_{m,D}}\right)^2$. Consequently, we deduce from Chebichev inequality that the convergence actually holds in probability:

$$\frac{Y_n}{n} \xrightarrow{(\mathrm{P})} \frac{m-1}{\beta_{m,D}}.$$

This implies the convergence in probability of $\frac{n}{Y_n+1}$ to $\frac{\beta_{m,D}}{m-1}$, and since this variable is bounded, we deduce the convergence:

$$\mathbb{E}\left[\frac{n}{Y_n+1}\right] \rightarrow \frac{\beta_{m,D}}{m-1}.$$

From the previous discussion, we obtain:

Lemma 9.4. *The numbers of rooted and pointed, and rooted only m-hypermaps or m-constellations are related by the following asymptotic relations, when n tends to infinity along multiples of d:*

$$h_{g,m,D}(n) \sim \frac{\beta_{m,D}}{(m-1)n} h^{\bullet}_{g,m,D}(n) \quad ; \quad c_{g,m,D}(n) \sim \frac{\beta_{m,D}}{(m-1)n} c^{\bullet}_{g,m,D}(n).$$

This last result completes the proof of Theorems 3.1 and 3.2, up to setting

$$t_g = \frac{c_g 3^g 2^{7-11g}}{(6g-3)\Gamma\left(\frac{5g-3}{2}\right)}.$$

The last thing to do is to check that t_g is indeed the same constant as in [4]: this will be done in the next and last subsection, where we examine some corollaries of the two theorems.

9.4. The case $D = \{k\}$.

In this subsection, we examine the case $D = \{k\}$. In this case, we have:

$$[(m-1)k-1]\binom{mk-1}{k} t_c^k = 1$$

which gives:

$$\begin{aligned} \beta_{m,k} &= \frac{(m-1)k}{(m-1)k-1} \\ \gamma_{m,k} &= (m-1)k \end{aligned}$$

We obtain the following:

Corollary 9.5. *Let $m \geq 2$ and $k \geq 2$ be integers. Then the number $c_{g,m,k}(n)$ of rooted m-constellations of genus g and size n, and whose all white faces have degree mk satisfies, when n tends to infinity along multiples of k:*

$$c_{g,m,k}(n) \sim t_g \frac{k}{2} \left(\frac{\sqrt{2}\sqrt{m-1}[(m-1)k-1]^{\frac{5}{2}}}{mk^2}\right)^{g-1} n^{\frac{5(g-1)}{2}} \left(z^{(c)}_{m,k}\right)^{-n}$$

where: $z_{m,k}^{(c)} = \left[\frac{(m-1)k}{(m-1)k-1}\right]^{1-m} \left[[(m-1)k-1]\binom{mk-1}{k}\right]^{-\frac{1}{k}}$.

For $m = 2$, we obtain the asymptotic number of bipartite $2k$-angulations with n edges:

$$c_{g,2,k}(n) \sim t_g \frac{k}{2} \left[\frac{1}{\sqrt{2}} \frac{(k-1)^{5/2}}{k^2}\right]^{g-1} n^{\frac{5(g-1)}{2}} z_{2,k}^{(c)\,-n}$$

If furthermore $k = 2$, we recover the asymptotic number of bipartite quadrangulations with $2n$ edges (which is also the number of maps with n edges, thanks to the classical bijection of Tutte), in accordance with [3, 9]):

Corollary 9.6. *The number $m_n^{(g)}$ of rooted maps on $\mathcal{S}_g$ with n edges satisfies:*

$$m_n^{(g)} \sim t_g n^{\frac{5(g-1)}{2}} 12^n$$

In particular, this proves that our constant t_g is indeed the same as the one introduced in [3]. Our last corollary concerns the number of all m-constellations of genus g (without degree restriction). The following lemma is classical and reduces the study of all m-constellations (without degree restriction) to the study of degree restricted $m+1$-constellations. See the proof of Corollary 2.4 in [6].

Lemma 9.7. *There is a bijection between rooted m-constellations with n black faces and rooted $m+1$-constellations with n black faces where all white faces have degree $m+1$.*

This implies

Corollary 9.8. *The number of all rooted m-constellations with n black faces on a surface of genus g is asymptotically equivalent to:*

$$\frac{t_g}{2} \left(\frac{\sqrt{2m}(m-1)^{5/2}}{m+1}\right)^{g-1} n^{\frac{5(g-1)}{2}} \left(\frac{m^{m+1}}{(m-1)^{m-1}}\right)^n.$$

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