Maps, symmetric functions, and decomposition equations.
Exercise sessions

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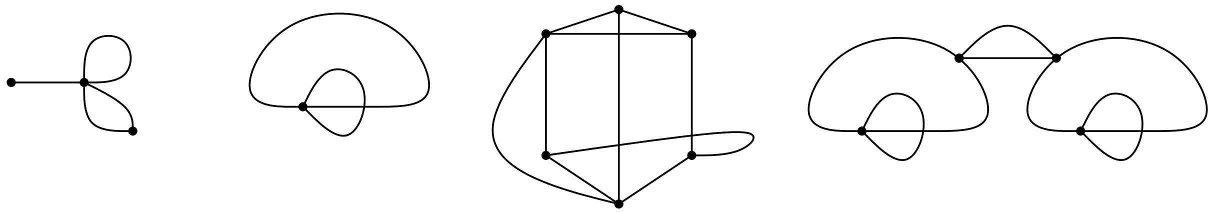
Exercise 0 – Warmup

1. How many faces do the following maps have? What is their genus?

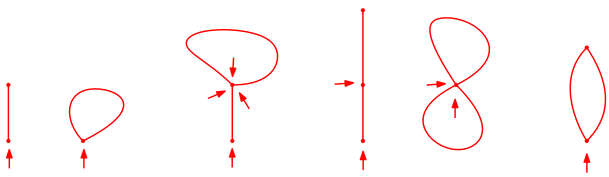
   Use your finger (or pen) for faces and Euler's formula $v + f - e = 2 - 2g$ for genus. $f = 3, 1, 2, 2$, $g = 0, 1, 2, 2$.



2. Can you add one edge (and no vertex) to the first map on the left to create a map of genus 1? a map with 4 faces? a map of genus 1 with 4 faces?
   yes (iff you link two corners in two different faces). yes (iff you link two corners in the same face). no (from the two iffs)

3. Can you add one edge to the second map to create a map with 2 faces? a map of genus 2?
   Yes, just add a diagonal in a face, this doesn't change the genus (in this case the diagonal will be a loop edge since there is a unique vertex). No, to increase the genus you would need to link two corners from different faces (by Euler's formula), but there is only one face.

4. Draw the 2 (resp. 9) rooted planar maps with 1 (resp. 2) edges.
   Here they are, each presented with all its inequivalent rootings:



Exercise 1 – Warmup - generating functions

1. Let $\mathcal{A}$ be a family of combinatorial objects with a size function $|\cdot| : \mathcal{A} \to \mathbb{N}$, and let $A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}$. Let $\mathcal{B}$ be a second such family and define $B(z)$ similarly. Show that $A(z)B(z) = \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} z^{|\alpha|+|\beta|}$. Explain to the members of your group why this can be useful to do combinatorics.
   The identity is clear. This is useful because it tells you that the o.g.f. of pairs of objects is the product of the o.g.f.

2. Let $\mathcal{A}, \mathcal{B}$ as in the previous question and let $\mathbf{A}(z) = \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n\geq 0} \frac{z^n}{n!} a_n$, where a_n is the number of objects of size n in $\mathcal{A}$. Define $\mathbf{B}(z)$ and b_n similarly. Show that $\mathbf{A}(z)\mathbf{B}(z) = \sum_{n\geq 0} \frac{z^n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)$. Explain to the members of your group why this can be useful to do combinatorics.
The identity is clear. This is useful because it tells you that the e.g.f. of pairs of *labelled* objects is the product of the e.g.f. (the binomials handle the redistribution of labels).

3. Let as in the lecture $C(z) = 1 + z + 2z^2 + 5z^3 + \cdots = \sum_n c_n z^n$ be the (ordinary) generating function of rooted planes trees by the number of edges. Deduce from the isomorphism $\mathcal{C} \approx \{\bullet\} + \{edge\} \times \mathcal{C}^2$ that $C(z) = 1 + zC(z)^2$.
direct consequence of question 1.

4. (confusing and unnatural) We consider rooted plane trees as in the previous question, but in which moreover the edges are labelled. Let $D(z) = \sum_n d_n \frac{z^n}{n!}$ be the (exponential) generating function of these objects, where d_n is the number of rooted plane trees with n *labelled* edges. Deduce from the isomorphism $\mathcal{D} \approx \{\bullet\} + \{edge\} \times \mathcal{D}^2$ that $D(z) = 1 + zD(z)^2$. Show that $d_n = n! c_n$ in *two* different ways (one bijective, one with generating functions).
direct consequence of question 2. From the equations we see that $C(z) = D(z)$, which implies $d_n = n! c_n$. The identity is also clear by saying that a rooted plane tree has no symmetry, so all the $n!$ labellings are inequivalent.

5. Think about this: in generating functions, everything works the same if I replace $\alpha \mapsto z^{|\alpha|}$ by $\alpha \mapsto z^{|\alpha|} w_\alpha$ where w_α is some weight associated to the objects in $\mathcal{A}$ which behaves multiplicatively under the cartesian product. What are typical examples of such weights?

Exercise 2 – The Tutte equation for planar maps.

The historical starting point of the decomposition equations we write in the lectures are the equations written by Tutte in a series of papers in the 1960's dedicated to the enumeration of planar maps. Let's look at the most famous one now.

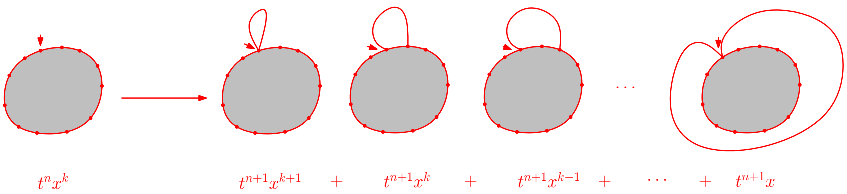
1. We let $a_{n,k}$ be the number of rooted (connected) planar maps with n edges whose root face (face containing the root corner) has degree k, and we let[1]

$$F(t,x) := \sum_{\substack{n\geq 0 \\ k\geq 0}} a_{n,k} t^n x^k = 1 + t(x + x^2) + \ldots$$

be the corresponding generating function. Show that

$$F(t,x) = 1 + tx^2 F(t,x)^2 + t\Delta F(t,x),$$

where Δ is the linear operator $\Delta : x^k \mapsto x + x^2 + \cdots + x^{k+1}$.
The map could be empty (first term). Otherwise, remove the first edge after the root corner. This could disconnect the map (second term: note that the number of corners aroud the root face increases by 2) or not (third term: to see this, obtain that starting from a map with root face degree k, there are $k+1$ ways to add a diagonal in this face starting from the root, giving rise to a new root face of size $1, \ldots, k+1$, hence the operator Δ. See figure:



2. Show that $\Delta G(t,x) = x\frac{xG(t,x) - G(t,1)}{x-1}$ for any $G \in \mathbb{Q}[x][[t]]$. Rewrite the previous equation for $F(t,x)$ in explicit form. Does this equation determine $F(t,x)$? Is it an algebraic equation?

[1] We conventionnally admit an "empty map" with 1 isolated vertex, 0 edge, and 1 (root) face of degree 0, hence the term 1 in the expansion.

We have $\Delta x = x + \cdots + x^{k+1} = x\frac{x^{k+1}-1}{x-1}$, then use linearity. We get, with $F = F(t,x)$ and $f = F(t,1)$,

$$F = 1 + tx^2F^2 + tx\frac{xF - f}{x-1}$$

This is not really an algebraic equation for $F(t,x)$ since it also involves the specialization $F(t,1)$. [Note: setting $x = 1$ in the equation is natural but won't work, since $d/dxF(t,x)_{x=1}$ will appear, and you're stuck].

3. Write the previous quadratic equation in the form $(g_1F + g_2)^2 = g_3$, with $F = F(t,x)$ and where g_1, g_2, g_3 may depend on the (unknown) univariate function $F(t,1)$. Show that there is a unique formal power series $X(t)$ such that $g_1F + g_2$ vanishes for $x = X(t)$.
You find g_1, g_2, g_3 by expanding the polynomial in F and equating the three coefficients (of F^0, F^1, F^2). Calculations done (I used Maple), the equation $g_1F + g_2 = 0$ (its numerator) can be written:

$$X(t) = 1 + 2F(t,X(t))tX(t)^3 - 2F(t,X(t))tX(t)^2 + tX(t)^2.$$

Even if we don't know $F(t,x)$ yet, this clearly defines the coefficients of a unique series $X(t)$ inductively ($F(t,x)$ is something well-defined and unique).

4. Deduce that $g_3\Big|_{x=X(t)}$ *and* $\left(\frac{\partial}{\partial x}g_3\right)\Big|_{x=X(t)}$ vanish. Explain how to compute $F(t,1)$ from there.
From the equation $(g_1F + g_2)^2 = g_3$, we clearly see that $X(t)$ is a double-root of g_3. Note that $g_3 = \frac{txf}{x-1} + 1/4\,\frac{t^2x^4 - 4\,tx^4 + 6\,tx^3 - 2\,tx^2 + x^2 - 2\,x + 1}{tx^2(x-1)^2}$ is a rational function in x. Therefore

$$\left(\frac{\partial}{\partial x}g_3 = 0 \quad , \quad \right)\Big|_{x=X(t)} = 0$$

are just two algebraic equations relating the two unknowns $F(t,1)$ and $x = X(t)$. We can just eliminate x and get an algebraic equation for $F(t,1)$!!! [Note: yes, you can eliminate a variable from two polynomials equations, and you will get a (higher) polynomial equation in the resulting variables. You can do this for example by typing "eliminate" in your favorite software. To understand the math, this is done (for example) by taking a resultant, `https://en.wikipedia.org/wiki/Resultant`.]

5. If you can, do it explicitly with Maple or you favourite computer algebra software. If you reach the end without mistake you will find the *amazing* formula, due to Tutte:

$$[t^n]F(t,1)\frac{2\cdot 3^n}{n+2}\mathrm{Cat}(n),$$

which is the founding stone of map enumeration! Or, equivalently, $27f^2t^2 - 18ft + f + 16t - 1$.

```
> # The equation, where we write F=F(t,x) and F=f(t,1)
  eq:=-F+1+t*x^2*F^2+t*x*(x*F-f)/(x-1);
```

$$eq := -F + 1 + t x^2 F^2 + \frac{t x (F x - f)}{x - 1}$$

```
> # Write it with g1,g2,g3 (we prefer to multiply the equation by t first,
  # so that the top coefficient is a square, to help Maple a little bit)
  -t*eq + (g1*F+g2)^2-g3;
  # solve for g1,g2,g3 by identifying coefficients:
  grels:=solve({seq(coeff(%,F,i),i=0..2)},{g1,g2,g3})[1];
```

$$-t\left(-F + 1 + t x^2 F^2 + \frac{t x (F x - f)}{x - 1}\right) + (F g1 + g2)^2 - g3$$

$$grels := \left\{g1 = t x, g2 = \frac{1}{2}\,\frac{t x^2 - x + 1}{x (x - 1)}, g3 = \frac{1}{4}\,\frac{4 f t^2 x^4 - 4 f t^2 x^3 + t^2 x^4 - 4 t x^4 + 6 t x^3 - 2 t x^2 + x^2 - 2 x + 1}{x^2 (x - 1)^2}\right\}$$

```
> # we look at the equation g1 F+ g2 , to see how it looks:
  numer(subs(grels, g1*F+g2));
```

$$2 F t x^3 - 2 F t x^2 + t x^2 - x + 1$$

```
> # we look at g3 and d/dx g3:
  subs(grels, g3);
  factor(diff(%,x));
  # And we just eliminate x, we obtain the wanted equation for f=F(t,1)!
  eliminate({%,%%},x)[2,1];
  # (note: the factor ft-1=0 is an artifact, clearly this branch is not the solution we want)
```

$$\frac{1}{4}\,\frac{4 f t^2 x^4 - 4 f t^2 x^3 + t^2 x^4 - 4 t x^4 + 6 t x^3 - 2 t x^2 + x^2 - 2 x + 1}{x^2 (x - 1)^2}$$

$$-\frac{1}{2}\,\frac{2 f t^2 x^4 - 2 f t^2 x^3 + t^2 x^4 - t x^4 + t x^3 + x^3 - 3 x^2 + 3 x - 1}{x^3 (x - 1)^3}$$

$$(27 f^2 t^2 - 18 f t + f + 16 t - 1)\,(f t - 1)$$

```
> # We have obtained the wanted equation for f:
  eqf:=27*f^2*t^2-18*f*t+f+16*t-1:
  # We can check its expansion and recognize the "9" maps of size two we struggled for earlier
  series(RootOf(eqf,f),t=0);
```

$$1 + 2 t + 9 t^2 + 54 t^3 + 378 t^4 + 2916 t^5 + \mathrm{O}(t^6)$$

```
> # And we can obtain the coefficient with the package gfun
  # convert algebraic to differential equation:
  gfun[algeqtodiffeq](eqf,f(t));
  # convert differential equation to recurrence
  gfun[diffeqtorec](%,f(t),u(n));
  # solve recurrence:
  rsolve(%,u(n));
  # ... and we have proved Tutte's formula!
```

$$2 + (6 t - 2) f(t) + (12 t^2 - t)\left(\frac{d}{dt} f(t)\right)$$

$$\{(6 + 12 n)\, u(n) + (-3 - n)\, u(n + 1), u(0) = 1\}$$

$$\frac{2\, 12^n\, \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma(3 + n)}$$

6. Refine the equation of question 1 by taking all face degrees into account (i.e. introduce a variable p_k for each $k \geq 1$, marking faces of degree k).
It is the same, with now $\Delta : x^k \mapsto p_1 x^{k+1} + p_2 x^k + \dots p_{k+1} x$. How to solve it is another story (cf the cultural note).

Cultural note 1: The equation of this exercise is a fundamental example of a polynomial "equation with one catalytic variable", i.e. an equation for a bivariate function $F(t, x)$ involving not only $F(t, x)$ but also $F(t, 1)$. The method of resolution we employed is the "quadratic method" of Tutte and Brown, which is a special case of the very general method of Bousquet-Mélou and Jehanne, which says that solutions are always algebraic for any such equation, under reasonable hypotheses. It also contains many examples, in particular the case of face-degrees of question 6 (first solved by Bender and Canfield). Reading (parts of) the BMJ paper is highly recommended! The solution of the present equation is also nicely presented in the book of Flajolet-Sedgewick, page 529 (also recommended!).

Cultural note2: One can write similar equations for the generating function $F_g(t; x_1, \dots, x_k)$ of maps of genus g having "k root faces", and compute them (in principle) by induction on $k + 2g$. This was done by Bender and Canfield in 1986 who proved in this way that the number of rooted maps with n edges and genus g grows as $t_g n^{\frac{5}{2}(g-1)} 12^n$ for some constant $t_g > 0$. The subject has been revived in mathematical physics with the Eynard-Orantin theory of topological recursion which is a very efficient way to compute these functions by induction, see e.g. the book "Counting surfaces" by Eynard. *Only* once you have solved all exercises you can try to write these equations (see my AEC lectures notes if you don't manage).

Exercise 3 – Permutation counting

1. Show that the number of labelled bipartite maps with n edges is $n!^2$.
   We want to count tripleso of permutations whose product is the identity. We can just choose the first two permutations arbitrarily and this fixes the third.

2. Show that there is a 1-to-$(n-1)!$ correspondence between *rooted*[2] bipartite maps with n edges, and *connected labelled* bipartite maps with n edges.
   Take a labelled map and declare the edge 1 (precisely the white corner following it c.w.) to be the root. You obtain each rooted map $(n-1)!$ times with this construction. Indeed, once a root is fixed all the labellings of edges are distinct (since any edge can be "specified" by a path from the root using "turn left/turn right" operations; here we use connectedness).

3. Deduce that the ordinary generating function of rooted (connected) bipartite maps by the number of edges is $\frac{td}{dt}\ln\sum_{n\geq 0} n!t^n$.
   By the exp-log principle [see e.g. Flajolet-Sedgewick's book], and question 1, $\ln\sum_{n\geq 0} n!t^n$ is the exponential g.f. of the labelled connected maps. From the previous question, we are done, since $d/dt\frac{t^n}{n!} = \frac{t^n}{(n-1)!}$.

Exercise 4 – Some leftovers from the lectures

1. As in the lectures, for a partition $\lambda \vdash n$, let $\mathcal{C}_\lambda$ be the set of permutations of cycle-type λ, and let $K_\lambda \in \mathbb{C}[\mathfrak{S}_n]$ defined by
   $$K_\lambda := \sum_{\sigma\in\mathcal{C}_\lambda} \sigma.$$
   Show that the K_λ for $\lambda \vdash n$ commute with any other element of $\mathbb{C}[\mathfrak{S}_n]$.
   $gK_\lambda g^{-1} = \sum_{\sigma\in\mathcal{C}_\lambda} g\sigma g^{-1} = K_\lambda$

2. For $1 \leq i \leq n$, let $\mathbb{C}[\mathfrak{S}_n] \ni J_i := (1,i) + (2,i) + \cdots + (i-1,i)$ be the i-th Jucys-Murphy element. Show that the J_i commute with each other. Show that
   $$\prod_{i=1}^{n}(u+J_i) = \sum_{\sigma\in\mathfrak{S}_n} u^{\ell(\sigma)}\sigma.$$
   For commutation, compute J_iJ_j and J_jJ_i. Induction on n. To create a permutation of size n, take one of size $n-1$ and either declare n a fixed point (number of cycles increase by 1), or multiply by (i,n) for some $i<n$ (number of cycles don't change).

3. Let $M \in Mat_{k\times k}$ and let $(M\cdot)$ be the linear operator acting on $Mat_{k\times k}$ by multiplication by M:
   $$(M\cdot) : A \longmapsto MA.$$
   Show that $\mathrm{Tr}(M\cdot) = k\mathrm{Tr}M$.
   Compute the trace in some base, for example elementary matrices $E_{i,j}$.

Exercise 5 – Symmetric functions

To learn symmetric function theory, a very good read is Stanley, Enumerative Combinatorics, vol 2, Chapter 7. This exercise contains the minimal steps to make the lectures self-contained. In particular we prove the rules to multiply/differentiate a Schur function by p_m, starting only from the SSYT definition.

[2] We recall that by convention a *rooted* map is always connected. Moreover by convention let us say that the root vertex of a rooted bipartite map is always white.

1. Prove that Schur functions, defined as generating functions of semistandard Young tableaux, are symmetric. *(Hint: show that they are symmetric under $x_i \leftrightarrow x_{i+1}$ for any $i \geq 1$.)*
   Look at the entries $(i, i+1)$, draw a picture, there are two types of them: the ones that form pairs $(i, i+1)$ which are incident vertically: among them there is the same number of i and $i+1$; other entries: these entries are arranged in horizontal chunks of the form $(i, \dots, i, i+1, \dots, i+1)$, and in any such chunk we can replace the pattern $i^a(i+1)^b$ with $i^b(i+1)^a$, since viewed from all entries $\neq i, i+1$, an i and an $i+1$ is the same.

2. Show that the Schur basis is triangular (in some sense) w.r.t. the monomial basis, and deduce that $(s_\lambda)_{\lambda \vdash n}$ is a basis of Λ_n, the space of homogeneous symmetric functions of degree n (and in fact even a $\mathbb{Z}$-basis, if one works with $\mathbb{Z}$ rather than $\mathbb{Q}$ or $\mathbb{C}$).
   Declare $\mu \leq \lambda$ if $\sum_{j<i} \mu_j \leq \sum_{j<i} \lambda_j$ for any i (this is called the dominance (partial) order). Then from the tableaux definition we have, after thinking about it, $s_\lambda = m_\lambda + \sum_{\mu<\lambda} (some\ coeff) m_\mu$.

3. In this question and the next we work with a finite number k of variables, i.e. we assume $\mathbf{x} = (x_1, \dots, x_k, 0, 0, \dots)$. For a sequence $(\beta_1, \dots, \beta_k)$ of integers we let $a_\beta := \det((x_i)^{\beta_j})_{1 \leq i,j \leq k}$. We let $\rho := (k-1, k-2, \dots, 0)$.
   Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition with at most k parts, and let $\tilde{s}_\mu := \frac{a_{\rho+\mu}}{a_\mu}$. Our goal is to prove that, as claimed in the lectures, $s_\mu(\mathbf{x}) = \tilde{s}_\mu(\mathbf{x})$, where we recall
$$s_\mu(\mathbf{x}) = \sum_{T \in SSYT(\mu)} \prod_i x_i^{m_i^T},$$
   where m_i^T is the number of entries equal to i in the tableau T. First prove that
$$a_\rho s_\mu = \sum_{T \in SSYT(\mu)} a_{\rho+m^T}.$$
   By definition we have
$$a_\rho s_\mu = \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \prod_i x_i^{k-\sigma_i} \sum_T x_i^{m_i^T}.$$
   But since Schur functions are symmetric this is equal to
$$\sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \prod_i x_i^{k-\sigma_i} \sum_T x_i^{m_{\sigma_i}^T} = \sum_T a_{\rho+m^T}.$$

4. Let T_μ be the tableau in which the i-th row is filled only with the letter i, for any $i \geq 1$. Prove that all the tableaux $T \in SSYT(\mu) \setminus \{T_\mu\}$ can be grouped together in pairs $\{T, \hat{T}\}$ such that
$$a_{\rho+m^T} + a_{\rho+m^{\hat{T}}} = 0.$$
   Hint: Scan the tableau from right to left and find the first column, then first row, where the tableau differs from T_μ. Then be creative!
   From left to right, look at the first column c that differs from T_μ, and let p be the smallest entry in that column that differs from T_μ. In all columns to the left of c, we exchange p and $p-1$ with the trick of question 1 (we leave c and columns to its right unchanged, so the construction is involutive – a pairing). Then we have $m_i^T = m_i^{\hat{T}}$ for $i \neq p, p-1$, and $m_p^{\hat{T}} = m_{p+1}^T - 1$, and $m_{p+1}^{\hat{T}} = m_p^T + 1$. This implies that the two corresponding determinants differ by permutin the columns p and $p+1$.

5. Deduce that $s_\mu = \tilde{s}_\mu$. *You have just proved that the "tableau definition" and "bialternant definition" of Schur functions are equivalent! Congratulations!*
   From the last question anything which is not T_μ doesn't survive in the sum, so we are done![Note: another simple proof is here (thanks Christian K.) *Equivalence of the combinatorial and the classical definitions of Schur functions Proctor, Robert A. J. Combin. Theory Ser. A 51 (1989), no. 1, 135-137.*]

6. Using the bialternant definition of Schur functions, prove the rule to multiply a Schur function by a powersum p_m, called the Murnaghan-Nakayama rule:

$$p_m s_\lambda = \sum_R \epsilon(R) s_{\lambda \uplus R}$$

where the sum is taken over all ribbons R of size m that can be added to λ to form a larger partition, and $\epsilon(R) = (-1)^{h(R)-1}$ where $h(R)$ is the number of rows of R.

(*Hint: what does adding a ribbon of size m do to the sequence $(\lambda_i - i)$??*)

If you draw a picture and think long engouh you will understand this: Adding a ribbon R of size m to λ has the following effect of the sequence $(\lambda_i - i)$: choose some i, replace $\lambda_i - i$ by $\lambda_i - i + m$, and reorder the sequence. Moreover, the sign associated to this reordering is equal to $\epsilon(R)$.

Therefore we only have to prove the following identity:

$$p_m a_\lambda = \sum_{i=1}^{k} a_{\lambda + m\delta_{i,\cdot}}.$$

But since $p_m = \sum_{i=1}^k x_i^m$, this identity is nothing by the linearity of the determinant according to the to i-th column!

7. Let $S \in \{0,1\}^{\mathbb{Z}}$ be an infinite word, such that $S_{-i} = 1$ and $S_i = 0$ for $i > n_0$, for some $n_0 \geq 0$. Let $m \geq 1$ and let $a = |\{j, S_j = 1, S_{j+m} = 0\}|$, $b = |\{j, S_j = 0, S_{j+m} = 1\}$. Show that $a - b = m$.

For $m = 1$ this is easy, read the word from right to left, you are counting the number of $0/1$ transitions minus the number of $1/0$ transitions, which equals 1 since you start with 0 and end with 1. For $m \geq 2$, apply the same argument on each congruence class modulo m.

8. (*) Deduce from the two previous questions the dual rule that we use in the lectures:

$$m \frac{\partial}{\partial p_m} s_\lambda = \sum_R \epsilon(R) s_{\lambda \setminus R},$$

where the sum is taken over all ribbons R of size m that can be removed from λ to form a smaller partition.

(*Hint: introduce the operators g_+, g_-, that act on Schur functions by adding/removing a ribbon of size m as in the RHS of the last two equations. Show (bijectively!) that $[g_-, g_+] = m$ (the multiplication by m). Show also that $[m\frac{\partial}{\partial p_m}, p_m] = m$ and conclude.*)

For convenience we write $s_\lambda = a_{\rho+\lambda}/a_\rho$ even when λ is not a partition. As already observed (adding a ribbon of size m is the same as adding m to one of the $\lambda_i - i$, and the signs work out well) we have $g_+ s_\lambda = \sum_i a_{\lambda + m\delta_i}/a_\rho$ and similarly $g_- s_\lambda = \sum_i a_{\lambda - m\delta_i}/a_\rho$, where δ_i is the indicator vector of the coordinate i.

Thus the coefficient of s_μ in $[g_-, g_+]s_\lambda$ is equal to $A - B$ where

- A is the (signed) number of ways to do this: take the sequence $\lambda_i - i$, add m to some part, remove m to some part, and get $\mu_i - i$. Everytime you add or remove a part, you get a sign given by the permutation for the reordering.
- B is the (signed) number of ways to do this: take the sequence $\lambda_i - i$, remove m to some part, add m to some part, and get $\mu_i - i$. Everytime you add or remove a part, you get a sign given by the permutation for the reordering.

If $\lambda \neq \mu$ we have $A = B$, indeed for any way to add/remove we can just remove/add at the same position, and the signs associated to the reorderings are the same. So the coefficient of s_μ is zero.

If $\lambda = \mu$, we only have to consider the case when we add/remove consecutively from the same part. Because determinants are antisymmetric, when we add/remove, we get a nonzero contribution only if we act on an integer $\lambda_i - i$ such that $\lambda_i - i + m$ is not part of the sequence

$\{\lambda_j - j\}$. Similarly, when we remove/add, we get a nonzero contribution only if we act on an integer $\lambda_i - i$ such that $\lambda_i - i - m$ is not part of the sequence $\{\lambda_j - j\}$. In both cases, the sign of the contribution is $+1$ (because after the two operations we go back to λ so the overall reordering is just the identity).

So to prove $[g-, g+]s_\lambda = ms_\lambda$ it suffices to prove the following lemma: Let λ be a partition and $S = \{\lambda_i - i\}$. Let $a = |\{j \in S, j+m \notin S\}|$ and $b = |\{j \in S, j-m \notin S\}|$. Then $a - b = m$. But we did this in the previous question! (To *truly* apply the previous question we have to take $k = \infty$, so that $\lambda_i - i$ is equal to $-i$ for $i < -n_0$. If you don't like this, just take k large enough with respect to m and λ, and adapt the statement of the previous question to "words starting with enough ones and ending with enough zeroes".).

Exercise 6 – Rooted maps, labelled maps

Recall that for partitions λ, μ, ν of the same integer n, $B_{\lambda,\mu,\nu}$ counts labelled bipartite maps with n edges of profile (λ, μ, ν). Equivalently,

$$B_{\lambda,\mu,\nu} = |\{(\sigma_\circ, \sigma_\bullet, \sigma_\diamond) \in \mathcal{C}_\lambda \times \mathcal{C}_\mu \times \mathcal{C}_\nu, \sigma_\circ\sigma_\bullet\sigma_\diamond = 1\}|.$$

As in the lecture we let $F \equiv F(t; \mathbf{p}, \mathbf{q}, \mathbf{r})$ be the exponential *generating function of these numbers:*

$$F := \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\lambda,\mu,\nu\vdash n} B_{\lambda,\mu,\nu} p_\lambda q_\mu r_\nu.$$

Here we recall that variables are extended multiplicatively from integers to partitions, i.e. $p_\lambda := \prod_i p_{\lambda_i}$, etc. In the generating function F, the variables p_i, q_j, r_k mark respectively the white vertices of degree i, black vertices of degree j, faces of degree $2k$ of the underlying bipartite maps, while t marks the edges.

1. Let $B^{conn}_{\lambda,\mu,\nu}$ counts the same (labelled) objects as $B_{\lambda,\mu,\nu}$, but with the additional condition that the map is connected, and let F^{conn} the corresponding generating function. Give a relation between F^{conn} and F.
$F^{conn} = \ln F$, as in exercise 3.

2. We let $B^{root}_{\lambda,\mu,\nu}$ be the number of *rooted (connected, and not labelled)* bipartite maps of profile λ, μ, ν, and F^{root} their *ordinary* generating function:

$$F^{root} := \sum_{n\geq 0} t^n \sum_{\lambda,\mu,\nu\vdash n} B^{root}_{\lambda,\mu,\nu} p_\lambda q_\mu r_\nu.$$

Show that

$$F^{root} = \frac{td}{dt} \ln F.$$

Deduce that $\tau^{root} = \frac{td}{dt} \ln \tau$, where τ, τ^{root} are obtained from F, F^{root} by the substitution $\mathbf{r} = (u, u, \dots)$.
This is as in exercise 3 (there is an $(n-1)!$ factor between labelled and unlabelled maps, provided they are rooted (and connected).

3. In the lecture we obtained an equation of the form $\frac{md}{dq_m}\tau = t^m B_m \tau$ for a certain operator B_m. Deduce from this that

$$\frac{td}{dt}\tau = \Lambda\tau$$

for an operator Λ explicitly expressed in terms of the B_m.
In all monomials the total degree in q variables (where q_i has degree i) is equal to the degree in t (because the sum of the white vertex degrees is the number of edges) so $\sum_m mq_m \frac{\partial}{\partial q_m}$ acts as td/dt. Therefore we just take $\Lambda = \sum_m t^m q_m B_m$ and we are good.

4. (*) In the lecture we show that the operators B_m are computable inductively by $mB_m = [\Omega, B_{m-1}]$ for an explicit operator Ω. Show that in fact B_1 and Ω belong to the Lie algebra generated by p_1 and the "cut and join" operator $D = \sum_{i,j\geq 1} ijp_{i+j}\partial^2/(\partial p_i \partial p_j) + (i+j)p_ip_j\partial/\partial p_{i+j}$.

Show that we have $\tau = \exp(\Gamma(t))$ where the operator $\Gamma(t)$ belongs to the formal Lie algebra generated by p_1 and D.
You can explicitly check that (if I'm not mistaken) $B_1 = up_1 + [D, p_1]$ and $\Omega = [D, B_1]$. The last property is a little bit subtle and I don't have time to write it!

Cultural note: p_1 and D both belong to a very special infinite dimensional Lie algebra of differential operators called $\widehat{gl(\infty)}$, therefore Γ also does. A function of the form $\exp(\Gamma)$ is called a *tau-function of the Kadomstev-Petiashvili (KP) hierarchy.* Such functions are characterised by an infinite set of partial differential equations. For example the first one is:

$$-\frac{\partial^2}{\partial p_3 \partial p_1} f + \frac{\partial^2}{\partial p_2 \partial p_2} f + \frac{1}{12} \frac{\partial^4}{(\partial p_1)^4} f + \frac{1}{2} \left(\frac{\partial^2}{(\partial p_1)^2} f \right)^2 = 0,$$

written here in terms of $f = \ln \tau$. This equation, in the case of our function τ counting bipartite maps, has stunning consequences, but no direct combinatorial interpretation!!! See e.g. C.-Carrell 2014 and the discussion in conclusion of that paper.

5. (*) Write a (very non linear) equation directly for τ^{conn}.
Let $\Gamma = \sum_i y_i \frac{i\partial}{\partial p_i}$, then $\Gamma\tau^{conn}$ is the generating function of rooted maps, where y_i marks the root face of degree i. Then

$$\Gamma\tau^{conn} = \sum_{m \geq 1} t^m q_m \left[Y_+ \left(\Lambda + u + \sum_{i,j \geq 1} y_{i+j-1} \left(ip_i \frac{\partial}{\partial p_i} \tau^{conn} \right) \frac{\partial}{\partial y_{j-1}} \right) \right]^m (y_0).$$

This comes from applying the decomposition of the lecture directly to connected maps (when we add an edge there is one more case, when we connect to a new connected component, this is the third term in the brackets, in which the factor $\left(ip_i \frac{\partial}{\partial p_i} \tau^{conn} \right)$ is "take a new connect component and mark a corner in a face of degree i inside it" – this face will be merged with the root face of degree $j-1$ of our current component, and this will create a new root face of degree $i+j$, which is what our operator records (recall $Y_+ y_{i+j-1} = y_{i+j}$.).