Maps, symmetric functions, and decomposition equations.
Exercise sessions

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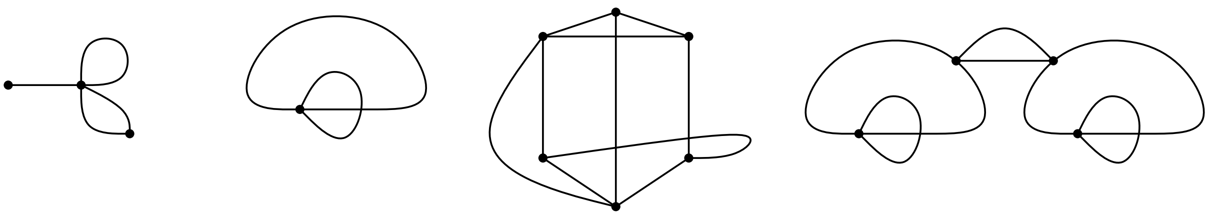
Exercise 0 – Warmup

1. How many faces do the following maps have? What is their genus?



2. Can you add one edge (and no vertex) to the first map on the left to create a map of genus 1? a map with 4 faces? a map of genus 1 with 4 faces?

3. Can you add one edge to the second map to create a map with 2 faces? a map of genus 2?

4. Draw the 2 (resp. 9) rooted planar maps with 1 (resp. 2) edges.

Exercise 1 – Warmup - generating functions

1. Let $\mathcal{A}$ be a family of combinatorial objects with a size function $|\cdot| : \mathcal{A} \to \mathbb{N}$, and let $A(z) = \sum_{\alpha\in\mathcal{A}} z^{|\alpha|}$. Let $\mathcal{B}$ be a second such family and define $B(z)$ similarly. Show that $A(z)B(z) = \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} z^{|\alpha|+|\beta|}$. Explain to the members of your group why this can be useful to do combinatorics.

2. Let $\mathcal{A},\mathcal{B}$ as in the previous question and let $\mathbf{A}(z) = \sum_{\alpha\in\mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n\geq 0} \frac{z^n}{n!} a_n$, where a_n is the number of objects of size n in $\mathcal{A}$. Define $\mathbf{B}(z)$ and b_n similarly. Show that $\mathbf{A}(z)\mathbf{B}(z) = \sum_{n\geq 0} \frac{z^n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)$. Explain to the members of your group why this can be useful to do combinatorics.

3. Let as in the lecture $C(z) = 1 + z + 2z^2 + 5z^3 + \cdots = \sum_n c_n z^n$ be the (ordinary) generating function of rooted planes trees by the number of edges. Deduce from the isomorphism $\mathcal{C} \approx \{\bullet\} + \{edge\} \times \mathcal{C}^2$ that $C(z) = 1 + zC(z)^2$.

4. (confusing and unnatural) We consider rooted plane trees as in the previous question, but in which moreover the edges are labelled. Let $D(z) = \sum_n d_n \frac{z^n}{n!}$ be the (exponential) generating function of these objects, where d_n is the number of rooted plane trees with n *labelled* edges. Deduce from the isomorphism $\mathcal{D} \approx \{\bullet\} + \{edge\} \times \mathcal{D}^2$ that $D(z) = 1 + zD(z)^2$. Show that $d_n = n!c_n$ in *two* different ways (one bijective, one with generating functions).

5. Think about this: in generating functions, everything works the same if I replace $\alpha \mapsto z^{|\alpha|}$ by $\alpha \mapsto z^{|\alpha|} w_\alpha$ where w_α is some weight associated to the objects in $\mathcal{A}$ which behaves multiplicatively under the cartesian product. What are typical examples of such weights?

Exercise 2 – The Tutte equation for planar maps.

The historical starting point of the decomposition equations we write in the lectures are the equations written by Tutte in a series of papers in the 1960's dedicated to the enumeration of planar maps. Let's look at the most famous one now.

1. We let $a_{n,k}$ be the number of rooted (connected) planar maps with n edges whose root face (face containing the root corner) has degree k, and we let[1]

$$F(t,x) := \sum_{\substack{n\geq 0\\ k\geq 0}} a_{n,k} t^n x^k = 1 + t(x+x^2) + \dots$$

be the corresponding generating function. Show that

$$F(t,x) = 1 + tx^2 F(t,x)^2 + t\Delta F(t,x),$$

where Δ is the linear operator $\Delta : x^k \mapsto x + x^2 + \cdots + x^{k+1}$.

2. Show that $\Delta G(t,x) = x\frac{xG(t,x)-G(t,1)}{x-1}$ for any $G \in \mathbb{Q}[x][[t]]$. Rewrite the previous equation for $F(t,x)$ in explicit form. Does this equation determine $F(t,x)$? Is it an algebraic equation?

3. Write the previous quadratic equation in the form $(g_1F+g_2)^2 = g_3$, with $F = F(t,x)$ and where g_1, g_2, g_3 may depend on the (unknown) univariate function $F(t,1)$. Show that there is a unique formal power series $X(t)$ such that $g_1F + g_2$ vanishes for $x = X(t)$.

4. Deduce that $g_3\Big|_{x=X(t)}$ *and* $\left(\frac{\partial}{\partial x} g_3\right)\Big|_{x=X(t)}$ vanish. Explain how to compute $F(t,1)$ from there.

5. If you can, do it explicitly with Maple or you favourite computer algebra software. If you reach the end without mistake you will find the *amazing* formula, due to Tutte:

$$[t^n]F(t,1)\frac{2\cdot 3^n}{n+2}\mathrm{Cat}(n),$$

which is the founding stone of map enumeration! Or, equivalently, $27f^2t^2 - 18ft + f + 16t - 1$.

6. Refine the equation of question 1 by taking all face degrees into account (i.e. introduce a variable p_k for each $k \geq 1$, marking faces of degree k).

Cultural note 1: The equation of this exercise is a fundamental example of a polynomial "equation with one catalytic variable", i.e. an equation for a bivariate function $F(t,x)$ involving not only $F(t,x)$ but also $F(t,1)$. The method of resolution we employed is the "quadratic method" of Tutte and Brown, which is a special case of the very general method of Bousquet-Mélou and Jehanne, which says that solutions are always algebraic for any such equation, under reasonable hypotheses. It also contains many examples, in particular the case of face-degrees of question 6 (first solved by Bender and Canfield). Reading (parts of) the BMJ paper is highly recommended! The solution of the present equation is also nicely presented in the book of Flajolet-Sedgewick, page 529 (also recommended!).

Cultural note2: One can write similar equations for the generating function $F_g(t; x_1, \dots, x_k)$ of maps of genus g having "k root faces", and compute them (in principle) by induction on $k+2g$. This was done by Bender and Canfield in 1986 who proved in this way that the number of rooted maps with n edges and genus g grows as $t_g n^{\frac{5}{2}(g-1)} 12^n$ for some constant $t_g > 0$. The subject has been revived in mathematical physics with the Eynard-Orantin theory of topological recursion which is a very efficient way to compute these functions by induction, see e.g. the book "Counting surfaces" by Eynard. *Only* once you have solved all exercises you can try to write these equations (see my AEC lectures notes if you don't manage).

Exercise 3 – Permutation counting

[1] We conventionnally admit an "empty map" with 1 isolated vertex, 0 edge, and 1 (root) face of degree 0, hence the term 1 in the expansion.

1. Show that the number of labelled bipartite maps with n edges is $n!^2$.

2. Show that there is a 1-to-$(n-1)!$ correspondence between *rooted*[2] bipartite maps with n edges, and *connected labelled* bipartite maps with n edges.

3. Deduce that the ordinary generating function of rooted (connected) bipartite maps by the number of edges is $\frac{td}{dt}\ln\sum_{n\geq 0} n!t^n$.

Exercise 4 – Some leftovers from the lectures

1. As in the lectures, for a partition $\lambda \vdash n$, let $\mathcal{C}_\lambda$ be the set of permutations of cycle-type λ, and let $K_\lambda \in \mathbb{C}[\mathfrak{S}_n]$ defined by
$$K_\lambda := \sum_{\sigma\in\mathcal{C}_\lambda} \sigma.$$
Show that the K_λ for $\lambda \vdash n$ commute with any other element of $\mathbb{C}[\mathfrak{S}_n]$.

2. For $1 \leq i \leq n$, let $\mathbb{C}[\mathfrak{S}_n] \ni J_i := (1,i)+(2,i)+\cdots+(i-1,i)$ be the i-th Jucys-Murphy element. Show that the J_i commute with each other. Show that
$$\prod_{i=1}^{n}(u+J_i) = \sum_{\sigma\in\mathfrak{S}_n} u^{\ell(\sigma)}\sigma.$$

3. Let $M \in Mat_{k\times k}$ and let $(M\cdot)$ be the linear operator acting on $Mat_{k\times k}$ by multiplication by M:
$$(M\cdot) : A \longmapsto MA.$$
Show that $\mathrm{Tr}(M\cdot) = k\mathrm{Tr}M$.

Exercise 5 – Symmetric functions

To learn symmetric function theory, a very good read is Stanley, Enumerative Combinatorics, vol 2, Chapter 7. This exercise contains the minimal steps to make the lectures self-contained. In particular we prove the rules to multiply/differentiate a Schur function by p_m, starting only from the SSYT definition.

1. Prove that Schur functions, defined as generating functions of semistandard Young tableaux, are symmetric. *(Hint: show that they are symmetric under $x_i \leftrightarrow x_{i+1}$ for any $i \geq 1$.)*

2. Show that the Schur basis is triangular (in some sense) w.r.t. the monomial basis, and deduce that $(s_\lambda)_{\lambda\vdash n}$ is a basis of Λ_n, the space of homogeneous symmetric functions of degree n (and in fact even a $\mathbb{Z}$-basis, if one works with $\mathbb{Z}$ rather than $\mathbb{Q}$ or $\mathbb{C}$).

3. In this question and the next we work with a finite number k of variables, i.e. we assume $\mathbf{x} = (x_1,\ldots,x_k,0,0,\ldots)$. For a sequence $(\beta_1,\ldots,\beta_k)$ of integers we let $a_\beta := \det((x_i)^{\beta_j})_{1\leq i,j\leq k}$. We let $\rho := (k-1,k-2,\ldots,0)$.

   Let $\mu = (\mu_1,\ldots,\mu_k)$ be a partition with at most k parts, and let $\tilde{s}_\mu := \frac{a_{\rho+\mu}}{a_\mu}$. Our goal is to prove that, as claimed in the lectures, $s_\mu(\mathbf{x}) = \tilde{s}_\mu(\mathbf{x})$, where we recall
$$s_\mu(\mathbf{x}) = \sum_{T\in SSYT(\mu)} \prod_i x_i^{m_i^T},$$
   where m_i^T is the number of entries equal to i in the tableau T. First prove that
$$a_\rho s_\mu = \sum_{T\in SSYT(\mu)} a_{\rho+m^T}.$$

[2] We recall that by convention a *rooted* map is always connected. Moreover by convention let us say that the root vertex of a rooted bipartite map is always white.

4. Let T_μ be the tableau in which the i-th row is filled only with the letter i, for any $i \geq 1$. Prove that all the tableaux $T \in SSYT(\mu) \setminus \{T_\mu\}$ can be grouped together in pairs $\{T, \hat{T}\}$ such that

$$a_{\rho+m^T} + a_{\rho+m^{\hat{T}}} = 0.$$

Hint: Scan the tableau from right to left and find the first column, then first row, where the tableau differs from T_μ. Then be creative!

5. Deduce that $s_\mu = \tilde{s}_\mu$. *You have just proved that the "tableau definition" and "bialternant definition" of Schur functions are equivalent! Congratulations!*

6. Using the bialternant definition of Schur functions, prove the rule to multiply a Schur function by a powersum p_m, called the Murnaghan-Nakayama rule:

$$p_m s_\lambda = \sum_R \epsilon(R) s_{\lambda \uplus R}$$

where the sum is taken over all ribbons R of size m that can be added to λ to form a larger partition, and $\epsilon(R) = (-1)^{h(R)-1}$ where $h(R)$ is the number of rows of R.

(*Hint: what does adding a ribbon of size m do to the sequence $(\lambda_i - i)$??*)

7. Let $S \in \{0,1\}^{\mathbb{Z}}$ be an infinite word, such that $S_{-i} = 1$ and $S_i = 0$ for $i > n_0$, for some $n_0 \geq 0$. Let $m \geq 1$ and let $a = |\{j, S_j = 1, S_{j+m} = 0\}|$, $b = |\{j, S_j = 0, S_{j+m} = 1\}$. Show that $a - b = m$.

8. (*) Deduce from the two previous questions the dual rule that we use in the lectures:

$$m \frac{\partial}{\partial p_m} s_\lambda = \sum_R \epsilon(R) s_{\lambda \setminus R},$$

where the sum is taken over all ribbons R of size m that can be removed from λ to form a smaller partition.

(*Hint: introduce the operators g_+, g_-, that act on Schur functions by adding/removing a ribbon of size m as in the RHS of the last two equations. Show (bijectively!) that $[g_-, g_+] = m$ (the multiplication by m). Show also that $[m\frac{\partial}{\partial p_m}, p_m] = m$ and conclude.*

Exercise 6 – Rooted maps, labelled maps

Recall that for partitions λ, μ, ν of the same integer n, $B_{\lambda,\mu,\nu}$ counts labelled bipartite maps with n edges of profile (λ, μ, ν). Equivalently,

$$B_{\lambda,\mu,\nu} = |\{(\sigma_\circ, \sigma_\bullet, \sigma_\diamond) \in \mathcal{C}_\lambda \times \mathcal{C}_\mu \times \mathcal{C}_\nu, \sigma_\circ\sigma_\bullet\sigma_\diamond = 1\}|.$$

As in the lecture we let $F \equiv F(t; \mathbf{p}, \mathbf{q}, \mathbf{r})$ be the exponential *generating function of these numbers:*

$$F := \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\lambda,\mu,\nu \vdash n} B_{\lambda,\mu,\nu} p_\lambda q_\mu r_\nu.$$

Here we recall that variables are extended multiplicatively from integers to partitions, i.e. $p_\lambda := \prod_i p_{\lambda_i}$, etc. In the generating function F, the variables p_i, q_j, r_k mark respectively the white vertices of degree i, black vertices of degree j, faces of degree $2k$ of the underlying bipartite maps, while t marks the edges.

1. Let $B^{conn}_{\lambda,\mu,\nu}$ counts the same (labelled) objects as $B_{\lambda,\mu,\nu}$, but with the additional condition that the map is connected, and let F^{conn} the corresponding generating function. Give a relation between F^{conn} and F.

2. We let $B^{root}_{\lambda,\mu,\nu}$ be the number of *rooted (connected, and not labelled)* bipartite maps of profile λ, μ, ν, and F^{root} their *ordinary* generating function:

$$F^{root} := \sum_{n \geq 0} t^n \sum_{\lambda,\mu,\nu \vdash n} B^{root}_{\lambda,\mu,\nu} p_\lambda q_\mu r_\nu.$$

Show that

$$F^{root} = \frac{td}{dt} \ln F.$$

Deduce that $\tau^{root} = \frac{td}{dt} \ln \tau$, where τ, τ^{root} are obtained from F, F^{root} by the substitution $\mathbf{r} = (u, u, \dots)$.

3. In the lecture we obtained an equation of the form $\frac{md}{dq_m}\tau = t^m B_m \tau$ for a certain operator B_m. Deduce from this that

$$\frac{td}{dt}\tau = \Lambda\tau$$

for an operator Λ explicitly expressed in terms of the B_m.

4. (*) In the lecture we show that the operators B_m are computable inductively by $mB_m = [\Omega, B_{m-1}]$ for an explicit operator Ω. Show that in fact B_1 and Ω belong to the Lie algebra generated by p_1 and the "cut and join" operator $D = \sum_{i,j\geq 1} ijp_{i+j}\partial^2/(\partial p_i \partial p_j) + (i+j)p_i p_j \partial/\partial p_{i+j}$. Show that we have $\tau = \exp(\Gamma(t))$ where the operator $\Gamma(t)$ belongs to the formal Lie algebra generated by p_1 and D.

   Cultural note: p_1 and D both belong to a very special infinite dimensional Lie algebra of differential operators called $\widehat{gl(\infty)}$, therefore Γ also does. A function of the form $\exp(\Gamma)$ is called a *tau-function of the Kadomstev-Petiashvili (KP) hierarchy.* Such functions are characterised by an infinite set of partial differential equations. For example the first one is:

$$-\frac{\partial^2}{\partial p_3 \partial p_1} f + \frac{\partial^2}{\partial p_2 \partial p_2} f + \frac{1}{12}\frac{\partial^4}{(\partial p_1)^4} f + \frac{1}{2}\left(\frac{\partial^2}{(\partial p_1)^2} f\right)^2 = 0,$$

   written here in terms of $f = \ln \tau$. This equation, in the case of our function τ counting bipartite maps, has stunning consequences, but no direct combinatorial interpretation!!! See e.g. C.-Carrell 2014 and the discussion in conclusion of that paper.

5. (*) Write a (very non linear) equation directly for τ^{conn}.