# Cartes non-orientables, polynômes de Jack, et $b$-positivité

Guillaume Chapuy
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joint work with

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Abstract. We introduce a one-parameter deformation of the 2-Toda tau-function of (weighted) Hurwitz numbers, obtained by deforming Schur functions into Jack symmetric functions. We show that its coefficients are polynomials in the deformation parameter $b$ with nonnegative integer coefficients. These coefficients count generalized branched coverings of the sphere by an arbitrary surface, orientable or not, with an appropriate $b$-weighting that "measures" in some sense their non-orientability.

Notable special cases include non-orientable *dessins d'enfants* for which we prove the most general result so far towards the Matching-Jack conjecture and the "$b$-conjecture" of Goulden and Jackson from 1996, expansions of the $\beta$-ensemble matrix model, deformations of the HCIZ integral, and $b$-Hurwitz numbers that we introduce here and that are $b$-deformations of classical (single or double) Hurwitz numbers obtained for $b = 0$.

A key role in our proof is played by a combinatorial model of non-orientable constellations equipped with a suitable $b$-weighting, whose partition function satisfies an infinite set of PDEs. These PDEs have two definitions, one given by Lax equations, the other one following an explicit combinatorial decomposition.

1. Introduction

**Hurwitz numbers and tau-functions.** Hurwitz numbers, in their most general sense, count the number of combinatorially inequivalent branched coverings of the sphere by an orientable surface with a given number of branchpoints and given ramification profiles. Hurwitz numbers and their variants (dessins d'enfants, weighted, monotone, orbifold Hurwitz numbers) have numerous connections to mathematical physics, combinatorics, and the moduli spaces of curves [Kon92, GJ97, ELSV01, GV03, GJV05, OP06, Mir07, GPH17].

Hurwitz himself [Hur91] showed that Hurwitz numbers can be expressed in terms of characters of the symmetric group. Equivalently, generating functions of Hurwitz numbers can be expressed explicitly in terms of Schur functions, which gives them a rich structure. A fundamental fact in the field, going back to Pandariphande [Pan00] and Okounkov [Oko00] and now understood in a wide generality (see e.g. [GJ08, GPH17]) is that Hurwitz numbers can be used to define a formal power series which is a tau-function of the KP, or more generally 2-Toda hierarchy [MJD00]. Explicitly, in the case of $k+2$ branchpoints, this tau-function has the form

$$(1)\quad \tau^{(k)}(t;\mathbf{p},\mathbf{q},u_1,\dots,u_k) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \left(\frac{f_\lambda}{n!}\right)^2 \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q})s_\lambda(\underline{u_1})\tilde{s}_\lambda(\underline{u_2})\dots\tilde{s}_\lambda(\underline{u_k}),$$

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. ERC-2016-STG 716083 "CombiTop"). MD is supported from *Narodowe Centrum Nauki*, grant UMO-2017/26/D/ST1/00186. Emails: guillaume.chapuy@irif.fr, mdolega@impan.pl.

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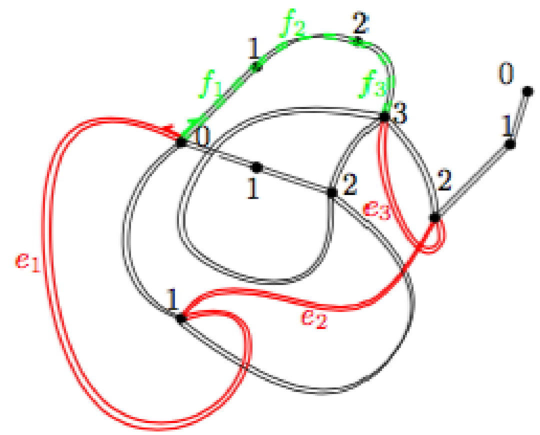
→ https://arxiv.org/abs/2004.07824




Séminaire CEA du LaBRI, avril 2020

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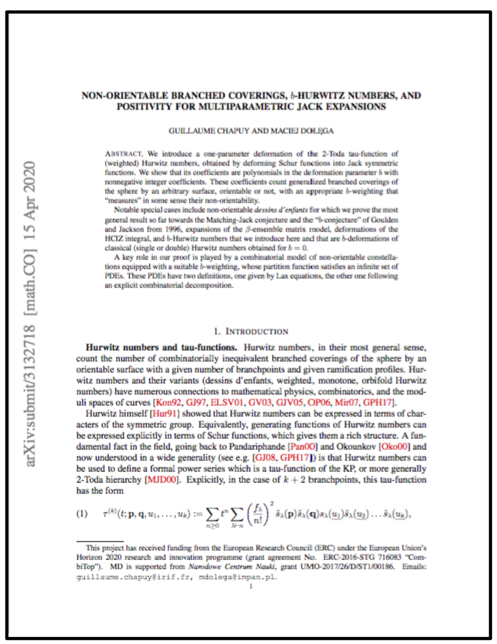

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Mes excuses pour les transparents en anglais

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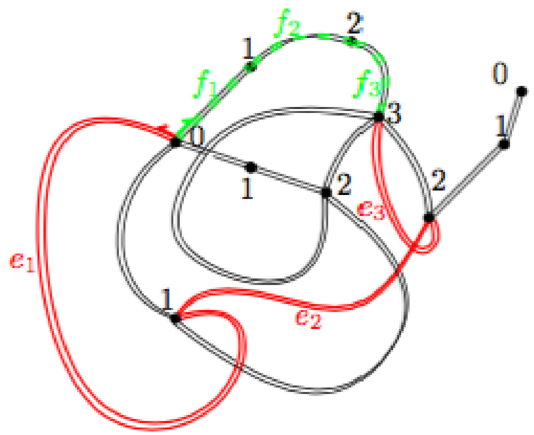


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# Symmetric functions

# Symmetric functions and their bases.

- Let $\Lambda_n$ be the vector space of formal power series in $x_1, x_2, \ldots$ which are symmetric, and homogeneous of degree $n$.

  Examples: $1 \in \Lambda_0$, $\sum_i x_i \in \Lambda_1$, $\sum_{i,j} x_i x_j \in \Lambda_2$, $\sum_i x_i^2 - 2\sum_{i,j} x_i x_j \in \Lambda_2$

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- $\Lambda_n$ has a basis which is naturally indexed by partitions of $n$

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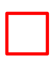
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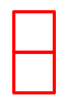
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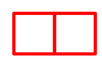
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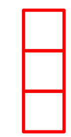
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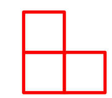
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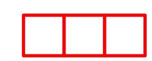
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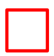
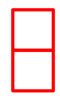
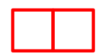
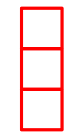
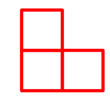
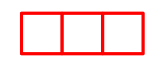
 $m_{[1]} = \sum_i x_i$

 $m_{[1,1]} = \sum_{i<j} x_i x_j$

 $m_{[2]} = \sum_i x_i^2$

 $m_{[1,1,1]} = \sum_{i<j<k} x_i x_j x_k$

 $m_{[2,1]} = \sum_{i,j} x_i^2 x_j$

 $m_{[3]} = \sum_i x_i^3$

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$$p_k(\mathbf{x}) = \sum_i x_i^k \quad \text{and} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_{\ell(\lambda)}}$$

($p_\lambda$: powersums)

Example: $p_{[2,2,1]} = p_2^2 p_1 = \sum_{i,j,k} x_i^2 x_j^2 x_k$
$= \ldots$

# Schur functions (I)

- If $\lambda$ is a partition, a semistandard Young tableau of shape $\lambda$ (SSYT) is a filling of $\lambda$ which is $\leq$ on rows and $\vee$ on columns.

$$\lambda = (4,4,2,1) \qquad T = \begin{array}{|c|c|c|c|} \hline 7 \\ \cline{1-1} 5 & 12 \\ \cline{1-2} 3 & 9 & 9 & 9 \\ \hline 2 & 2 & 3 & 7 \\ \hline \end{array}$$

$$x^T = x_2^2 x_3^2 x_5 x_7^2 x_9^3 x_{12}$$

$$s_{[2]} = \sum_{i \leq j} x_i x_j \qquad \begin{array}{|c|c|} \hline i & j \\ \hline \end{array}$$

$$s_{[1,1]} = \sum_{i < j} x_i x_j \qquad \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array}$$

$$s_{[3,1]} = \sum_{\substack{i \leq j \leq k \\ m > i}} x_i x_j x_k x_m \qquad \begin{array}{|c|c|c|} \hline m \\ \cline{1-1} i & j & k \\ \hline \end{array}$$

The Schur function $s_\lambda$ is the generating function of SSYT's of shape $\lambda$.

$$s_\lambda(x) = \sum_{T:SSYT(\lambda)} x^T$$

**Thm:** The $s_\lambda$ for $\lambda \vdash n$ are a basis of $\Lambda_n$.

(yes, in particular they are symmetric functions)

# Schur functions (bis)

- Cool fact: Viewed as polynomials in the powersums $\mathbf{p} = (p_k)_{k\geq 1}$, Schur functions generate characters of the symmetric group

$$\boxed{s_\lambda =: s_\lambda(\mathbf{p}) = \frac{1}{n!}\sum_{\mu\vdash n} |C_\mu|\chi^\lambda(\mu)p_\mu}$$

(here $n = |\lambda|$)

$\chi^\lambda(\mu)$: trace of a permutation of type $\mu$ acting on the representation $V^\lambda$ of $\mathfrak{S}_n$

- We use powersums to equip $\Lambda_n$ with the Hall scalar product:

$$\boxed{\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda,\mu} \quad z_\lambda = \frac{n!}{|\mathcal{C}_\lambda|}}$$

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- A characterisation of Schur functions:

$$\boxed{\begin{cases} \text{orthonormal for Hall: } \langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu} \\ \text{triangular w.r.t. to monomials: } s_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu}m_\mu \end{cases}}$$

[recall dominance order: $\mu \leq \lambda : \mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i \quad \forall i$]

# Jack polynomials

- Piotr Śniady: "*Jack polynomials are Schur functions under steroids.*"
- More precisely: we deform the Hall scalar product and keep triangularity

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}$$

- Jack polynomials:

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for example $J_{3,1}^{(\alpha)}(\mathbf{p}) = {p_1}^4 + (3\alpha - 1) p_2 {p_1}^2 + (2\alpha^2 - 2\alpha) p_3 p_1 - 2\alpha^2 p_4 - \alpha {p_2}^2$

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- For $\alpha = 1$, Jacks are (normalized) Schur: $J_\lambda^{(1)} = H_\lambda s_\lambda =: \tilde{s}_\lambda$

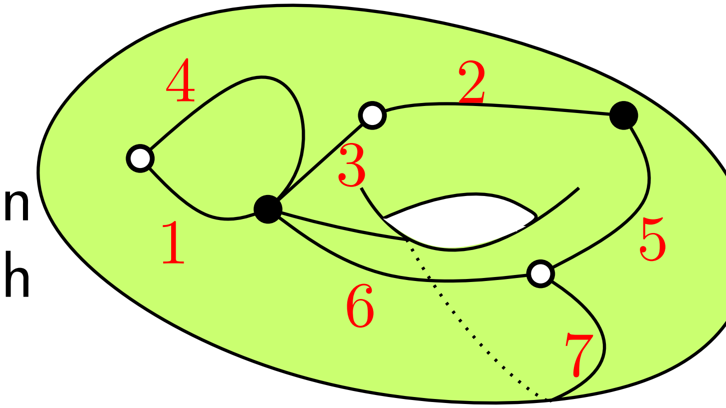
hook product

For $\alpha = 2$, Jacks are zonal polynomials: $J_\lambda^{(2)}$ is related to representation theory of the Gelfand pair $(\mathfrak{S}_{2n}, B_n)$ "in the same way as" $\tilde{s}_\lambda$ is to representation theory of $\mathfrak{S}_n$.

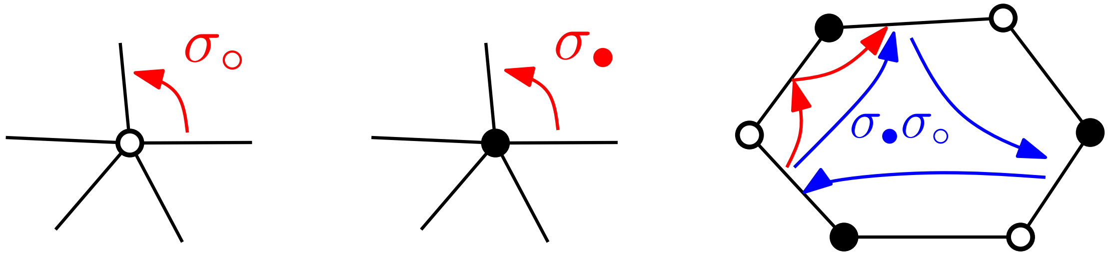
(a bit more later)

# Maps and factorizations

# Maps on orientable surfaces




- Bipartite map: bipartite ($\circ/\bullet$) graph embedded on an oriented surface with edges labelled $\{1, 2, \ldots, n\}$, with simply connected faces, considered up to homeomorphism.

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$\rightarrow$ the same as a triple of permutations $(\sigma_\circ, \sigma_\bullet, \sigma_\diamond)$ such that $\sigma_\circ\sigma_\bullet\sigma_\diamond = id$.

cf. [Cori-Machí 80s]

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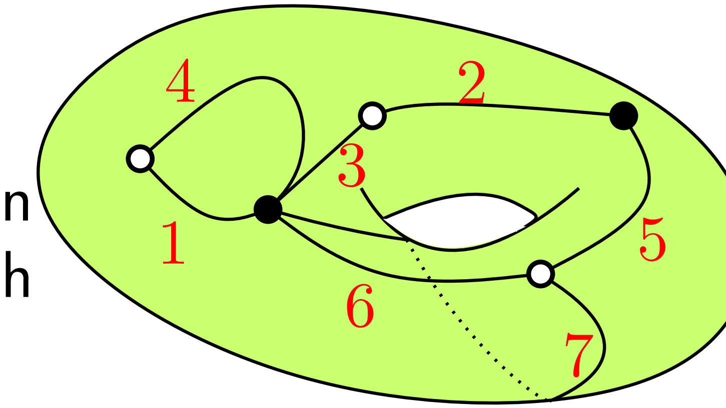


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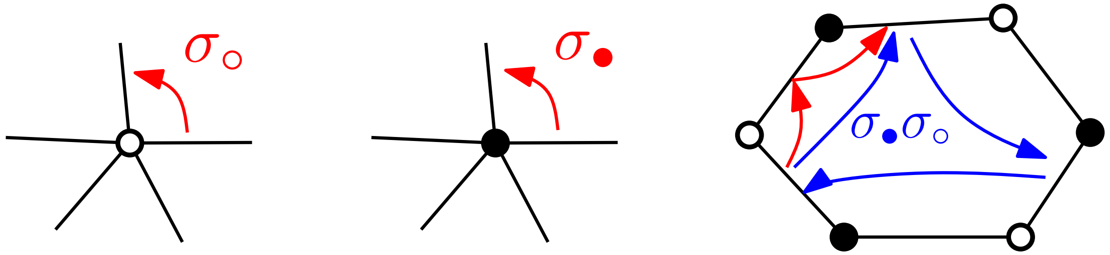


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- Cool fact (Frobenius). The number of factorizations of the identity in a finite group into factors of given conjugacy classes, can be expressed in terms of irreducible characters of the group. As a consequence we have:

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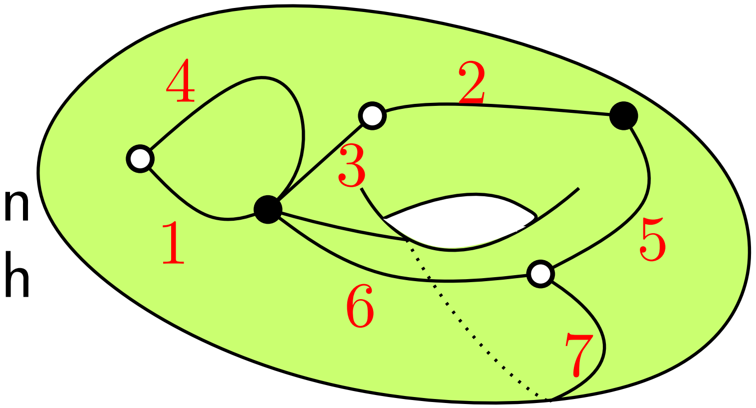


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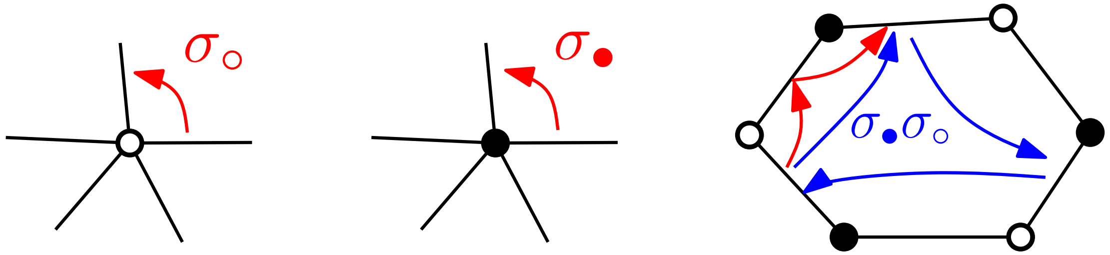


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$\sigma_\diamond^{-1} = \sigma_\bullet\sigma_\circ = (1)(2,4,6,3,5,7)$

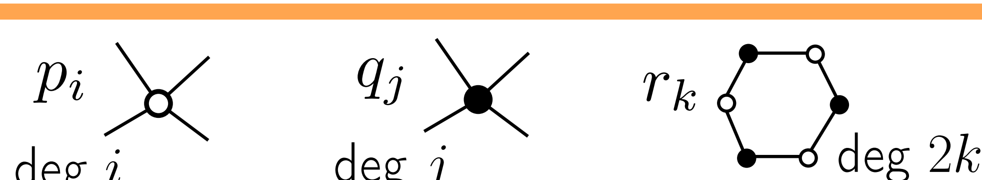
$\rightarrow$ the same as a triple of permutations $(\sigma_\circ, \sigma_\bullet, \sigma_\diamond)$ such that $\sigma_\circ\sigma_\bullet\sigma_\diamond = id$.

cf. [Cori-Machí 80s]

- Cool fact (Frobenius). The number of factorizations of the identity in a finite group into factors of given conjugacy classes, can be expressed in terms of irreducible characters of the group. As a consequence we have:

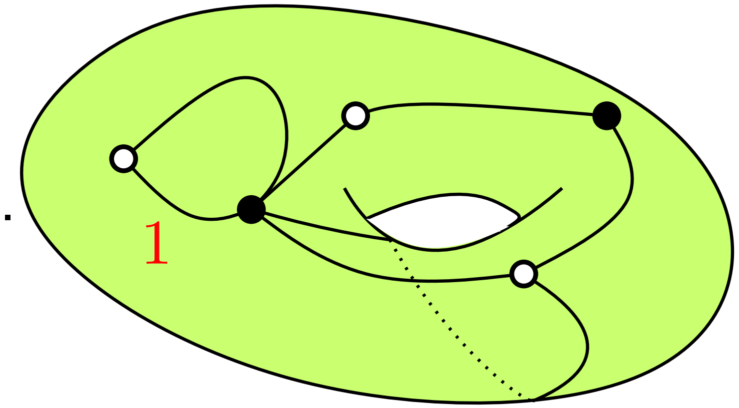
**"Character formula" for map generating function**

$$\sum_{\substack{\mathbf{m}:\\ bip.map}} \frac{t^n}{n!} p_{\lambda^\circ(\mathbf{m})} q_{\lambda^\bullet(\mathbf{m})} r_{\lambda^\diamond(\mathbf{m})} = \sum_{\lambda \in \mathcal{P}} t^{|\lambda|} H_\lambda s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}) s_\lambda(\mathbf{r})$$




[proof: put the two cool facts together]

## Variant: rooted maps




- We only remember the position of the label $1$ ("root edge").

We ask the surface to be connected.

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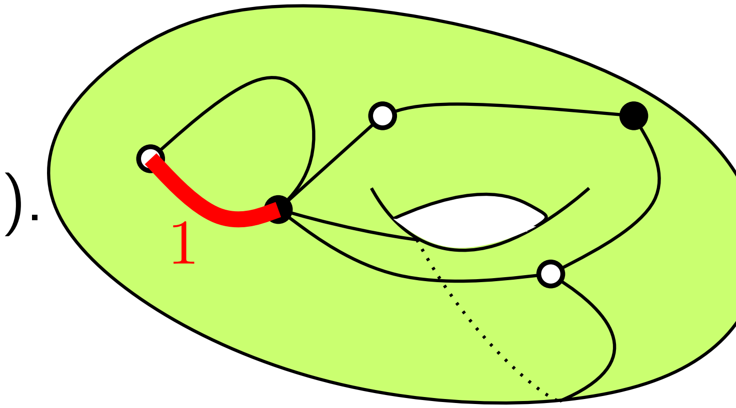


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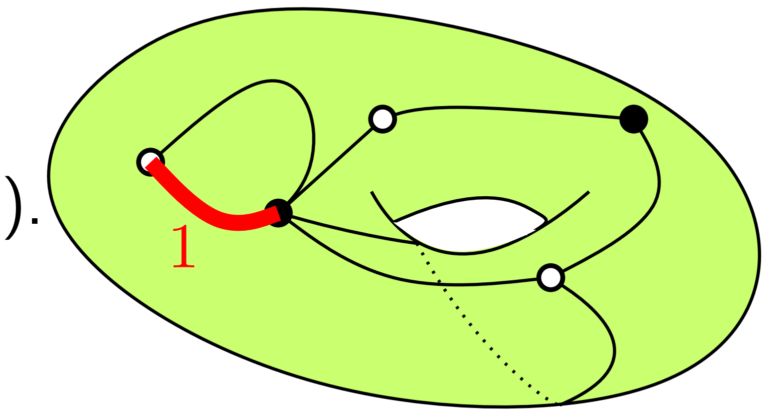


- We only remember the position of the label $1$ ("root edge").
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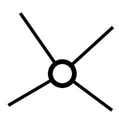
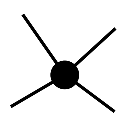
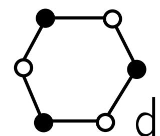
**"Character formula" for rooted maps**

$$\sum_{\substack{\mathbf{m}:\\ \textit{rooted bip. map}}} t^n p_{\lambda^\circ(\mathbf{m})} q_{\lambda^\bullet(\mathbf{m})} r_{\lambda^\diamond(\mathbf{m})} = t\frac{\partial}{\partial t} \log \sum_{\lambda \in \mathcal{P}} t^{|\lambda|} H_\lambda s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}) s_\lambda(\mathbf{r})$$

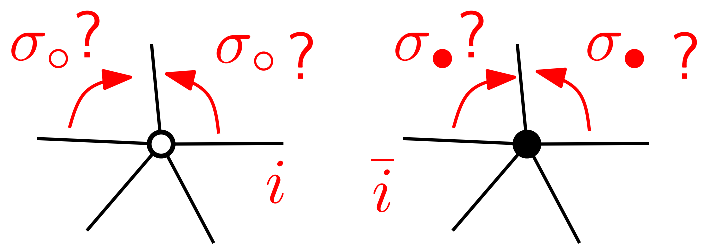
$p_i$  $q_j$  $r_k$ 


Note: we now have coefficients in $\mathbb{N}$ (no more labels, usual g.f. instead of exponential g.f.)

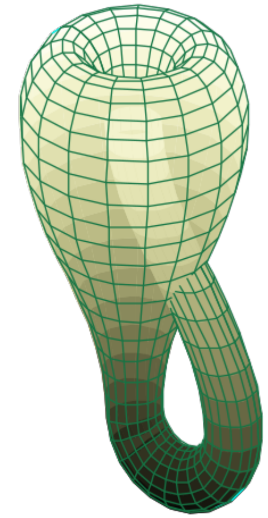
(this is not obvious from the RHS!)

# Maps on non (necessarily) orientable surfaces

- We now want to look at bipartite maps on non-necessarily orientable surfaces.




An encoding by permutations still works but everything is defined up to change of local orientation around each vertex.... so it's more complicated



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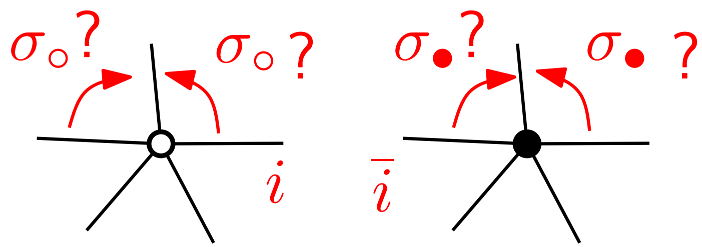


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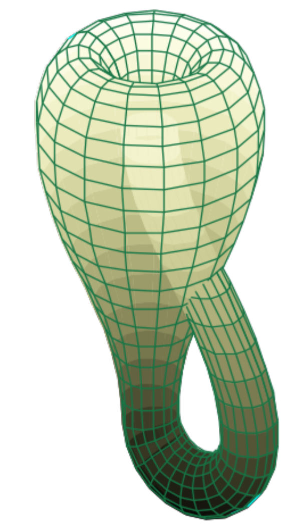

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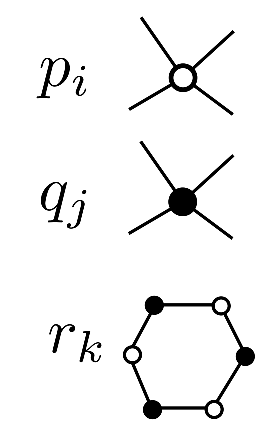

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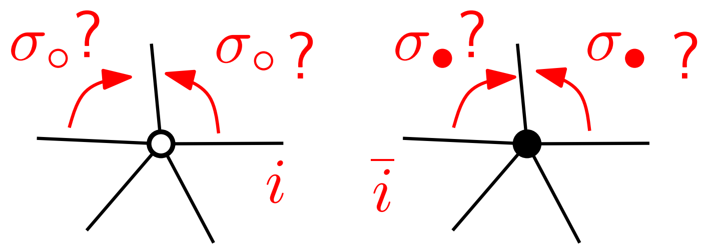


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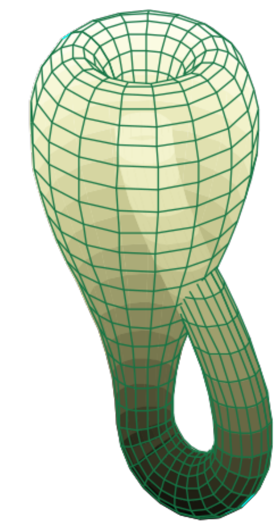

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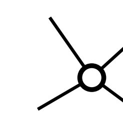
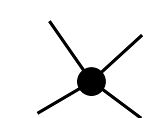
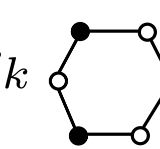
$p_i$ 

$q_j$ 

$r_k$ 

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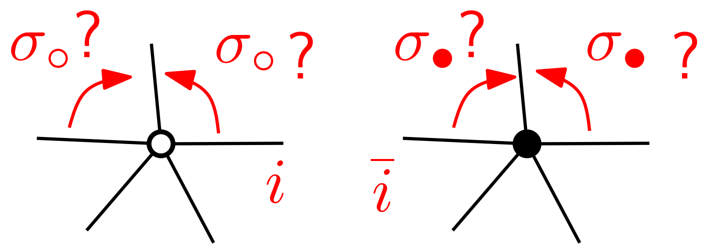


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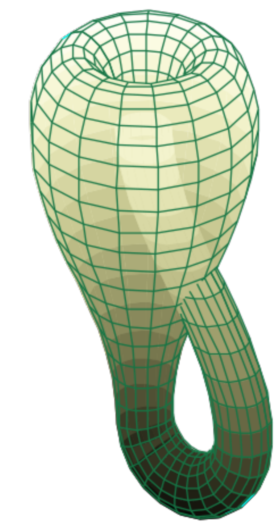

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Zonal

$\alpha = 2$

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??

$\alpha = 1$

Schur

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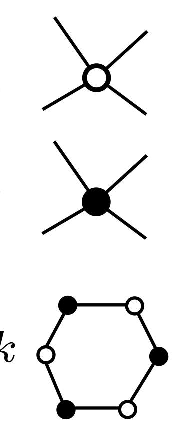

# $b$-positivity

# The Goulden-Jackson $b$-conjecture (1996)

- Conjecture: The generating function

$$(1+b)t\frac{\partial}{\partial t}\log\sum_{\lambda\in\mathcal{P}} t^{|\lambda|}\frac{J_\lambda^{(1+b)}(\mathbf{p})J_\lambda^{(1+b)}(\mathbf{q})J_\lambda^{(1+b)}(\mathbf{r})}{j_\lambda^{(1+b)}}$$

is $b$-positive, with integer coefficients!! It counts bipartite maps!!!

## The Goulden-Jackson $b$-conjecture (1996)

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- Why is it so interesting? Because we have (had?) NO tools to attack it!

for $b\notin\{0,1\}$ there is no "character" technology. Progress is rare.

[Lacroix '10] $\rightarrow$ OK if we keep ONE full set of variables ("times"):

$$\mathbf{p}=(p_i)_{i\geq 1},\ \mathbf{q}=(\delta_{i,2})_{i\geq 1},\ \mathbf{r}=\underline{u}=(u,u,\dots)$$

Uses [Okounkov'97] about (linear) expectations of Jacks under $\beta$-ensembles.

Other cases proved for some particular coefficients [Kanunnikov,Promyslov,Vassilieva '18] [Dołęga '17]
That coefficients are in $\mathbb{Q}[b]$ (not $\mathbb{Q}(b)$) is proved in [Dołęga-Féray '17]

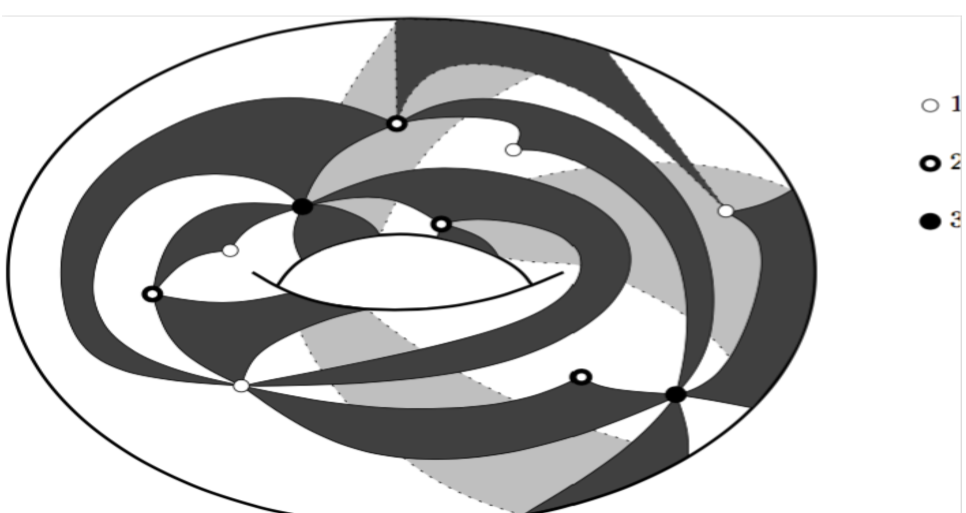
- One of our results: $\rightarrow$ OK if we keep TWO sets of times.

$$\mathbf{p}=(p_i)_{i\geq 1},\ \mathbf{q}=(q_i)_{i\geq 1},\ \mathbf{r}=\underline{u}=(u,u,\dots)$$

# Even better: constellations and the tau-function

- Factorisations of the form $\sigma_\circ\sigma_\bullet\sigma_1\sigma_2\dots\sigma_k = id$ in $\mathfrak{S}_n$ are in bijection with generalizations of bipartite maps called $k$-constellations.

The character approach is still valid to count them.



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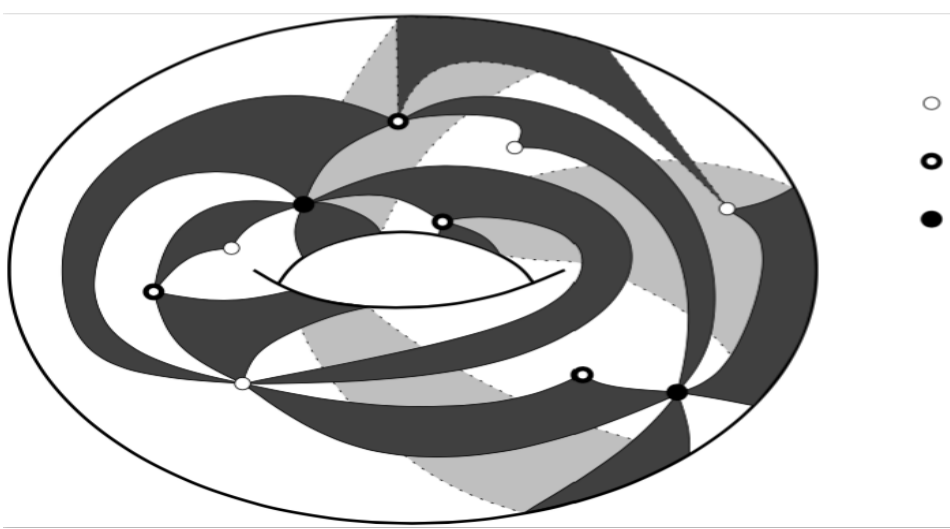
- Fact: The coolest object in the orientable $(b = 0)$ literature is

$$\tau^{(k)}(t; \mathbf{p}, \mathbf{q}, u_1, \dots, u_k) = \sum_\lambda t^{|\lambda|} \frac{\tilde{s}_\lambda(\mathbf{p}) \tilde{s}_\lambda(\mathbf{q}) \tilde{s}_\lambda(\underline{u_1}) \tilde{s}_\lambda(\underline{u_2}) \dots \tilde{s}_\lambda(\underline{u_k})}{j_\lambda^{(1)}}$$

$$= \sum_{\mathfrak{m}} \frac{t^n}{n!} p_{\sigma_\circ} q_{\sigma_\bullet} u_1^{\ell(\sigma_1)} \dots u_k^{\ell(\sigma_k)}$$

This is a tau-function of the $2$-Toda (and KP) hierarchy.

[Goulden-Jackson'09,Okounkov'00]




This is a central object in enumerative geometry (it counts branched coverings of the sphere).

# Our main result

- Theorem[Chapuy-Dołęga'20] Consider the $b$-deformed tau-function

$$\tau_b^{(k)}(t;\mathbf{p},\mathbf{q},u_1,\dots,u_k) = \sum_{\lambda\in\mathcal{P}} t^{|\lambda|} \frac{J_\lambda^{(1+b)}(\mathbf{p}) J_\lambda^{(1+b)}(\mathbf{q}) J_\lambda^{(1+b)}(\underline{u_1}) \dots J_\lambda^{(1+b)}(\underline{u_k})}{j_\lambda^{(1+b)}}$$

  Then $\quad (1+b)t\dfrac{\partial}{\partial t}\log\tau_b^{(k)} \quad$ is $b$-positive.

  Its coefficients count (properly defined) $k$-constellations on non-orientable surfaces with a weight $b^{\nu(\mathbf{m})}$ where $\nu(\mathbf{m}) = 0$ iff $\mathbf{m}$ is orientable.

- our result has three sets of parameters $\mathbf{p} = (p_i)_{i\geq 1}$, $\mathbf{q} = (q_i)_{i\geq 1}$, $\mathbf{u} = (u_i)_{i\leq k}$
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- The case $b=0$ is the classical tau function.
- By letting $k\to\infty$ we can do $b$-analogues of Hurwitz numbers (factorisations in transpositions) and in fact, general weighted Hurwitz numbers ($k=\infty$).

## Elements of proof (?)

# Proof structure

- Our proof has three halves:

$1/2$ – If you are a map expert, you can, in principle, write some sort of linear PDE for the g.f. of constellations that reflects a "root-edge" decomposition. You can hope to do it by controlling the variables $\mathbf{p}$ and $\mathbf{q}$ and $(u_i)$.

– do it. And do it also for the non-orientable case.

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$3/2$ – Suffer proving that the combinatorial (explicit) PDEs and the recursively defined (Lax type) PDEs are in fact the same.

This part of the proof is long *and* difficult, at least in the way to do it. For $b \in \{0, 1\}$ we have a combinatorial proof. We develop some sort of operator calculus that "lifts the combinatorial proof" to the world of differential operators, and the lifted proof works for general $b$.

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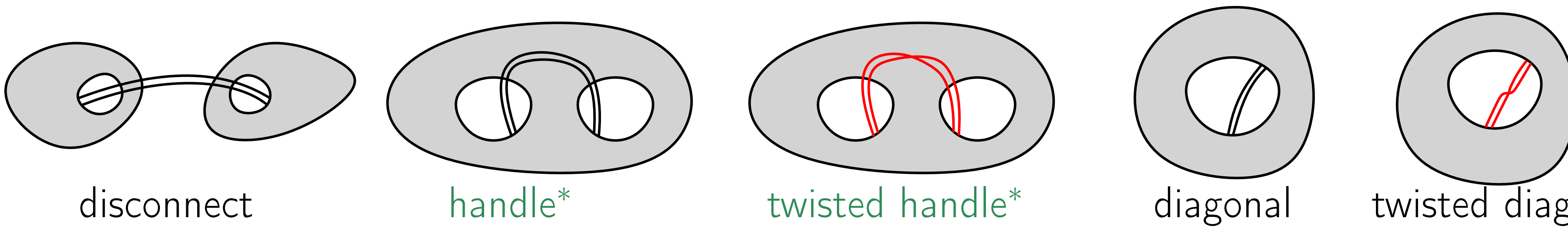


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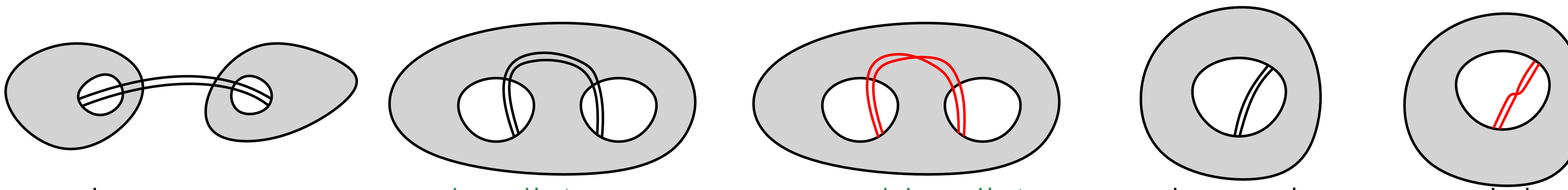

disconnect — handle* — twisted handle* — diagonal — twisted diagonal

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- Claim: the following operator takes all these cases into account:

$$\Lambda_Y := (1+b)\sum_{i,j\geq 1} y_{i+j-1}\frac{i\partial^2}{\partial p_i \partial y_{j-1}} + \sum_{i,j\geq 1} y_{i-1}p_j \frac{\partial}{\partial y_{i+j-1}} + b\cdot\sum_{i\geq 0} y_i \frac{i\partial}{\partial y_i},$$

($y_i$: root-face of degree $i$; $p_i$: non-root-face of degree $i$; also extra weight $1/(1+b)$ per cc.)

# The combinatorial equations for $k=1$ (bipartite maps) - II

$$\Lambda_Y := (1+b)\sum_{i,j\geq 1} y_{i+j-1}\frac{i\partial^2}{\partial p_i \partial y_{j-1}} + \sum_{i,j\geq 1} y_{i-1}p_j\frac{\partial}{\partial y_{i+j-1}} + b\cdot\sum_{i\geq 0} y_i \frac{i\partial}{\partial y_i},$$

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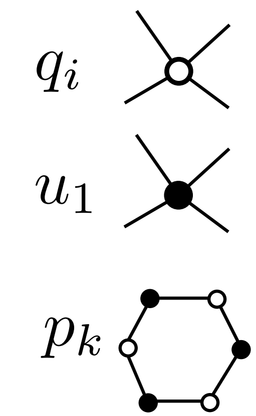
mark a vertex of degree $m$



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take a smaller map

build a "fake" root face of degree 0 (track root-face degree with variables $y_i$)

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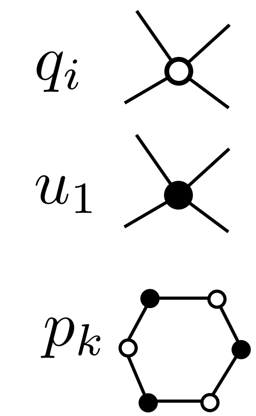


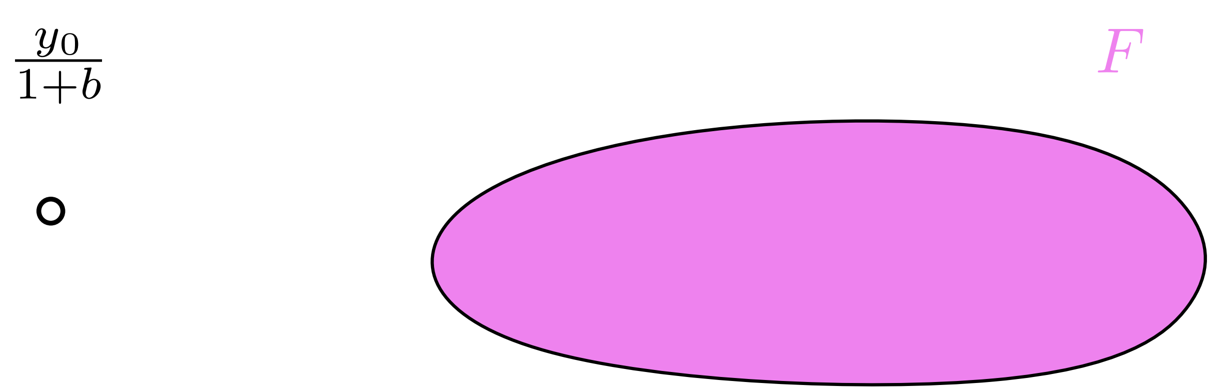

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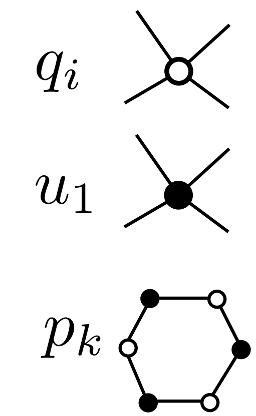


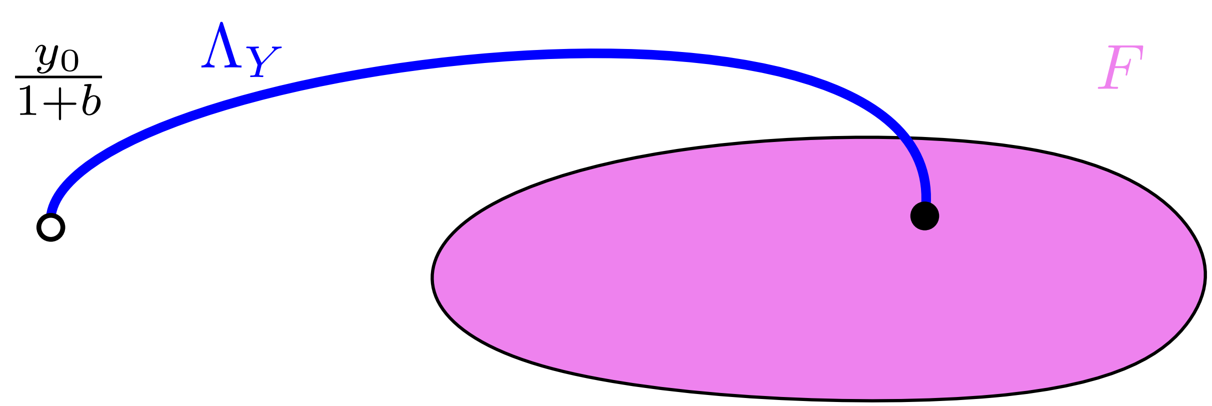

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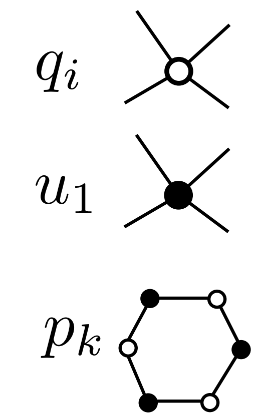


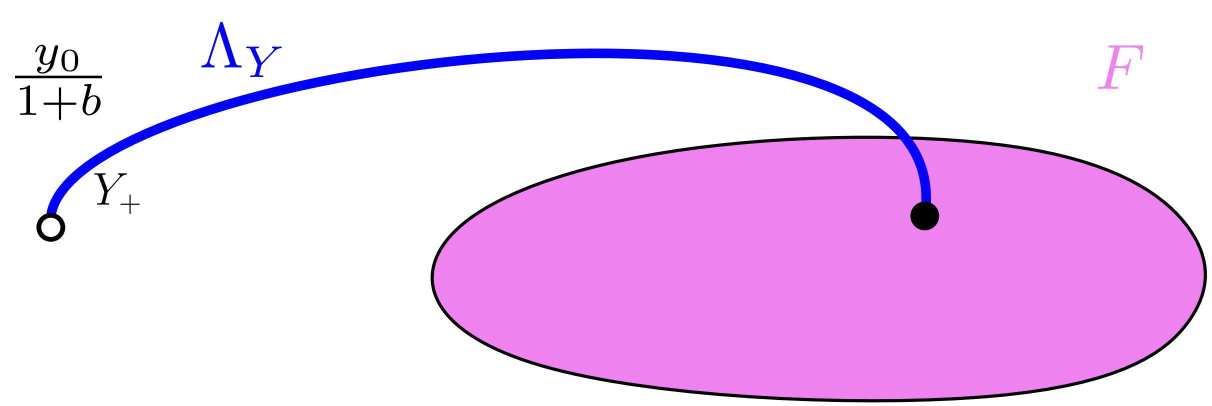

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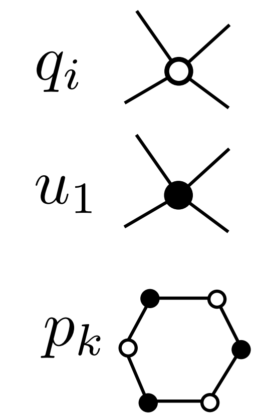


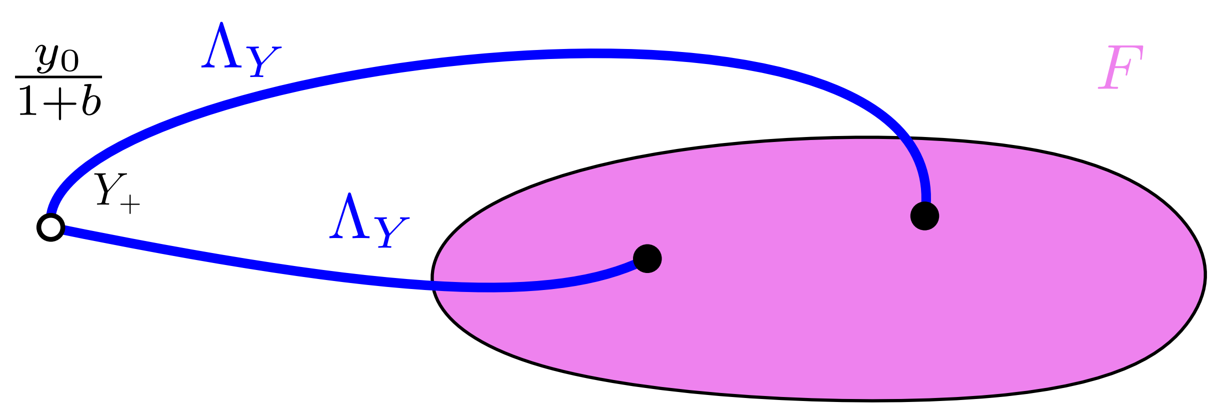

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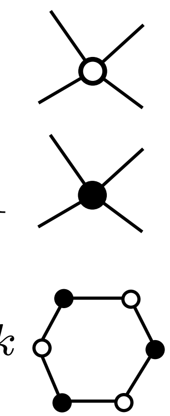
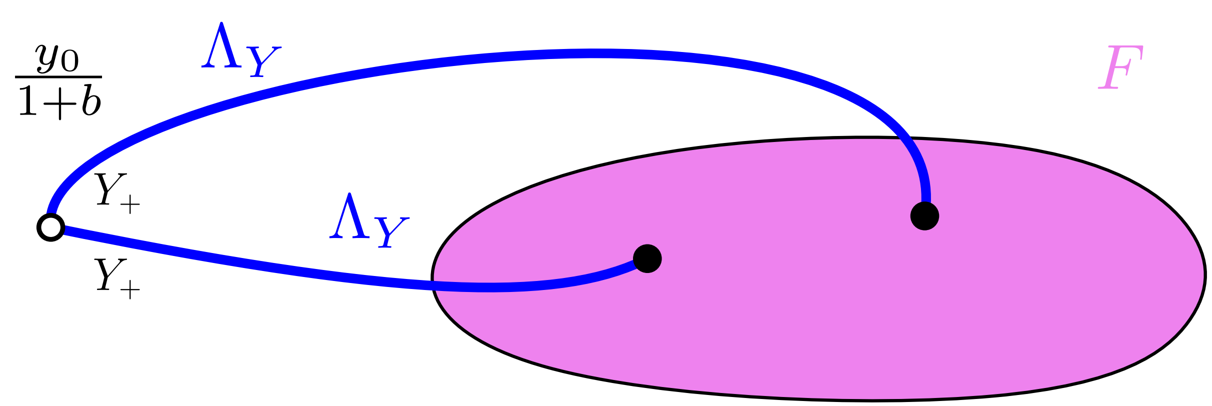
build a "fake" root face of degree $0$ (track root-face degree with variables $y_i$)

$q_i$

$u_1$

$p_k$





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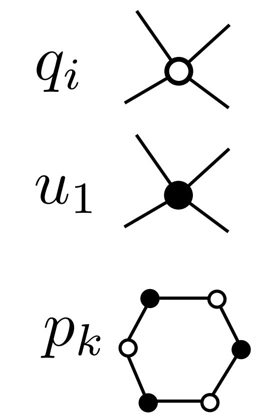


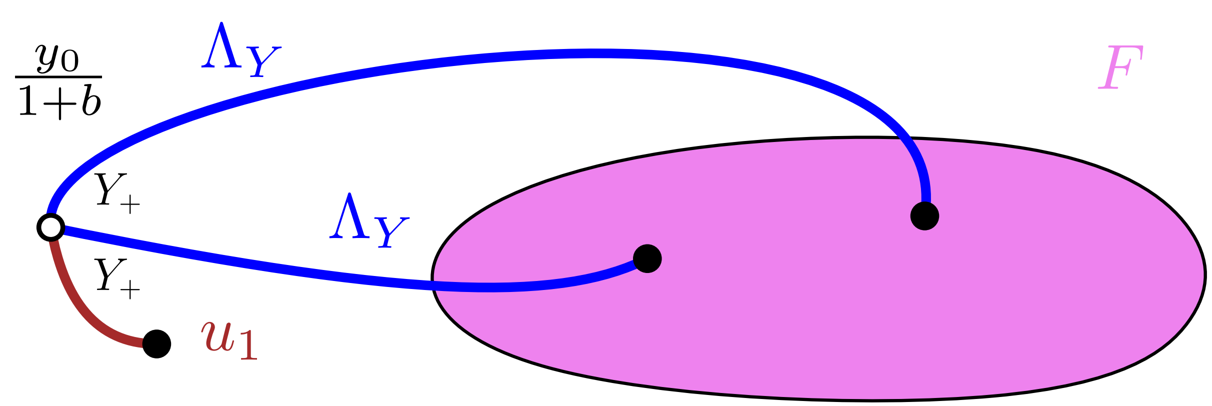

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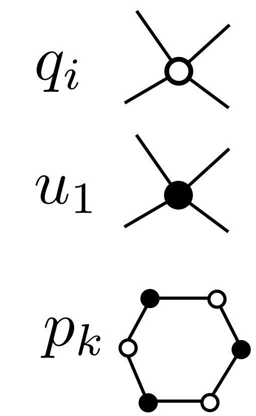


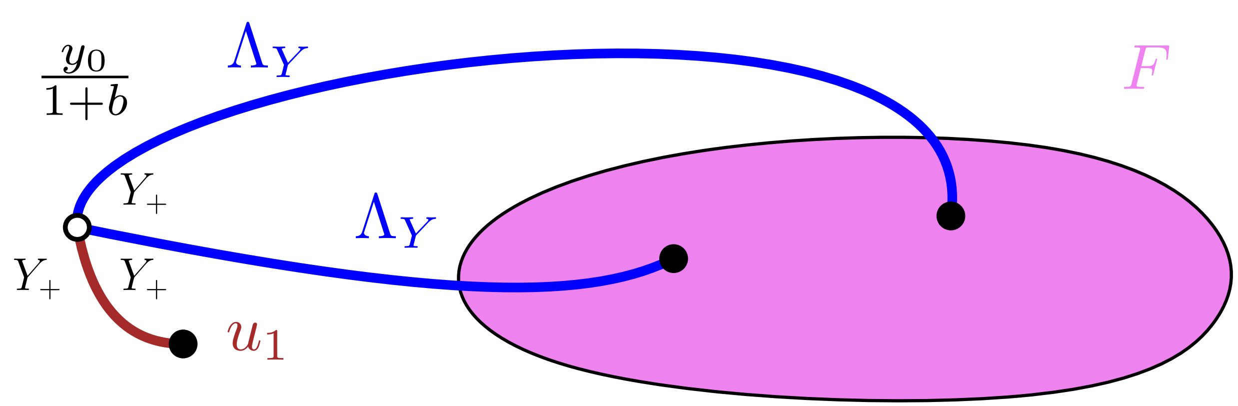

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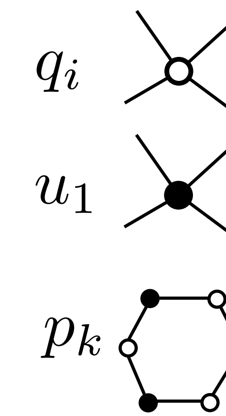
take a smaller map

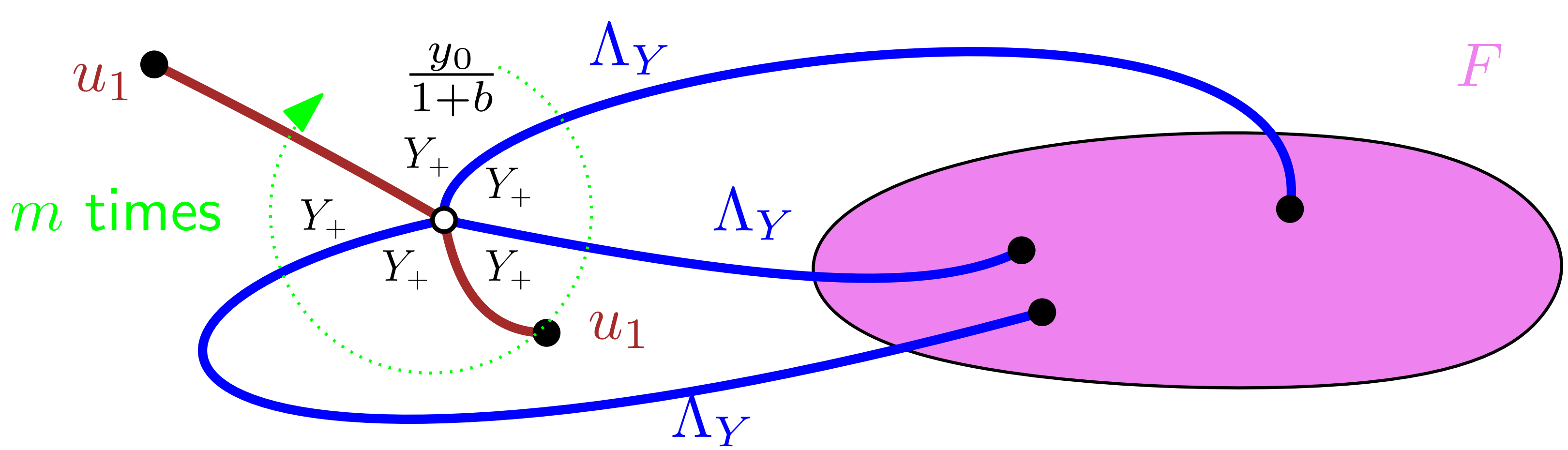
mark a vertex of degree $m$

build a $k$-hyperedge inside the root face

do it $m$ times to create "root" vertex of degree $m$

build a "fake" root face of degree 0 (track root-face degree with variables $y_i$)



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mark a vertex of degree $m$

anonymise root face

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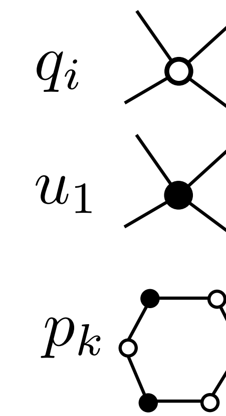


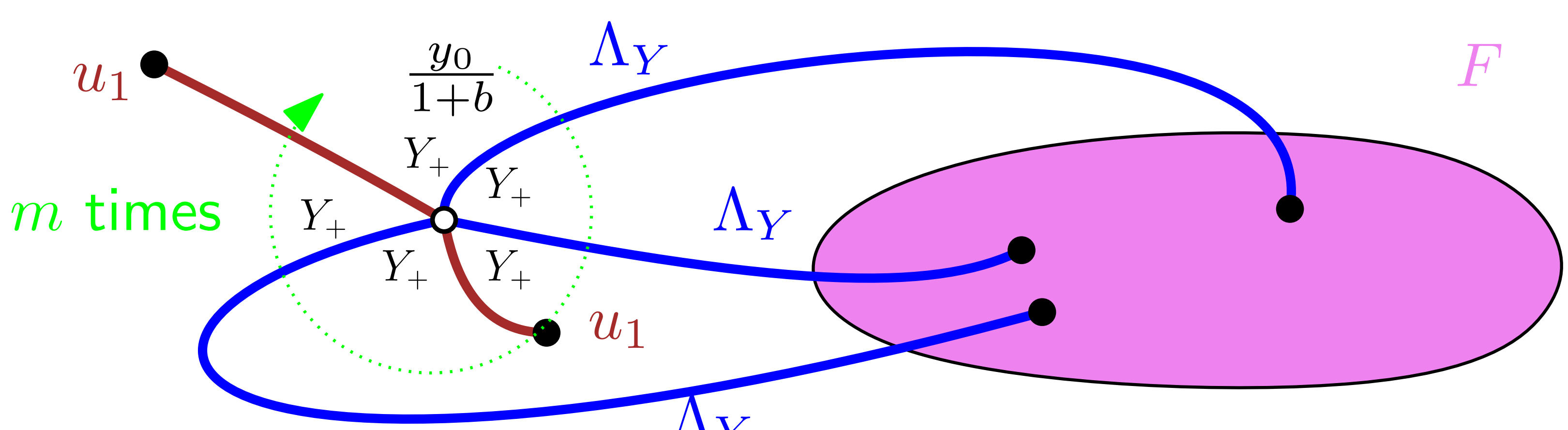

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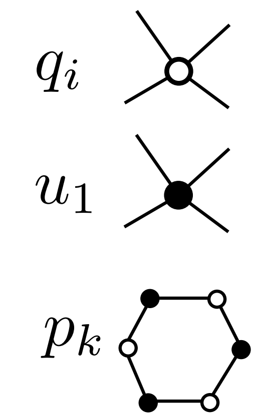


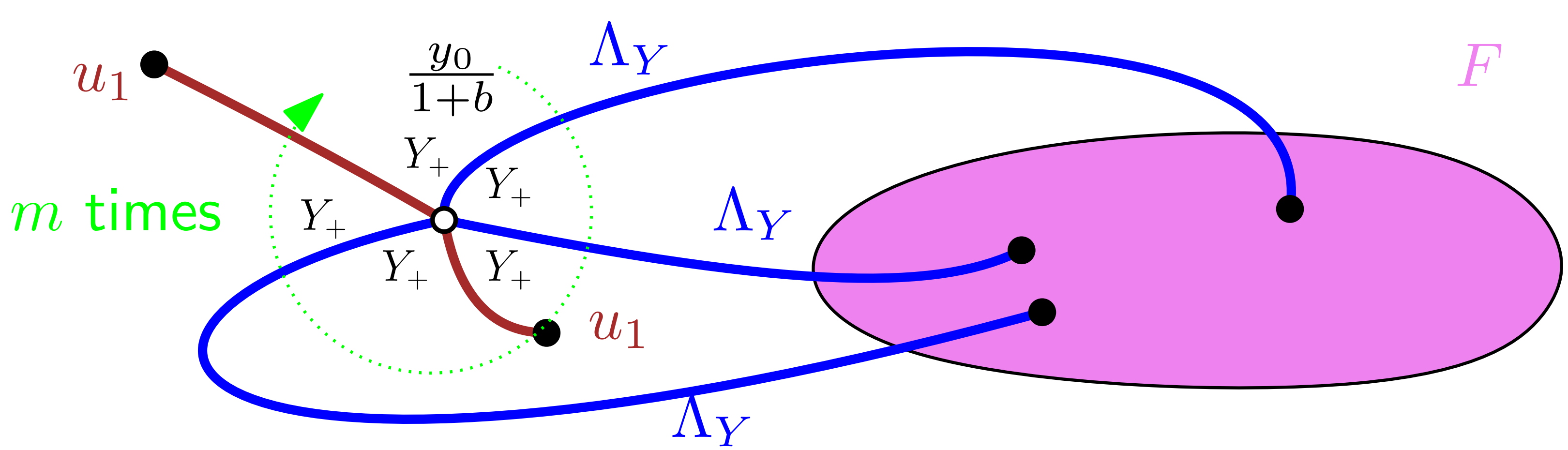


Note: this corresponds to a randomized decomposition and proves only rational weights. One can be more precise and do a deterministic rooted decomposition instead.

# How do the “Lax pair” equations look like?

- Define the Laplace-Beltrami operator:

$$D_\alpha = \frac{1}{2}\left( (1+b)\sum_{i,j\geq 1} p_{i+j}\frac{ij\partial^2}{\partial p_i \partial p_j} + \sum_{i,j\geq 1} p_i p_j \frac{(i+j)\partial}{\partial p_{i+j}} + b\cdot \sum_{i\geq 1} p_i \frac{i(i-1)\partial}{\partial p_i}\right).$$

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- Define the operators:

$$A_1 = p_1/(1+b) \quad , \quad A_{j+1} = [D_\alpha, A_j], \text{ , for } j \geq 1$$

$$\Omega_Y^{(k)} = \textstyle\sum_{1\leq i \leq k+1} A_{i+1} e_{k+1-i}(u_1, \ldots, u_k)$$

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- Then one has the equation, for $m \geq 1$:

$$m! \frac{\partial}{\partial q_m} \tau_b^{(k)} = t^m B_m^{(k)} \tau_b^{(k)}$$

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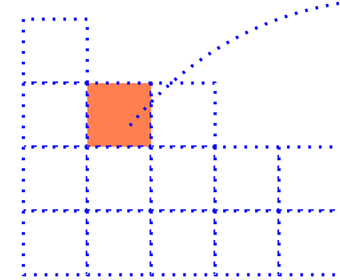
# How to construct PDE's for $\tau_b^{(k)}$? ($m = 1, k = 1$, Schur case).

- Hook content formula:

$$\tilde{s}_\lambda(\underline{u}) = \prod_{\square \in \lambda} (u + c(\square))$$



$c(\square) := x(\square) - y(\square) = -1$

$$\longrightarrow B(\mathbf{p}, \mathbf{q}, u) = \sum_{\lambda} \Big( \prod_{\square \in \lambda} (u + c(\square)) \Big) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q})$$

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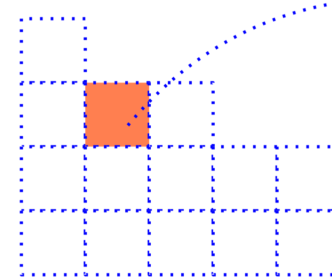

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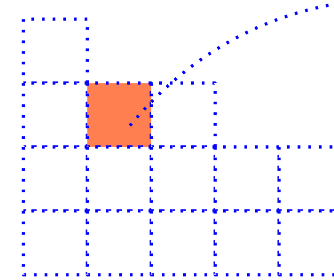

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- Another rule $D_\alpha s_\lambda = \Big( \sum_{\square\in\lambda} c(\square) \Big) s_\lambda$

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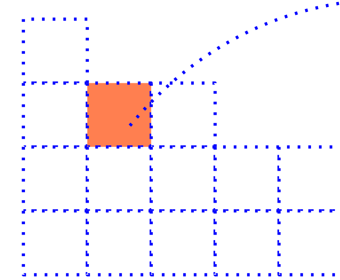

$$c(\square) := x(\square) - y(\square) = -1$$

$$\tilde{s}_\lambda(\underline{u}) = \prod_{\square\in\lambda}(u + c(\square))$$

$$\longrightarrow B(\mathbf{p},\mathbf{q},u) = \sum_\lambda \Big(\prod_{\square\in\lambda}(u+c(\square))\Big) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q})$$

- Pieri rule: multiply by $p_1$ adds a box to a Schur function: $p_1 s_\lambda(\mathbf{p}) = \sum_{\square} s_{\lambda\uplus\square}(\mathbf{p})$.
  $\partial/\partial p_1$ removes a box.

- Another rule $D_\alpha s_\lambda = \Big(\sum_{\square\in\lambda} c(\square)\Big) s_\lambda$

$$\longrightarrow [D_\alpha, p_1] s_\lambda = \sum_{\square} c(\square) s_{\lambda\uplus\square}(\mathbf{p})$$

- From this we obtain a differential equation for our function!!!

$$\frac{\partial}{\partial q_1} B(\mathbf{p},\mathbf{q},u) = (u p_1 + [D_\alpha, p_1]) B(\mathbf{p},\mathbf{q},u)$$

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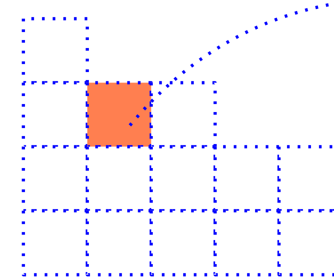

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The miracle is that (by computing the commutator explicitly) this is the same equation as the one we wrote for maps in the prevous slides!

La preuve que le miracle a lieu et que les opérateurs sont les mêmes pour tout $k$ et $m$ est une partie de l'histoire que j'omets...

Merci!