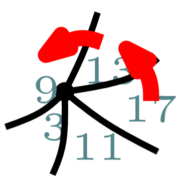
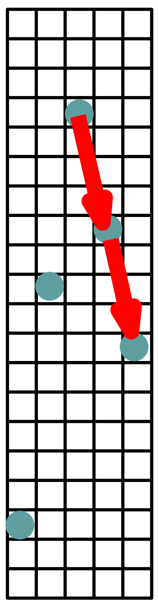
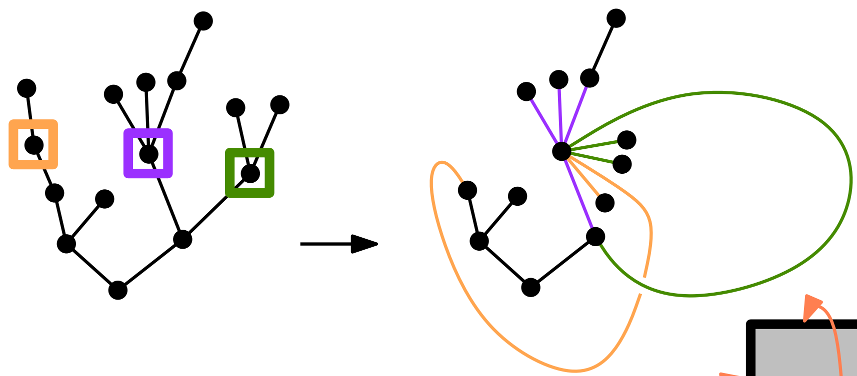
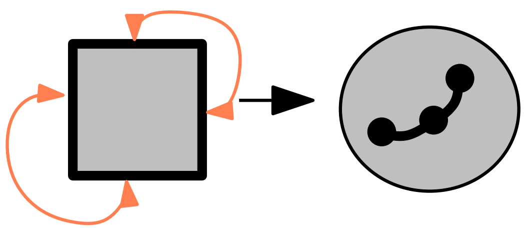
A new combinatorial identity for unicellular maps, via a direct bijective approach

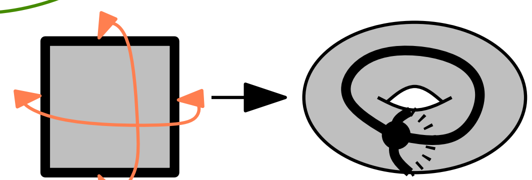
Guillaume Chapuy,
École Polytechnique (France)











FPSAC, Hagenberg, 2009.

Unicellular maps as polygon gluings

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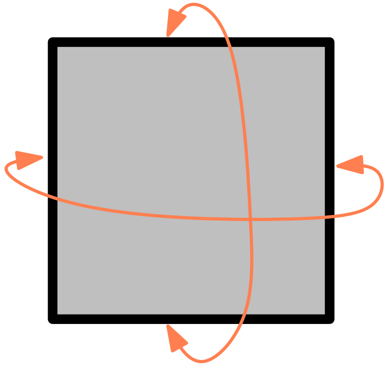

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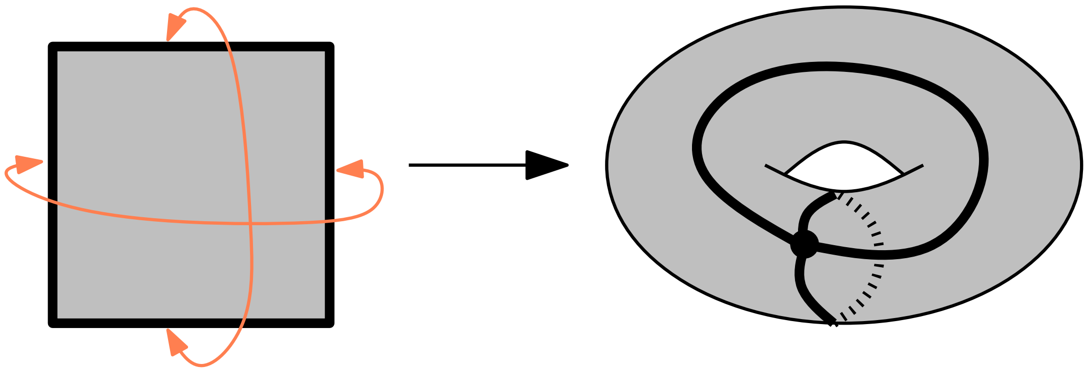

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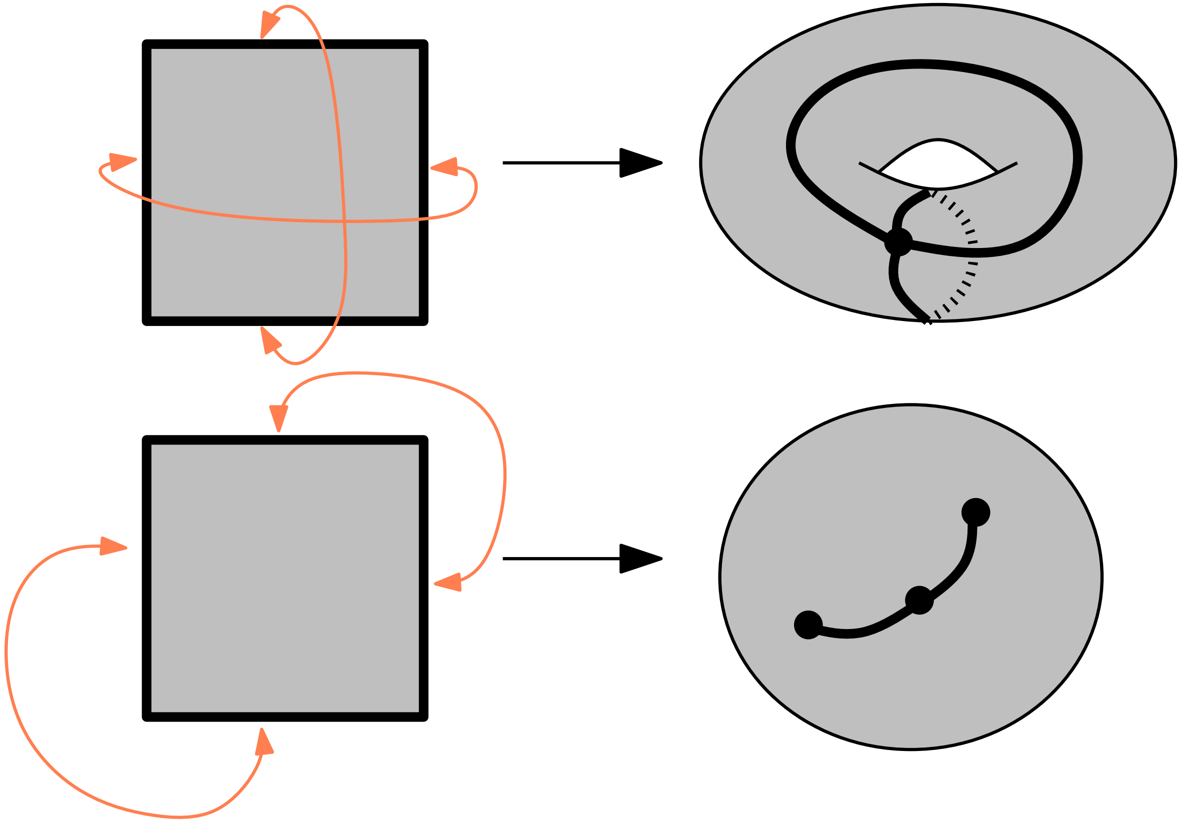

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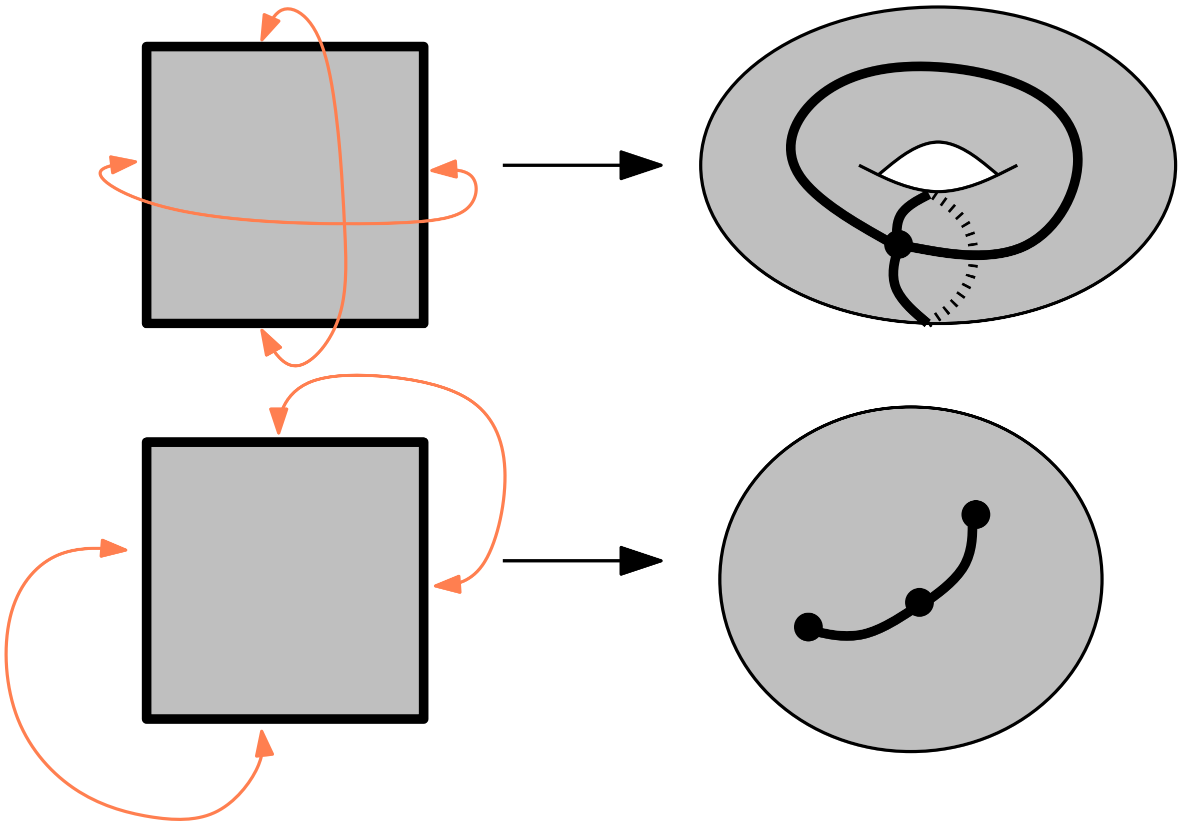

The image of the polygon forms the drawing of an n-edge graph on the surface.

Euler's formula relates the number of vertices to the genus of the surface :

$$v = n + 1 - 2g$$

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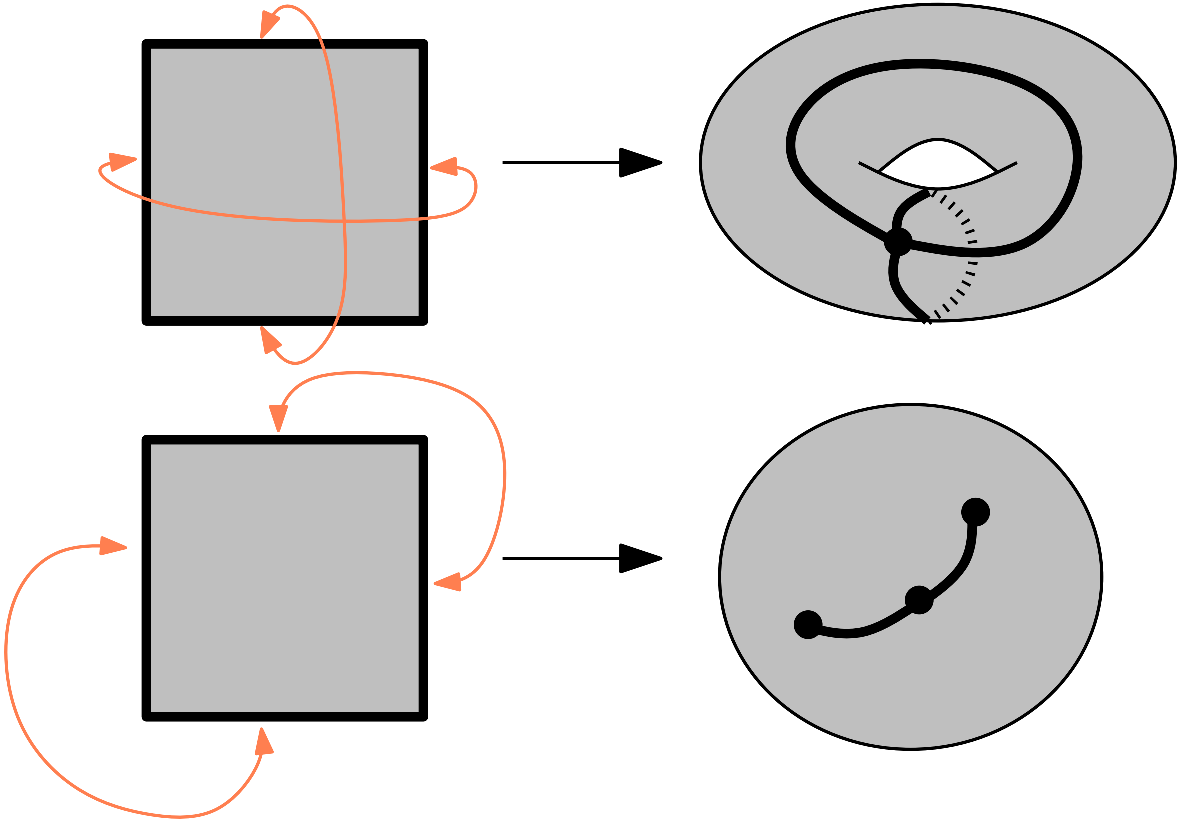

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3 vertices, genus 0

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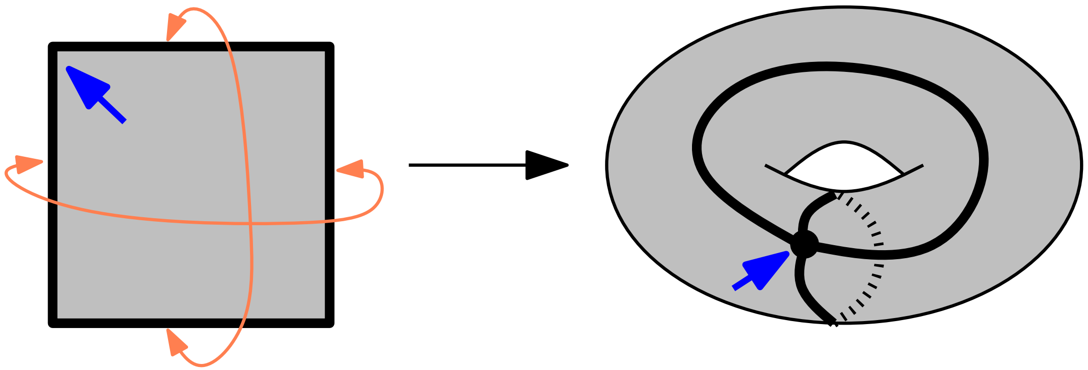

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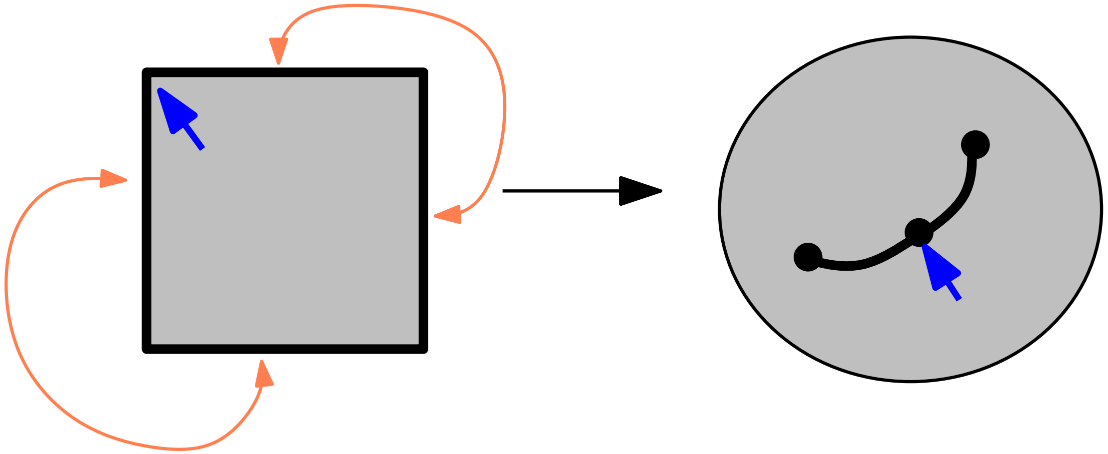

3 vertices, genus 0

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Counting

The number of unicellular maps with n edges is equal to the number of distinct matchings of the edges : $\dfrac{(2n)!}{2^n n!}$.

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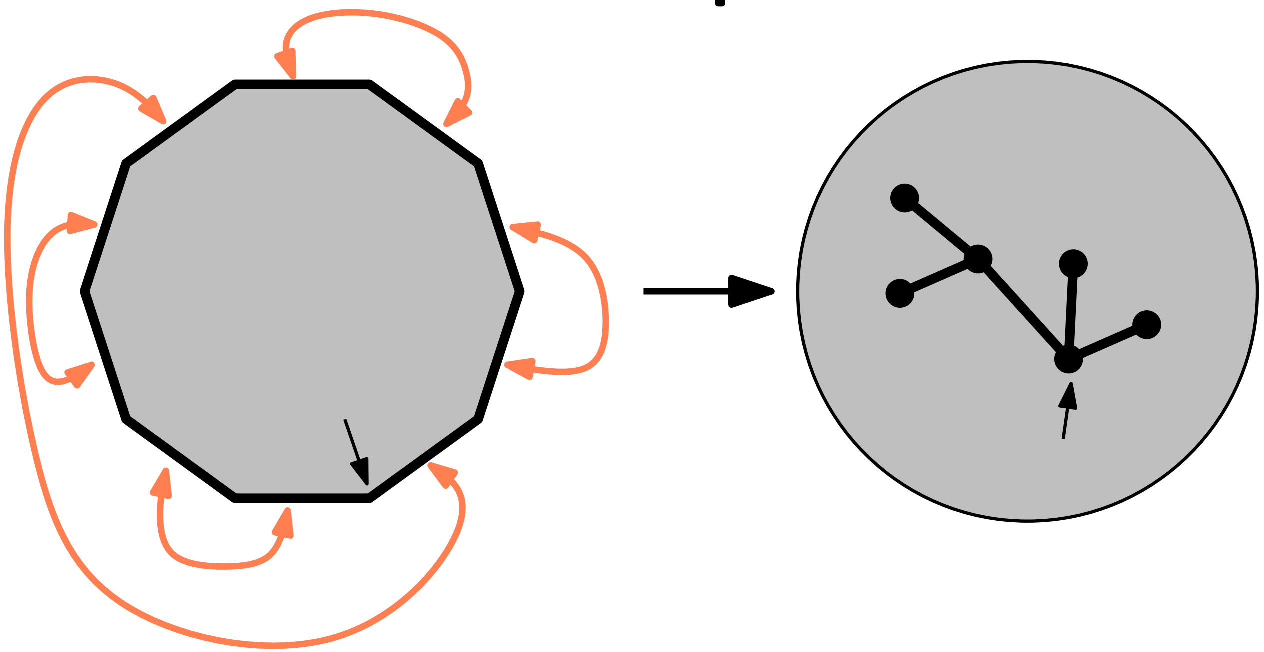
Aim: count unicellular maps of fixed genus.

For instance, in the planar case...



Unicellular maps are exactly plane trees.

Therefore the number of n-edge unicellular maps of genus 0 is :

$$\epsilon_0(n) = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

Higher genus ?

For each g the number of n-edge unicellular maps of genus g has the (beautiful) form :

$$\epsilon_g(n) = \text{(some polynomial)} \times \mathrm{Cat}(n)$$

For instance :

$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12}\mathrm{Cat}(n)$$

$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440}\mathrm{Cat}(n)$$

References : Lehman and Walsh 72 (formal power series), Harer and Zagier 86 (matrix integrals).

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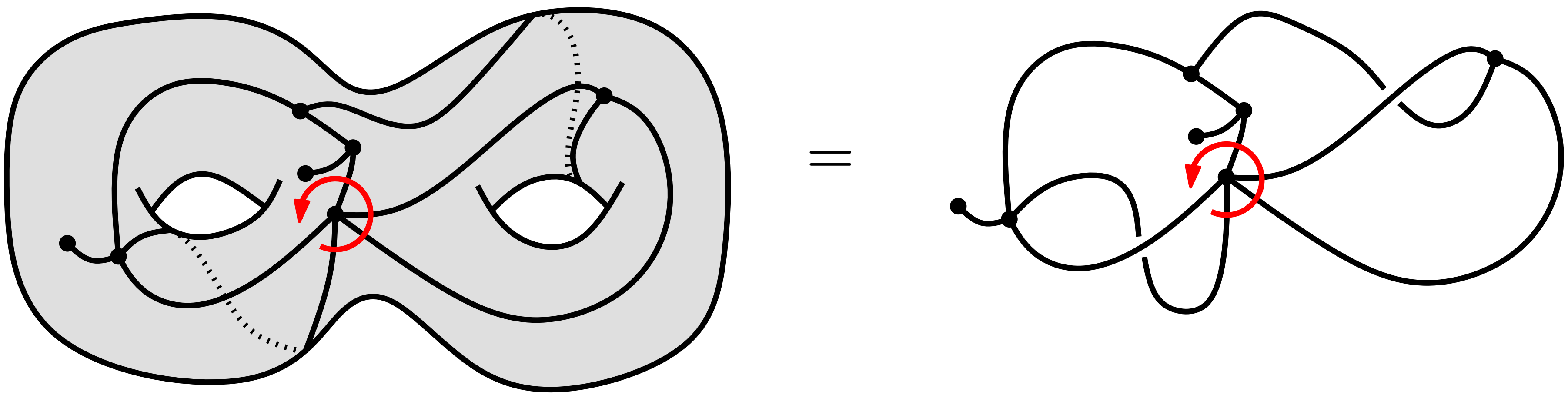
Note for experts: the Goulden-Nica bijection does not solve the same problem (it solves a "Poissonized" version of the problem).

Map = graph + rotation system



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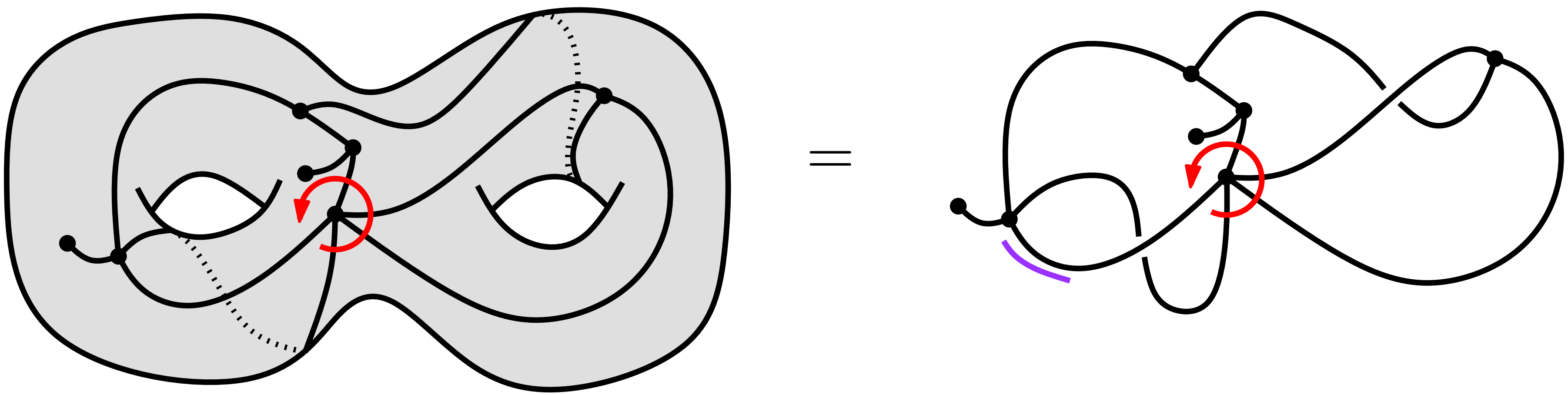

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We started from one single polygon
$\Rightarrow$ the graph has only one border

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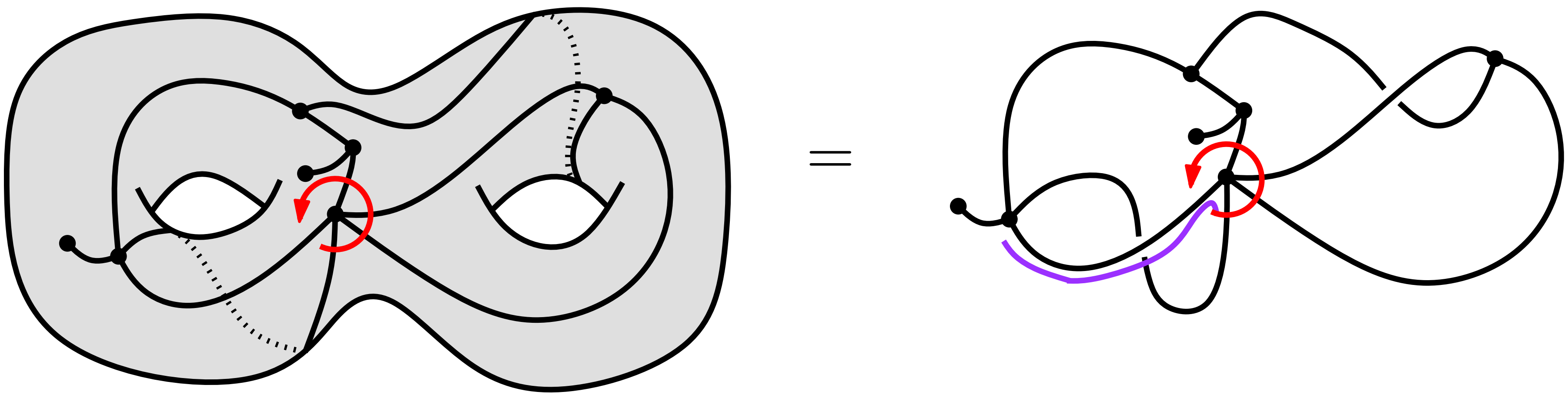

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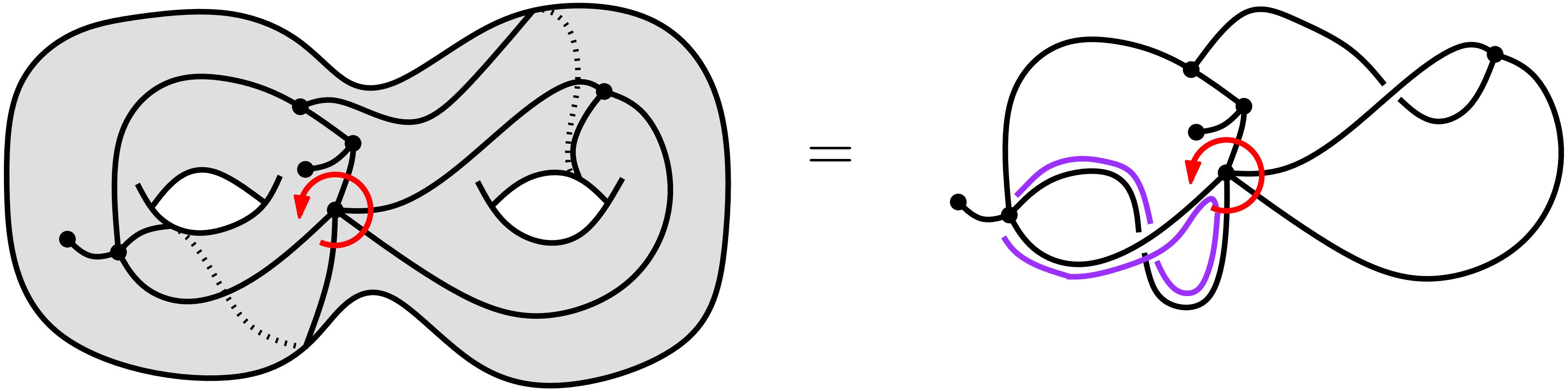

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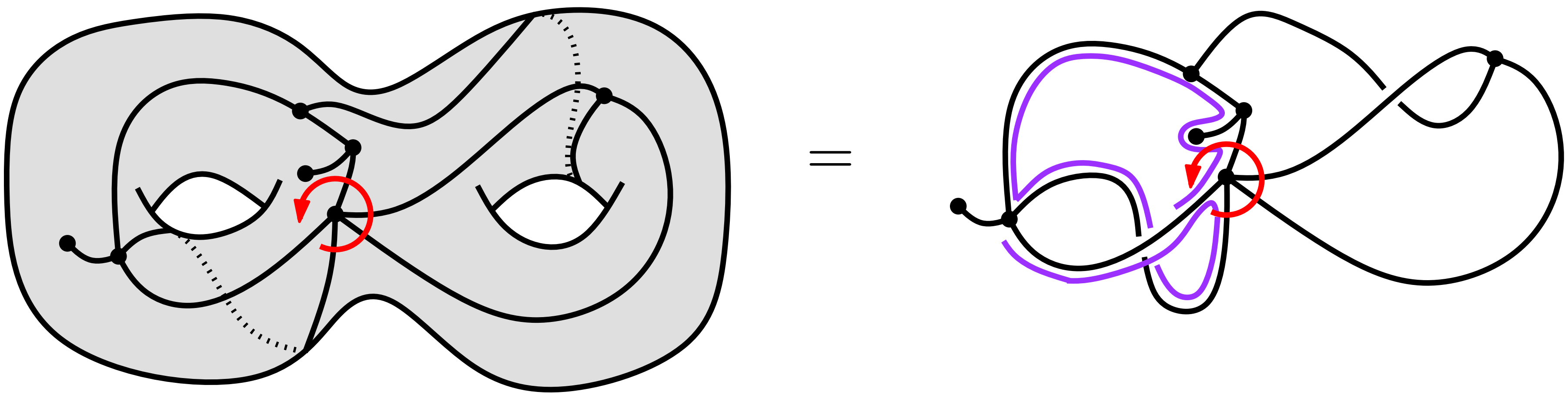

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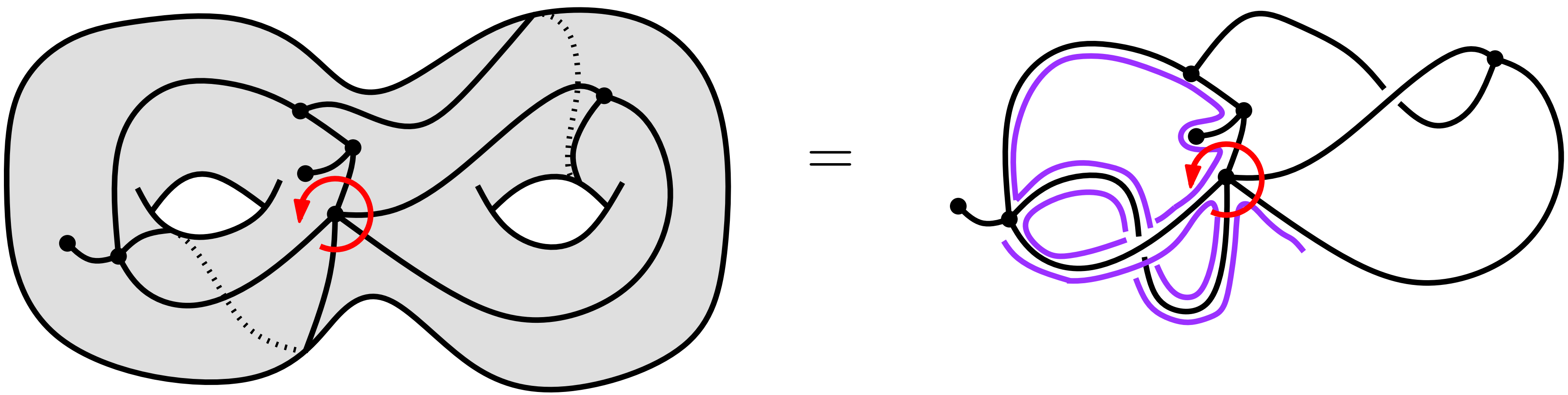

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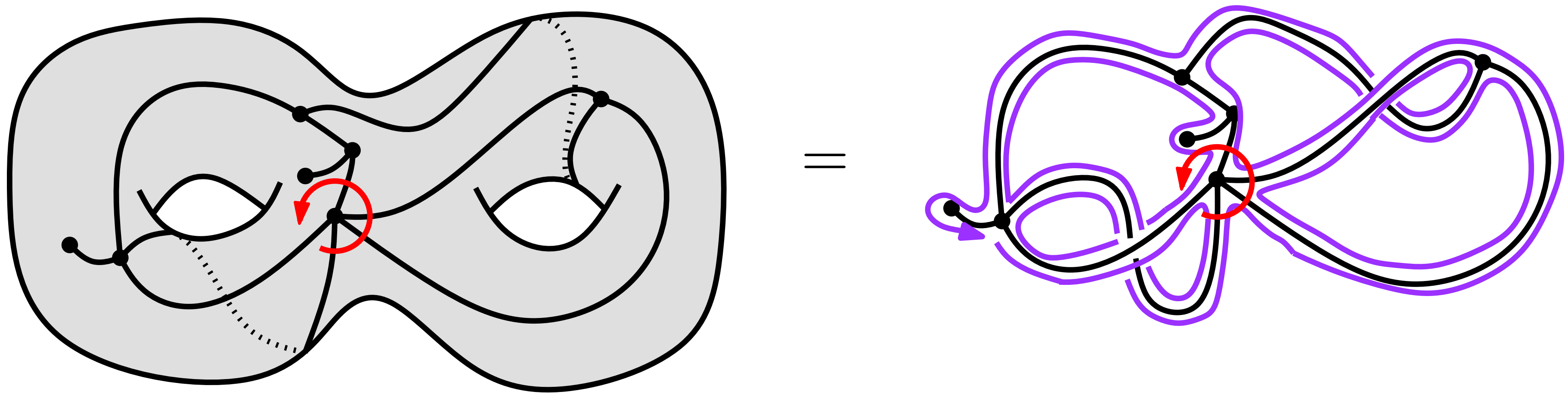

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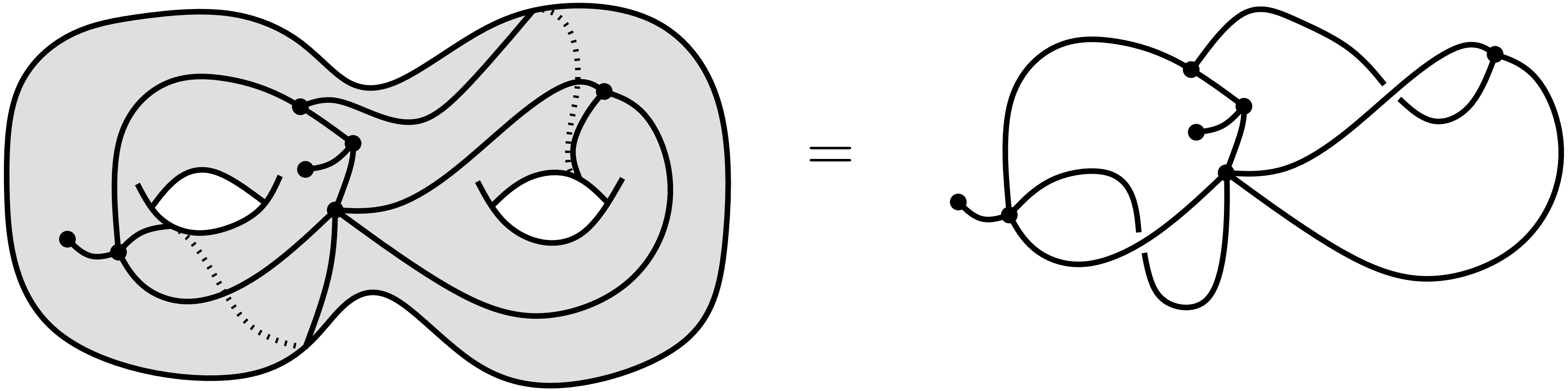

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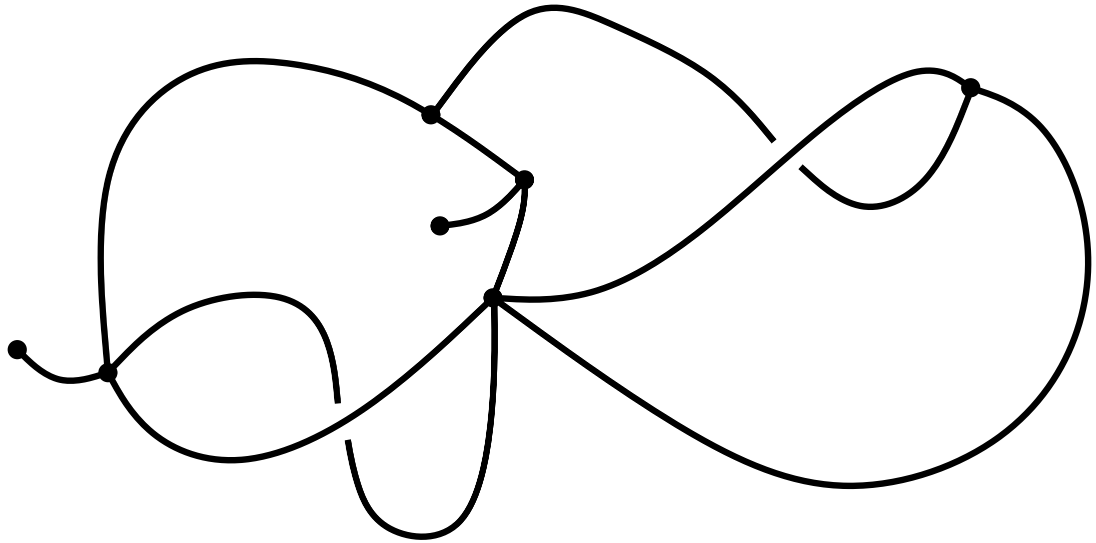
To do: cut the 2g independant cycles of this graph in order to obtain a tree. Problem: where to cut ?

Numbering the corners.

We follow the border of the map starting from the root, and we number the corners from 1 to $2n$.



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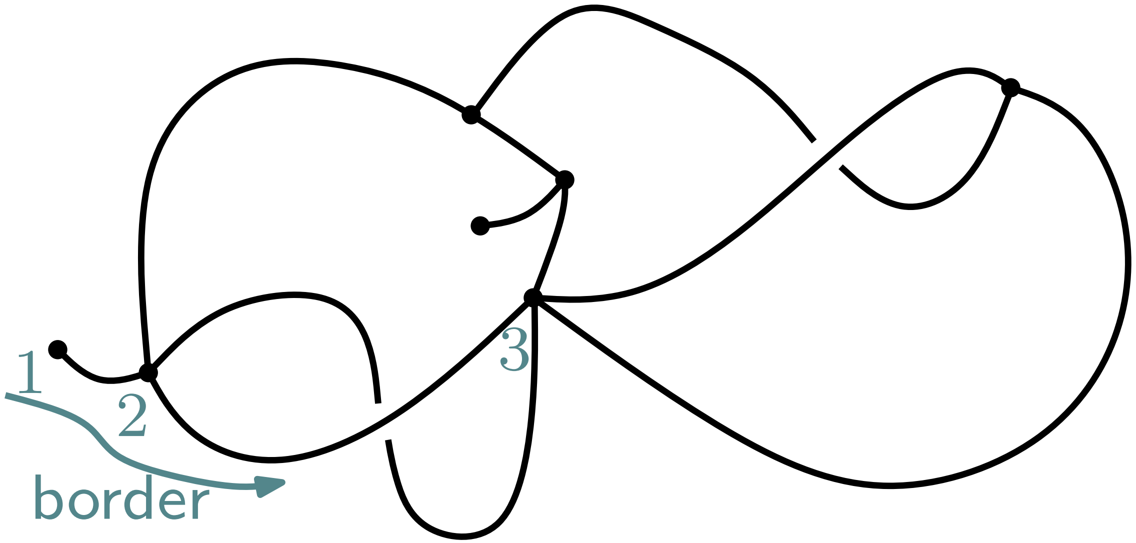

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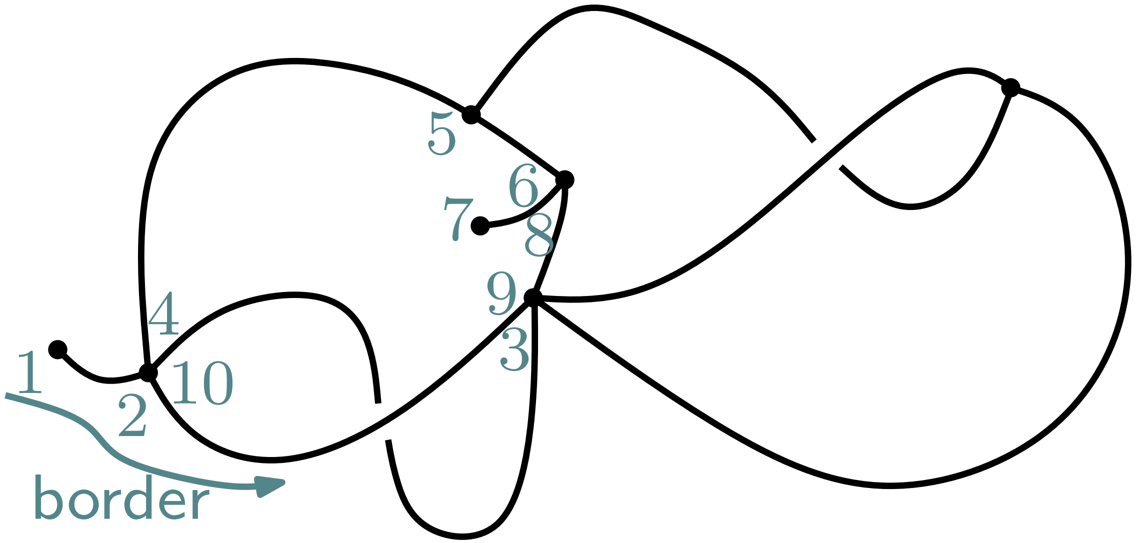

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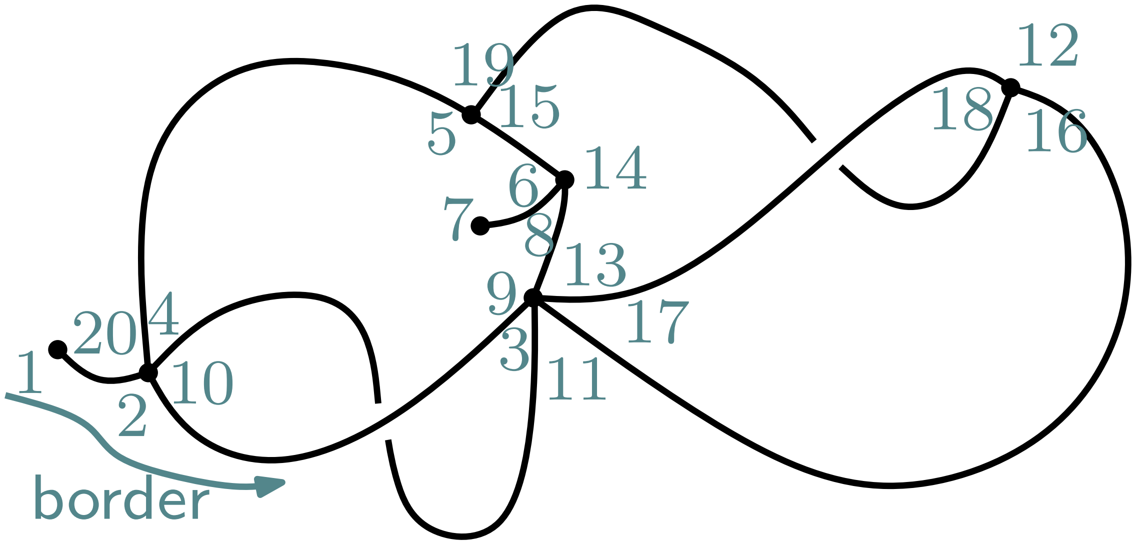

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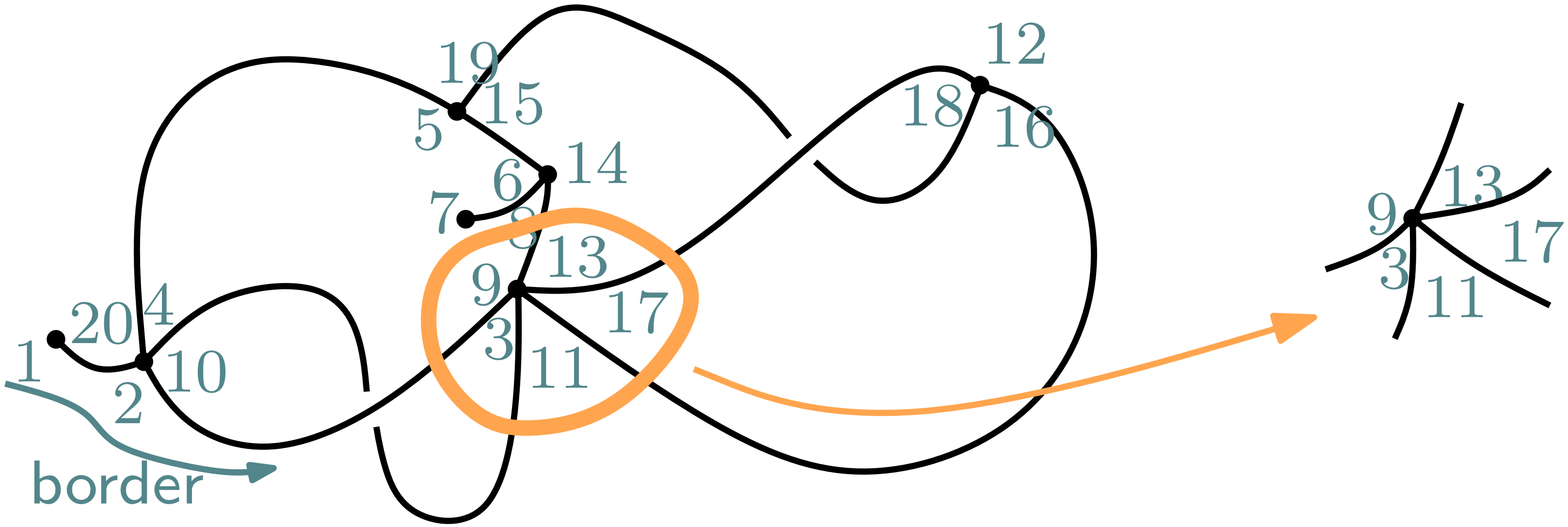

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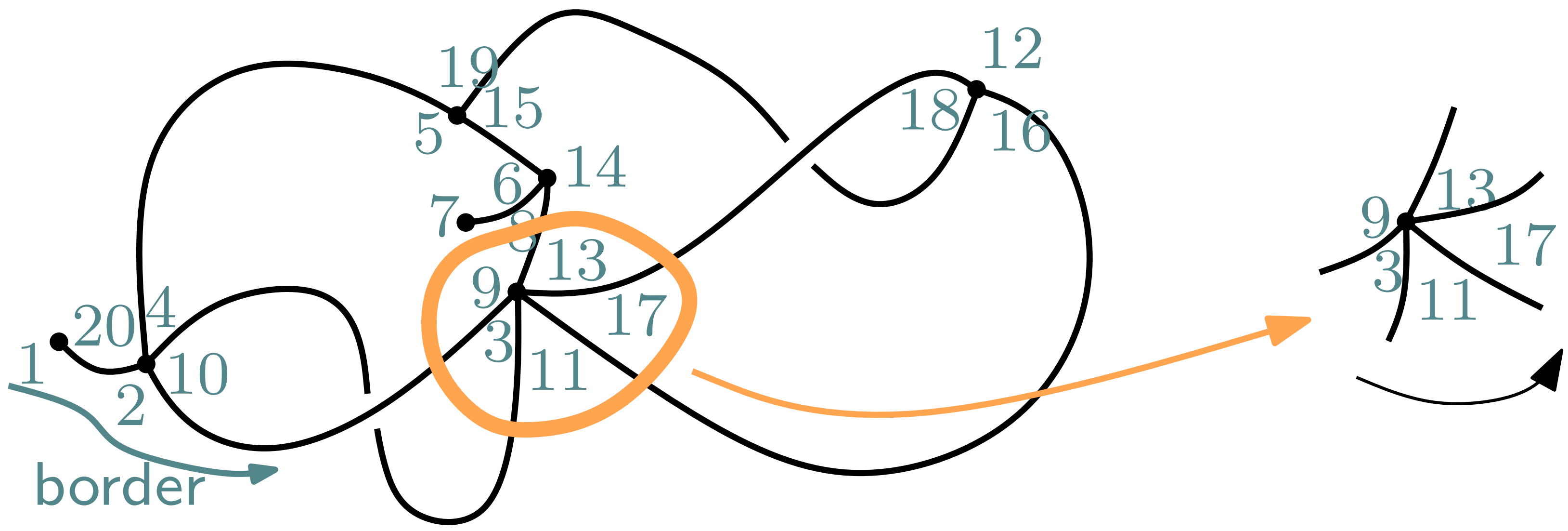

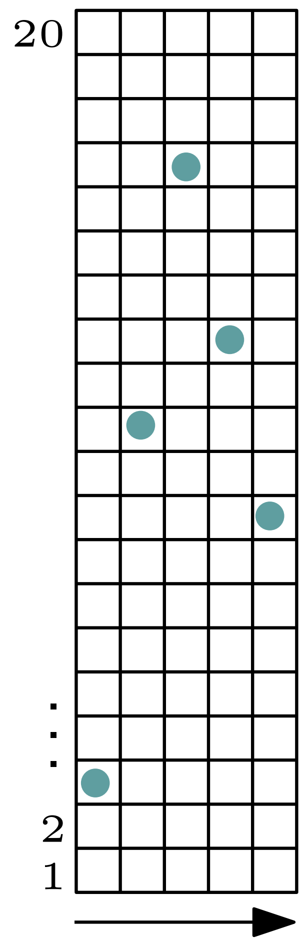

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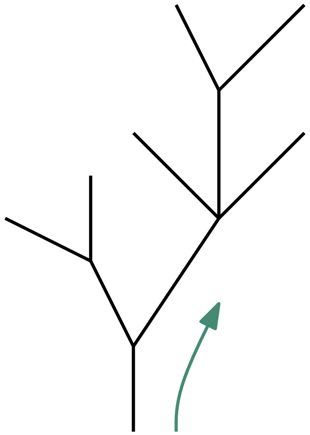
Planar case

In the planar case, the border-numbering and the cyclic ordering always coincide:



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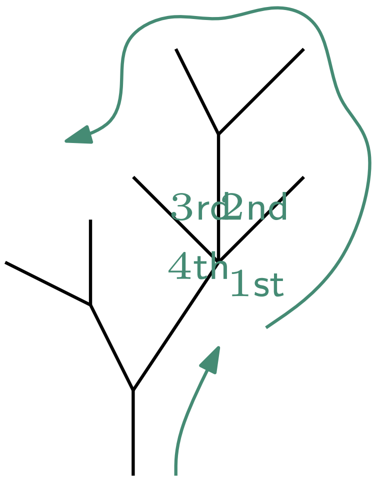

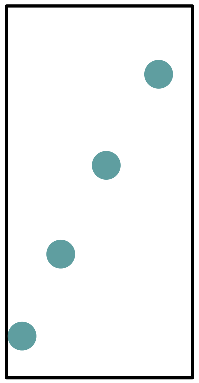

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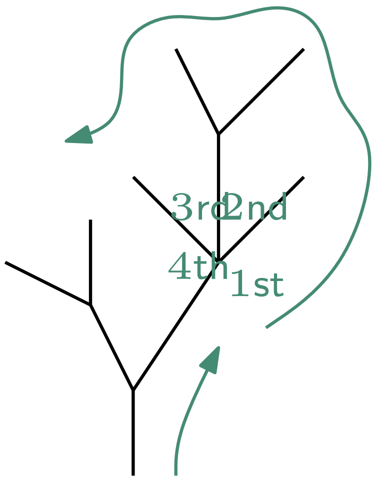
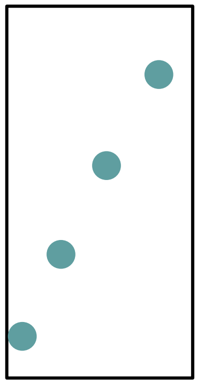
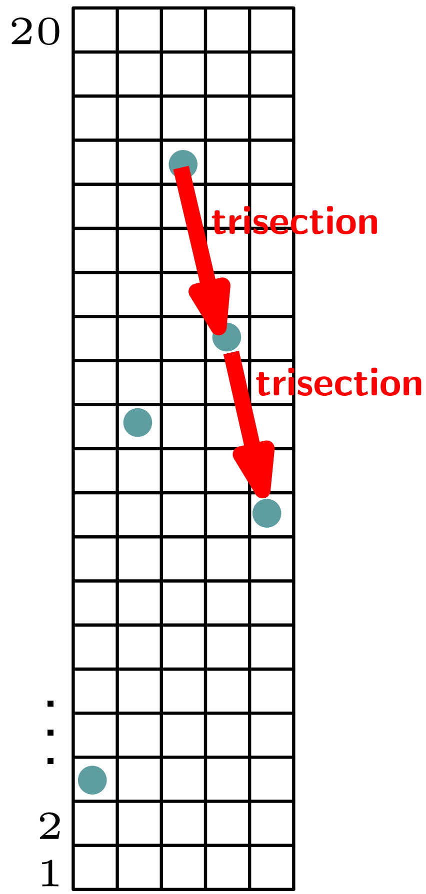
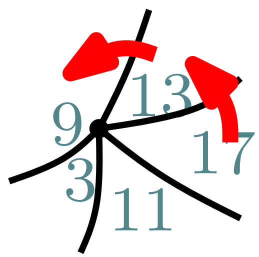






Higher genus

Around each vertex, a decrease in the diagram is called a trisection.



The trisection lemma

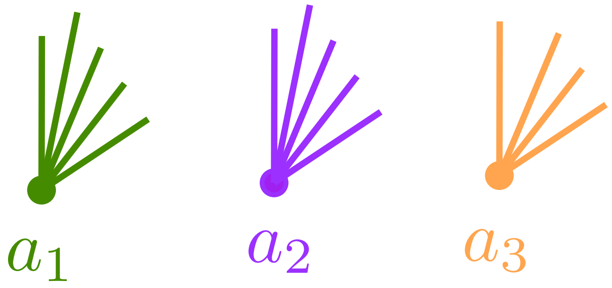
A unicellular map of genus g always has exactly $2g$ trisections.

Proof: simple counting argument.

$\longrightarrow$ It is an equivalent problem to count unicellular maps with a distinguished trisection.

How to build a trisection : first method.

- Start with a map of genus $(g-1)$ with three marked vertices.
- Let $a_1 < a_2 < a_3$ be the labels of their minimal corners.
- Glue these three corners together as follows :



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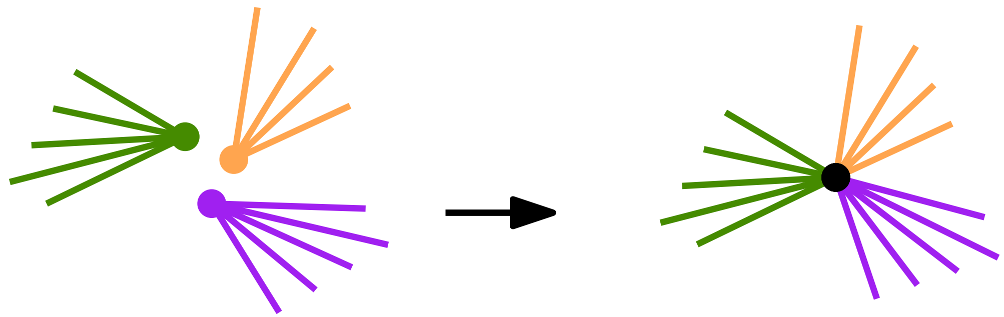

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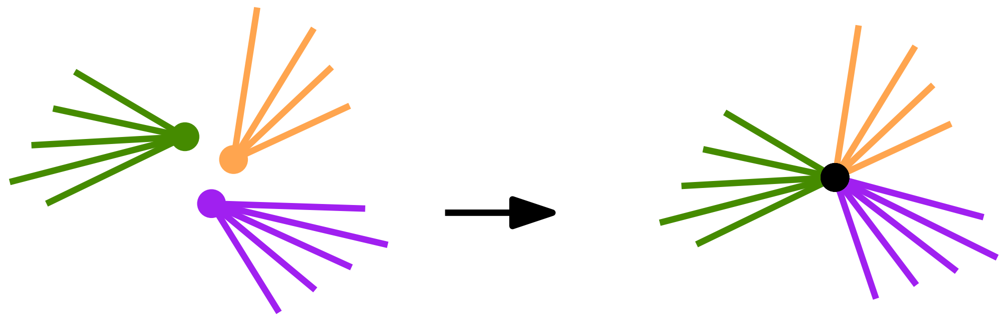

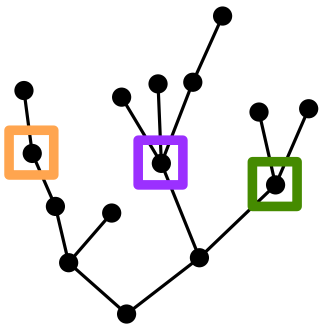

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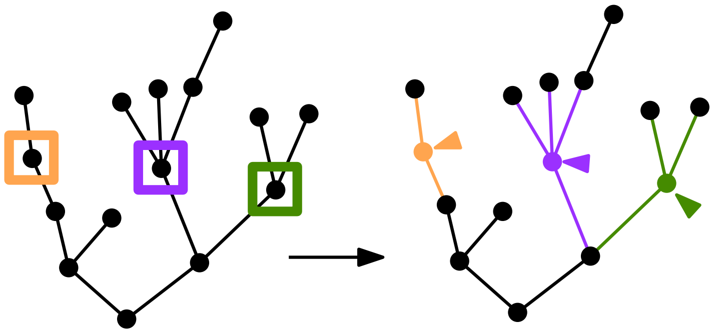

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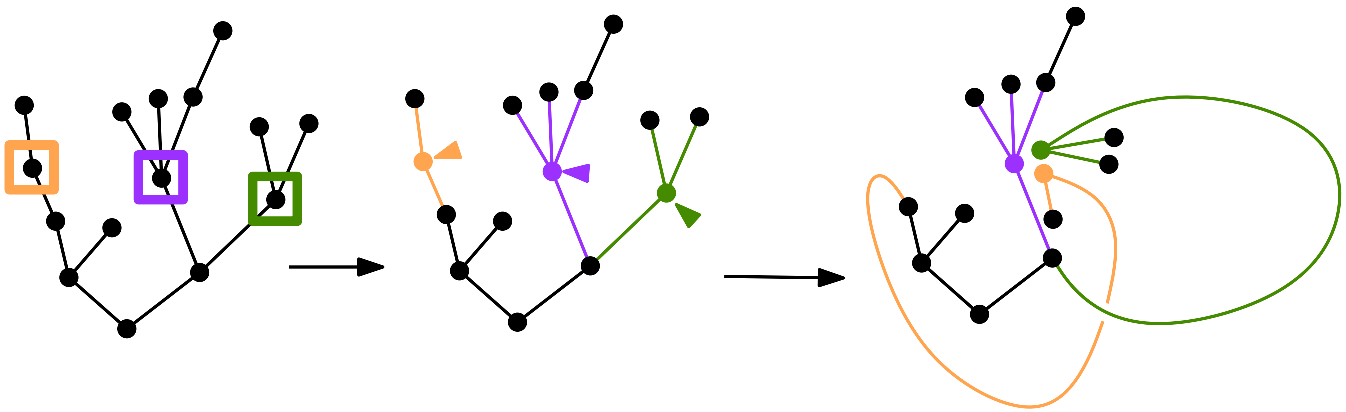

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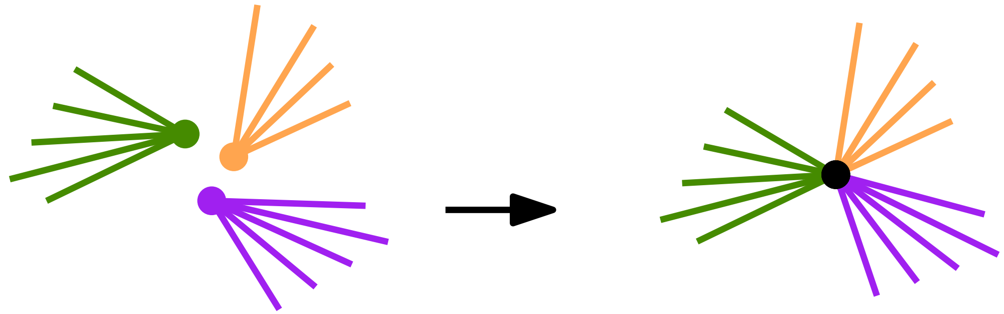

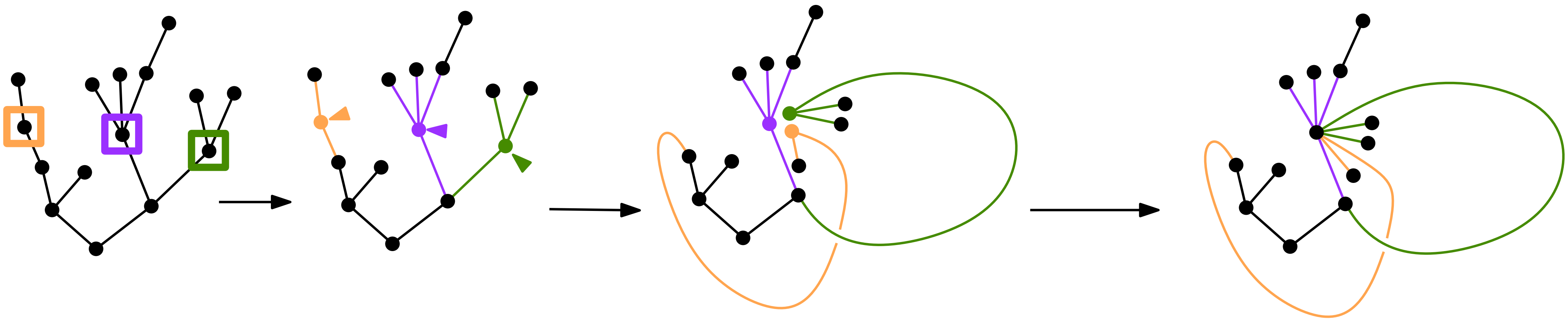

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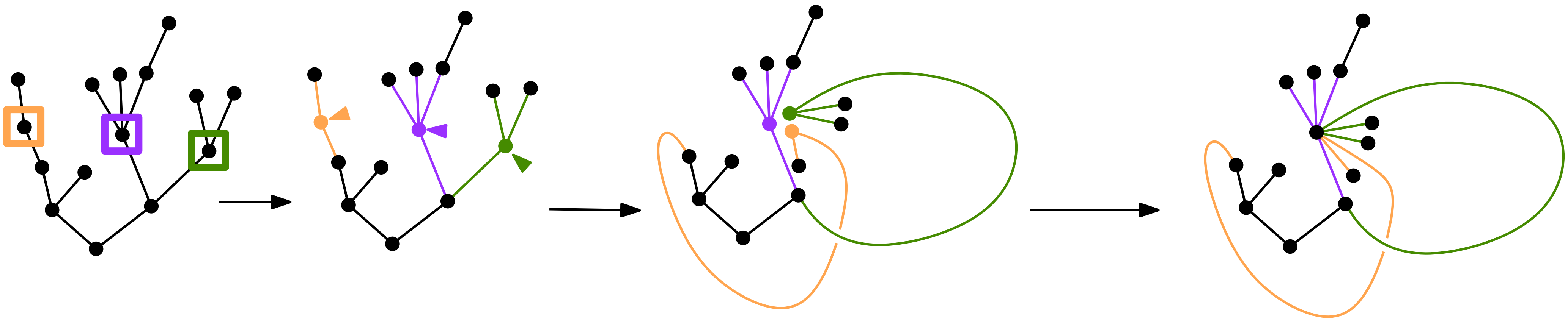
- The resulting map has only one border :

$$1 \to 2 \to \ldots \to a_1 \to \ldots \to a_2 \to \ldots \to a_3 \to \ldots \to 2n$$



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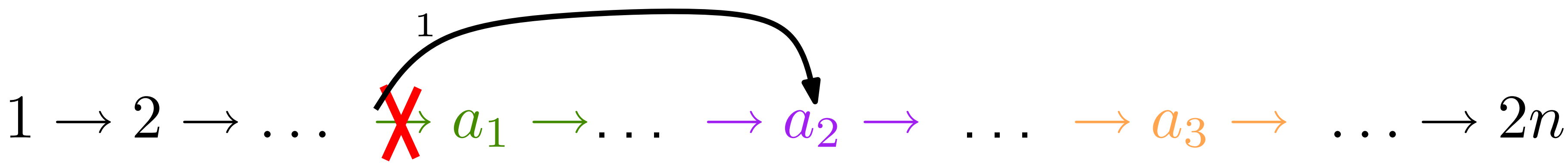

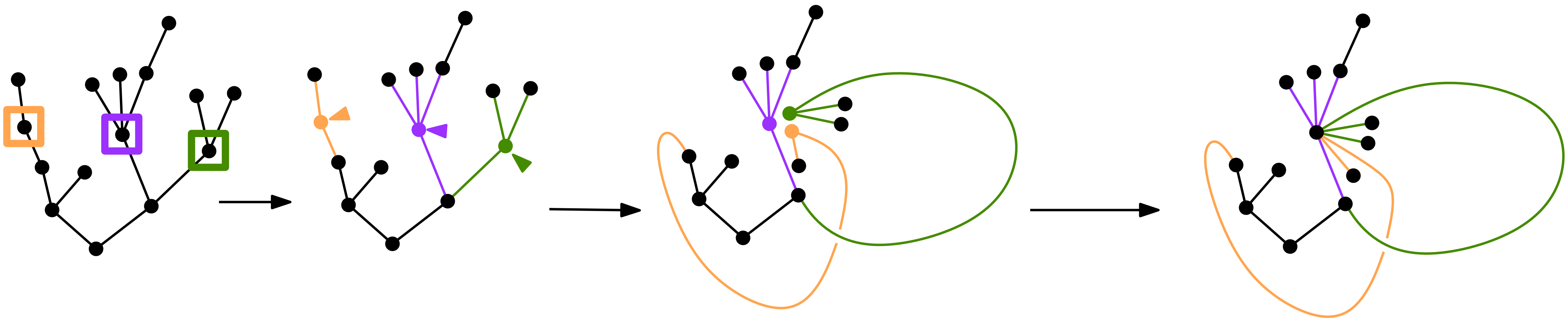

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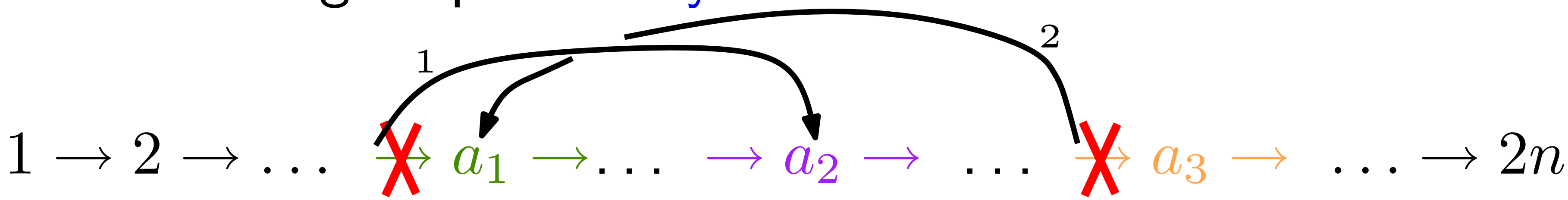

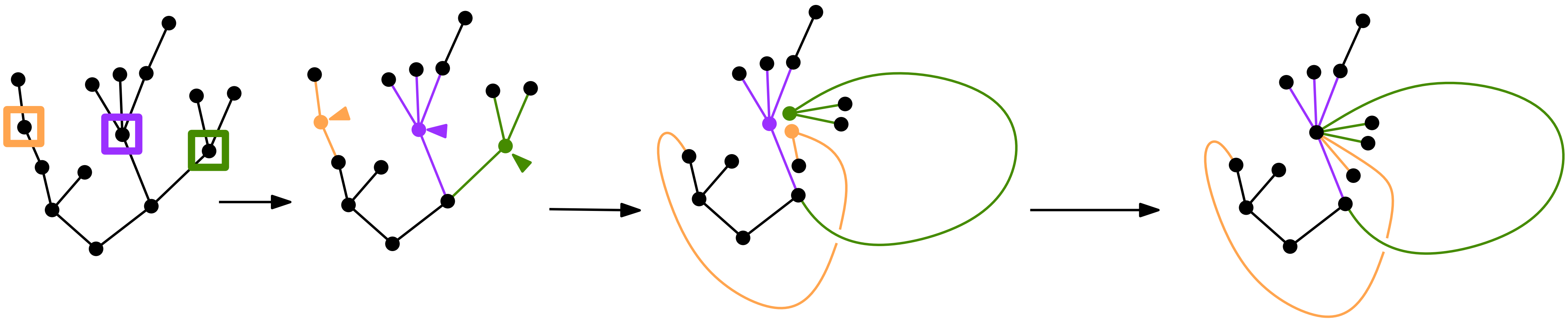

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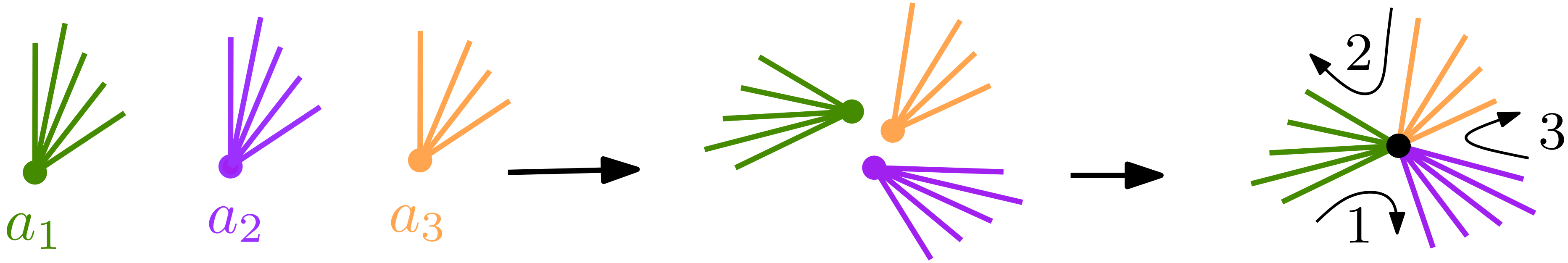

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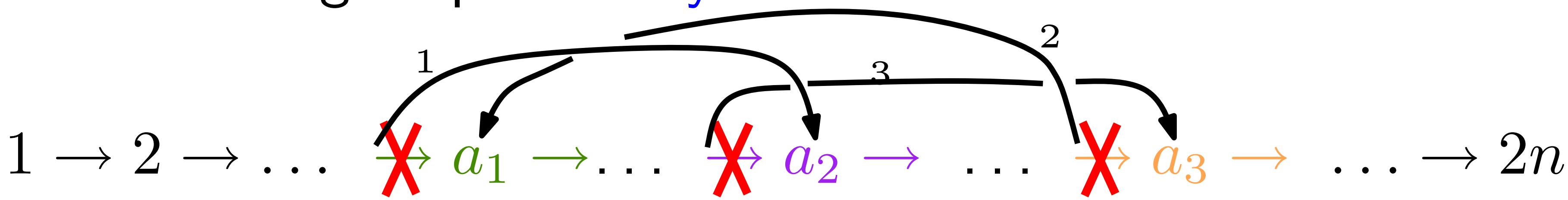

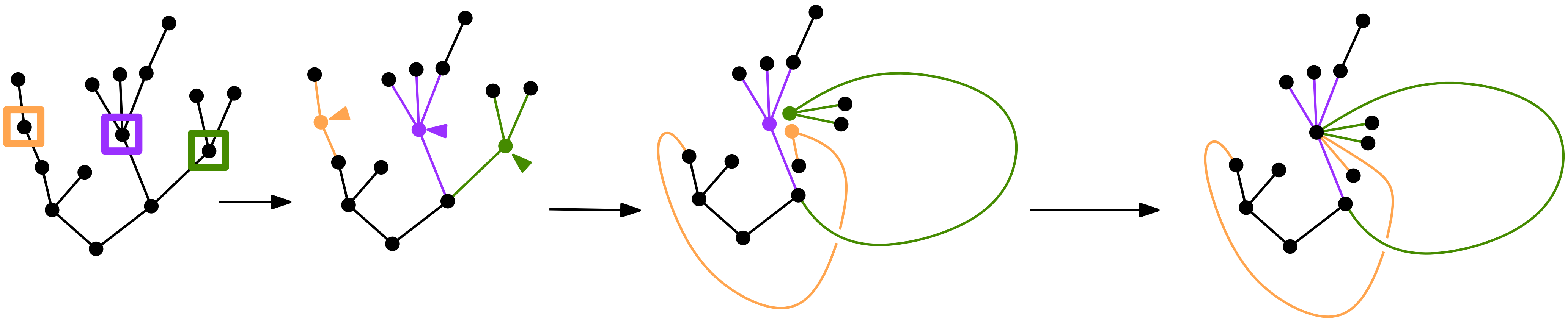

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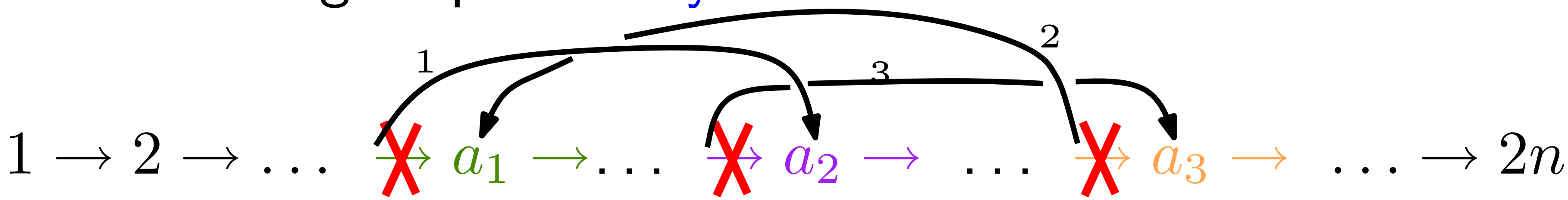

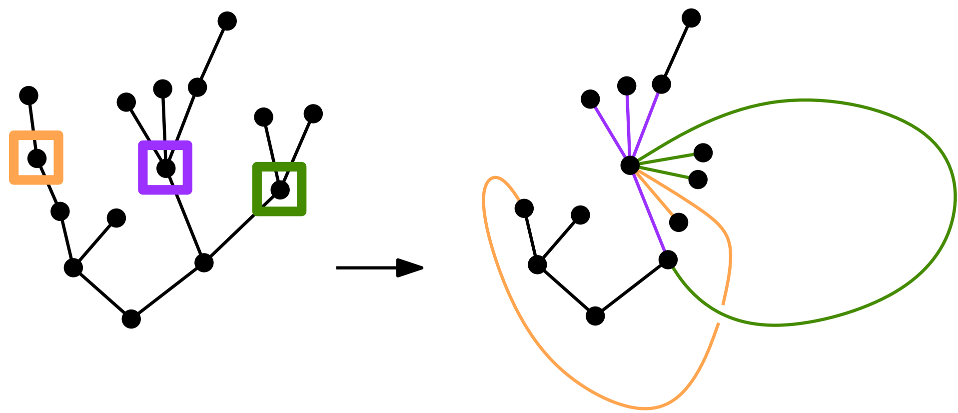

How to build a trisection : first method.

- Start with a map of genus $(g-1)$ with three marked vertices.
- Let $a_1 < a_2 < a_3$ be the labels of their minimal corners.
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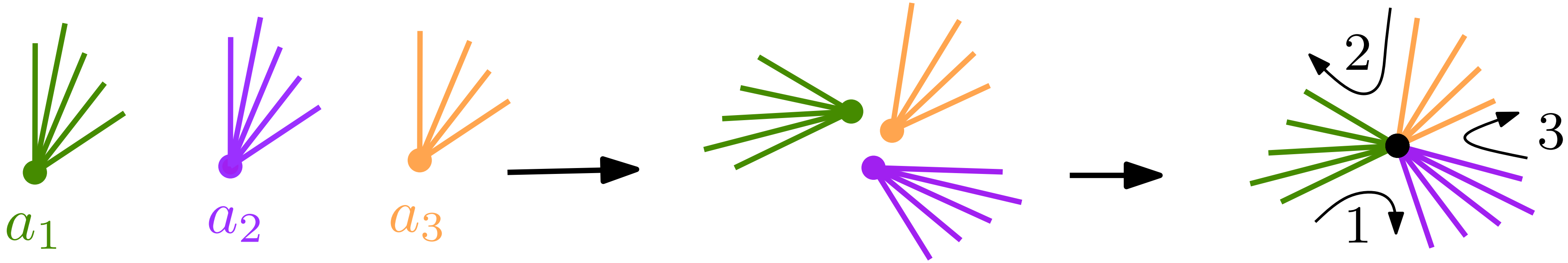

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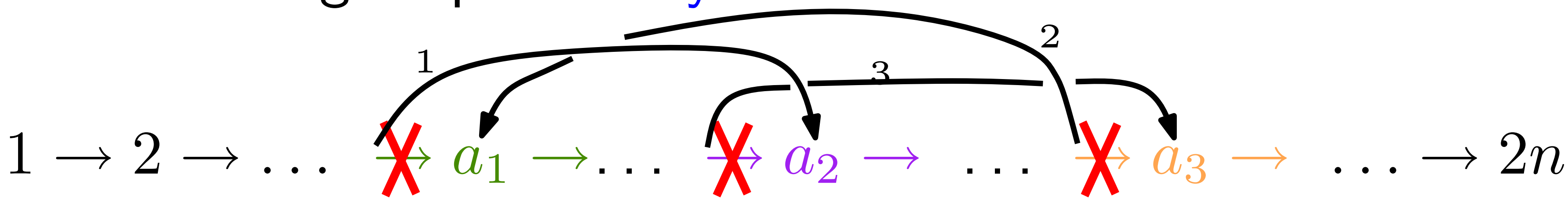

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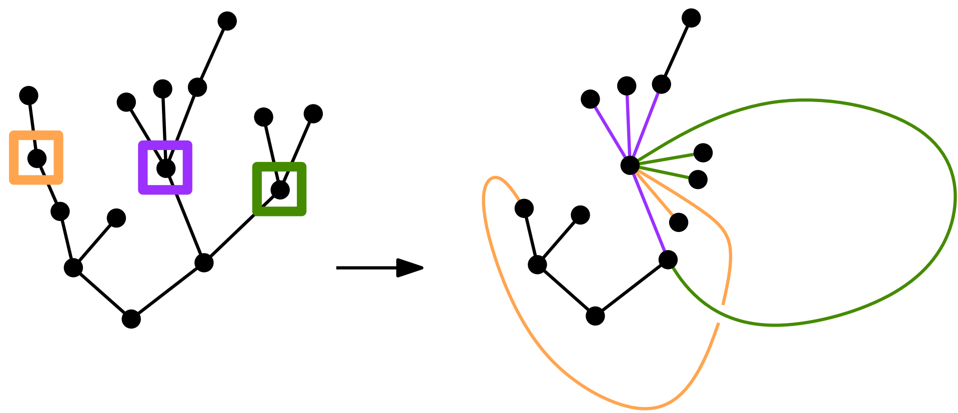

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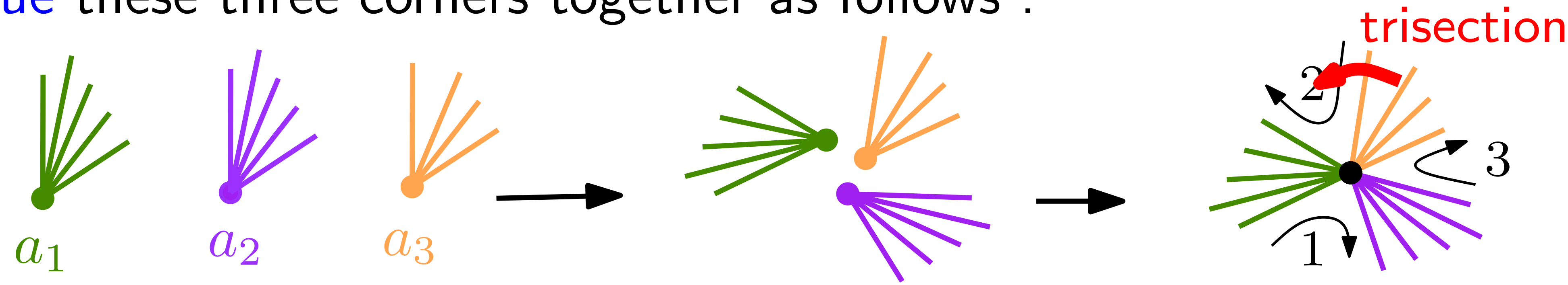

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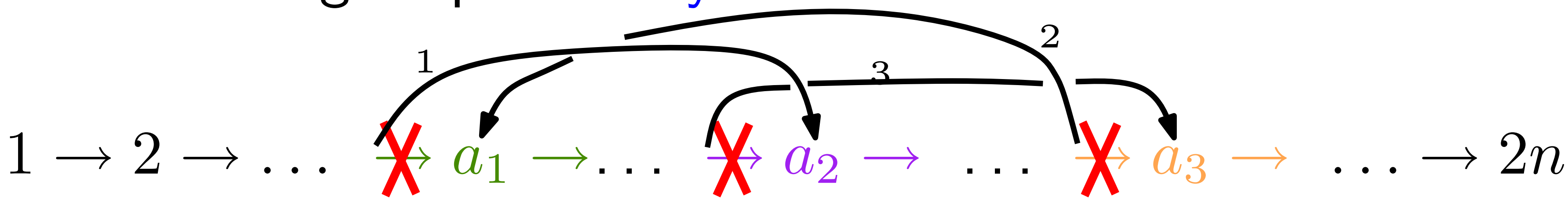

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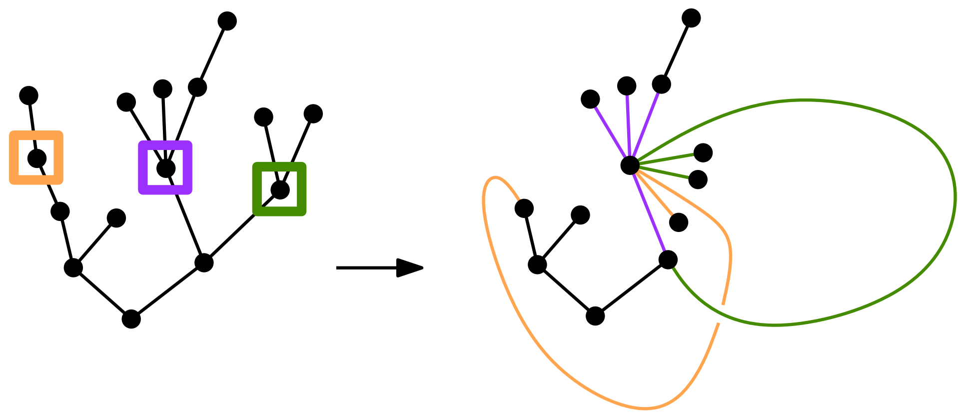
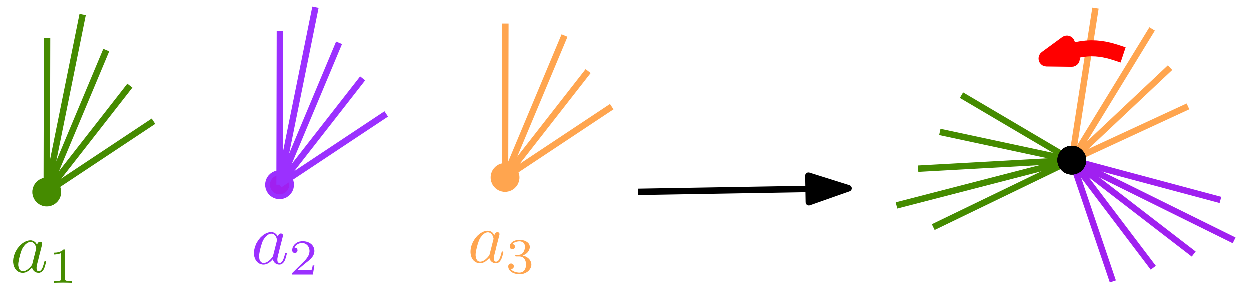
- Moreover we have built a trisection.



Therefore we have a mapping :



genus $g-1$, three marked vertices

genus g, one marked trisection

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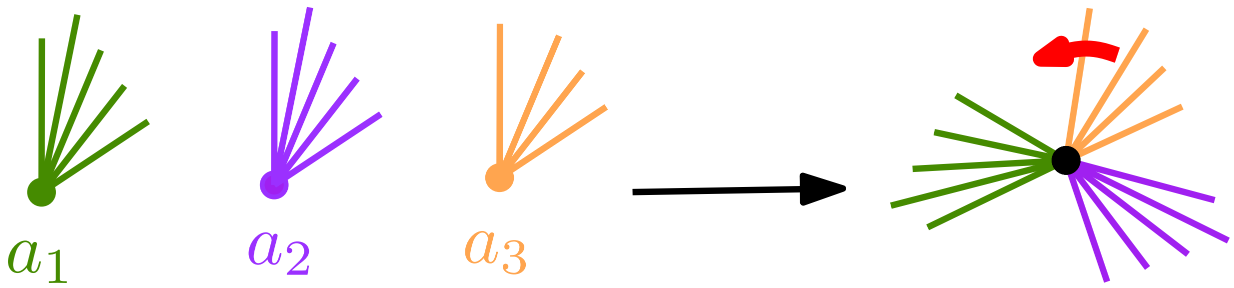

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The mapping is injective because we can retrieve the three corners, and cut the vertex back.

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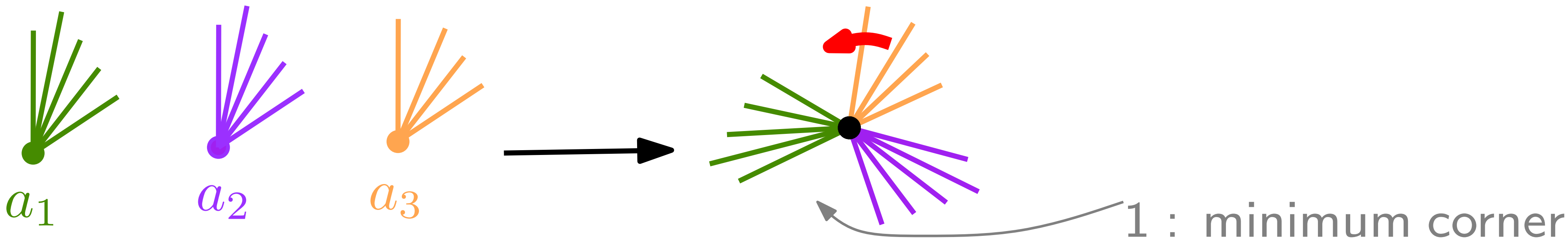

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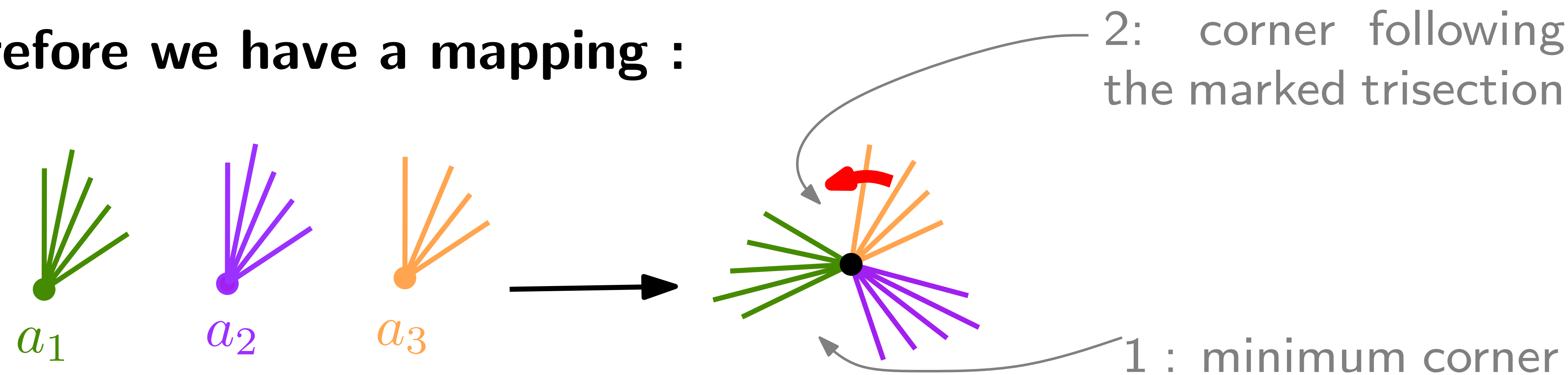

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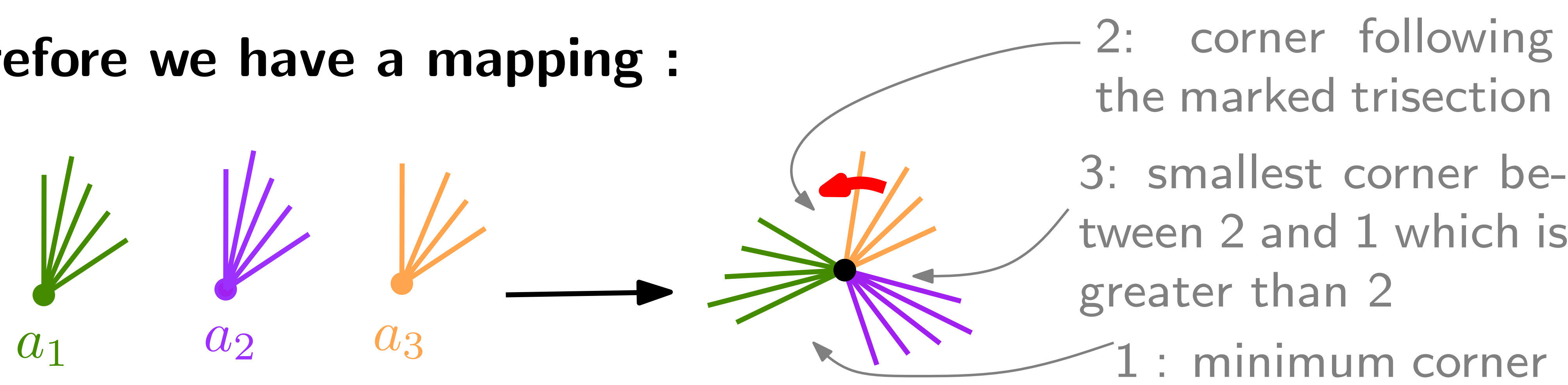

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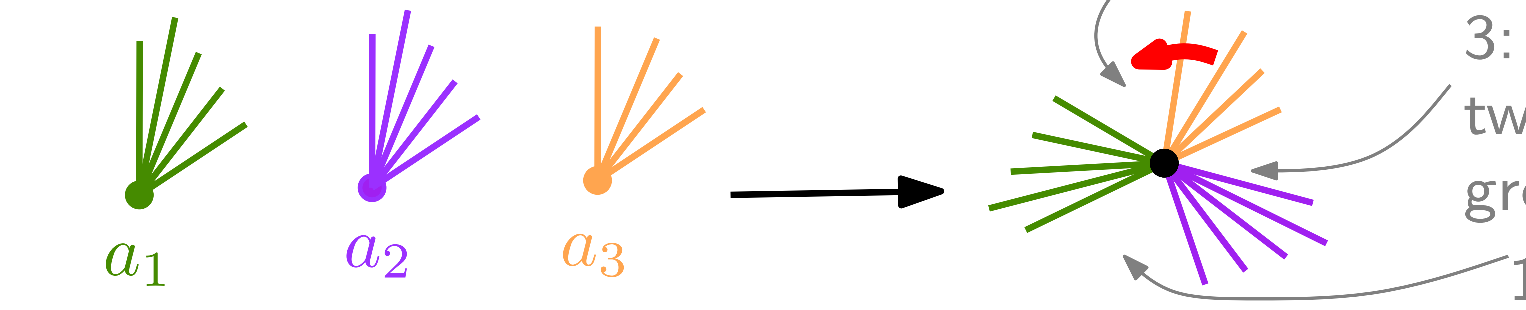

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Hence :

$$2g \cdot \epsilon_g(n) = \binom{n+3-2g}{3} \epsilon_{g-1}(n) + \dots$$

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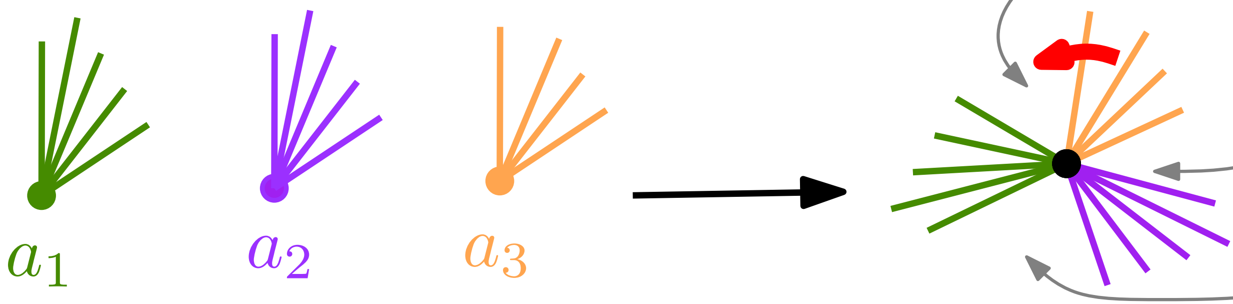

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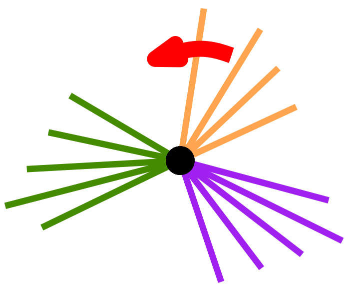

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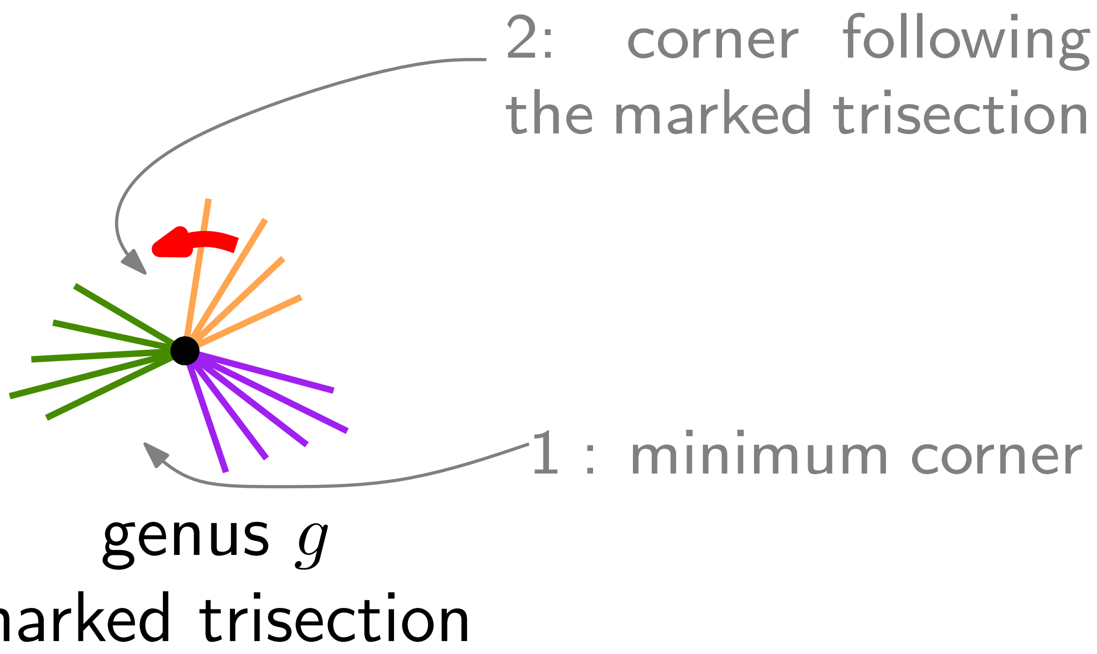

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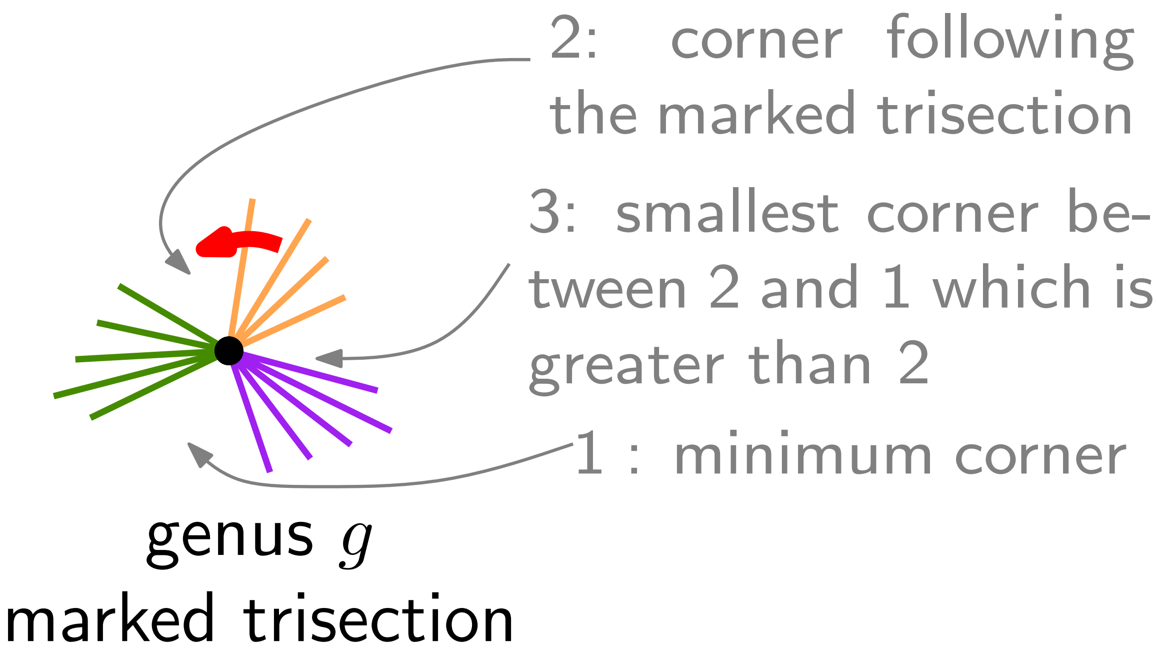

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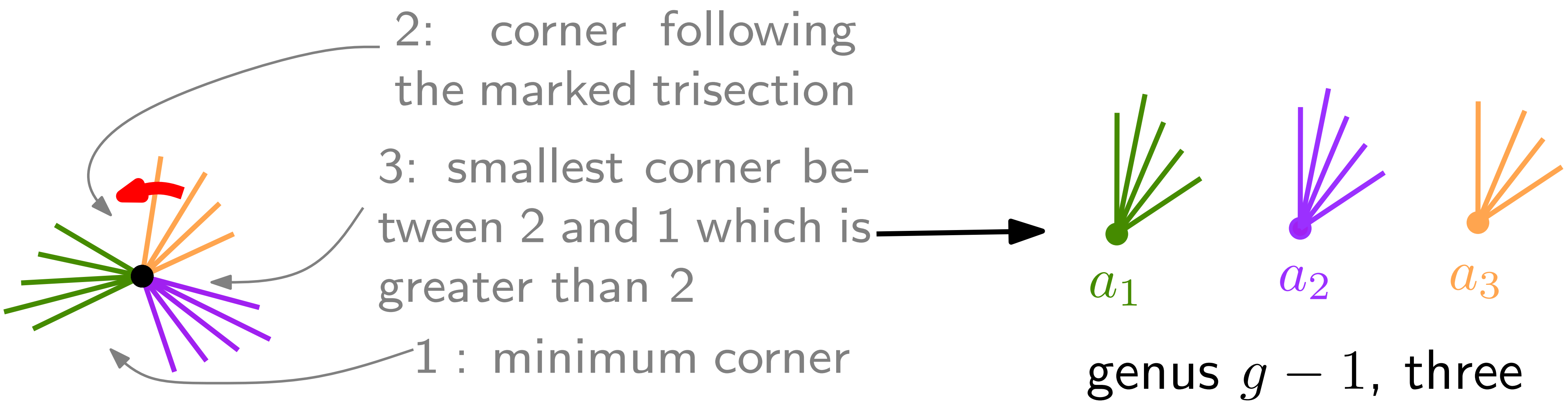

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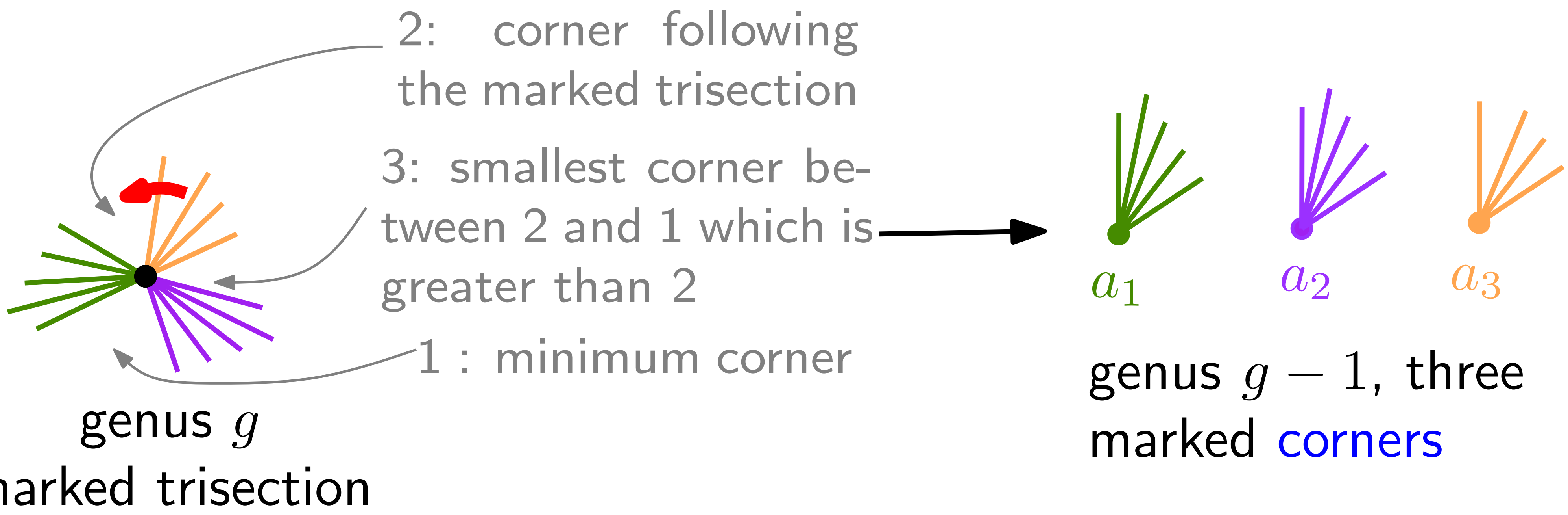

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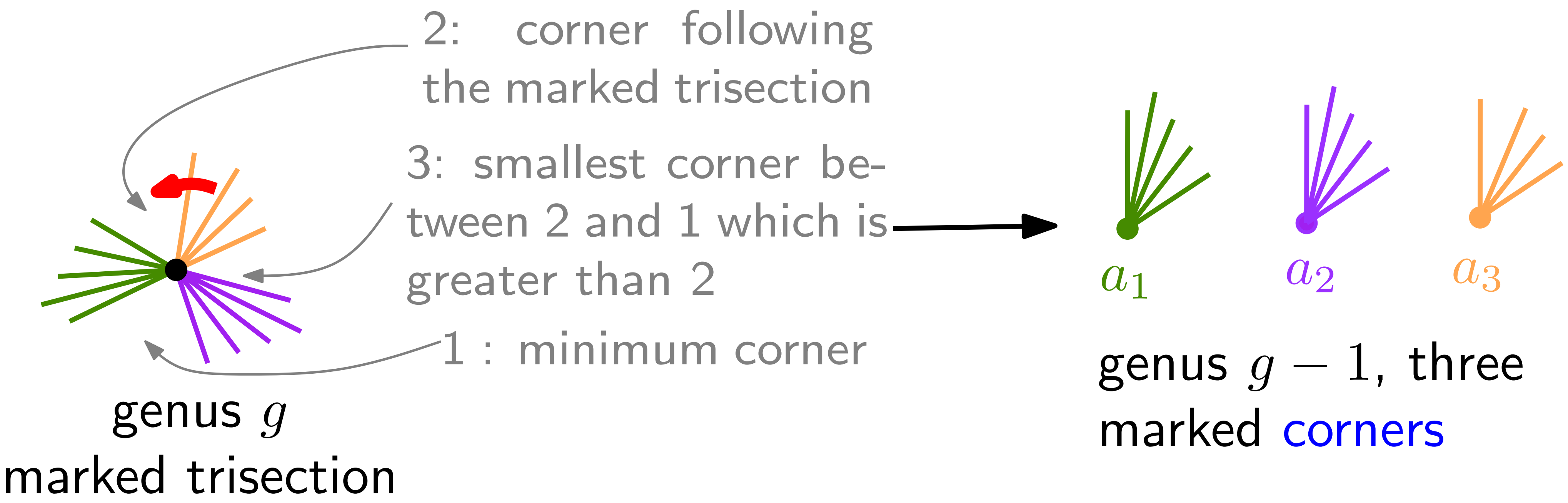

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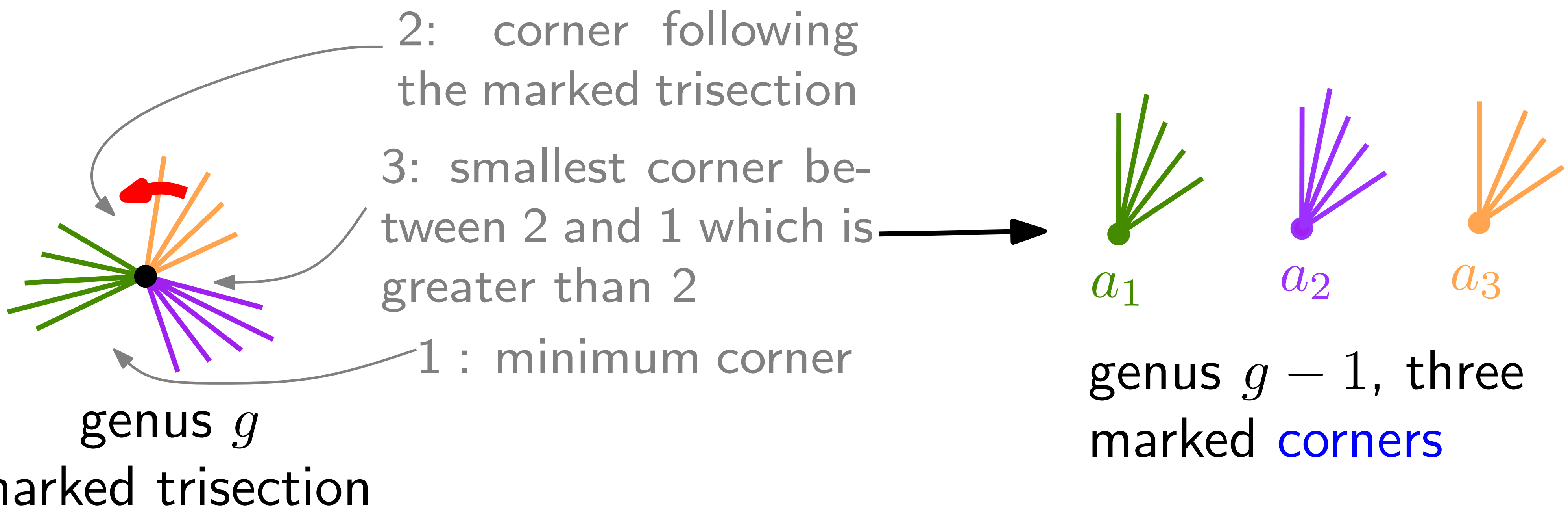

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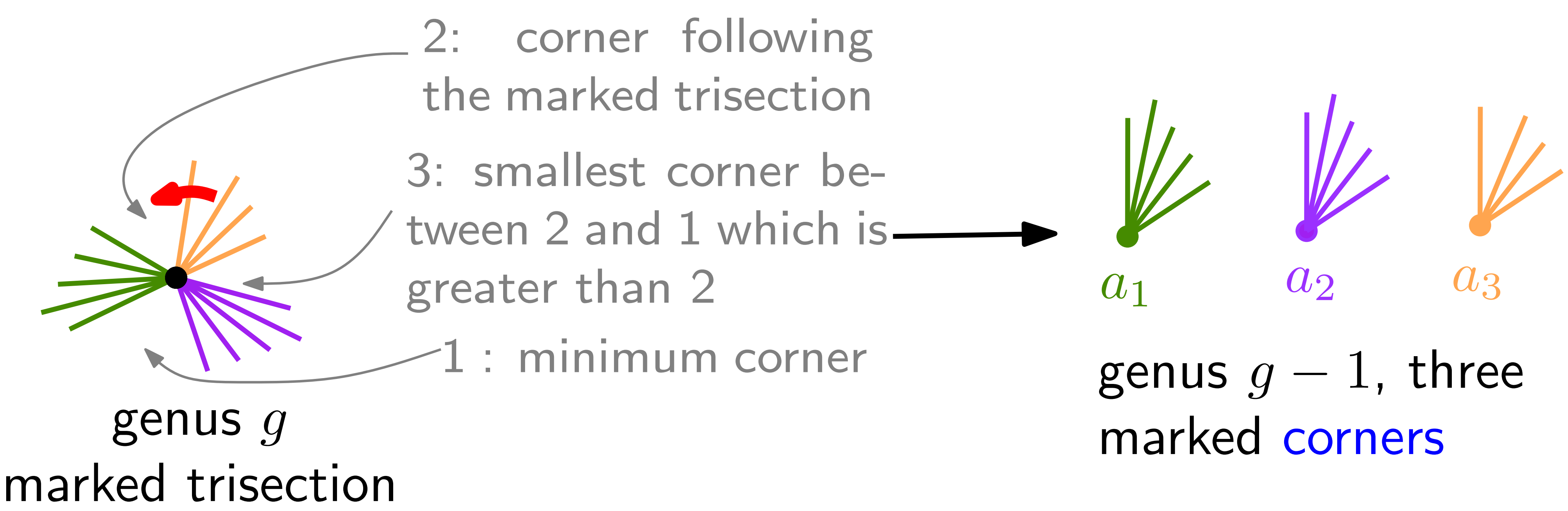

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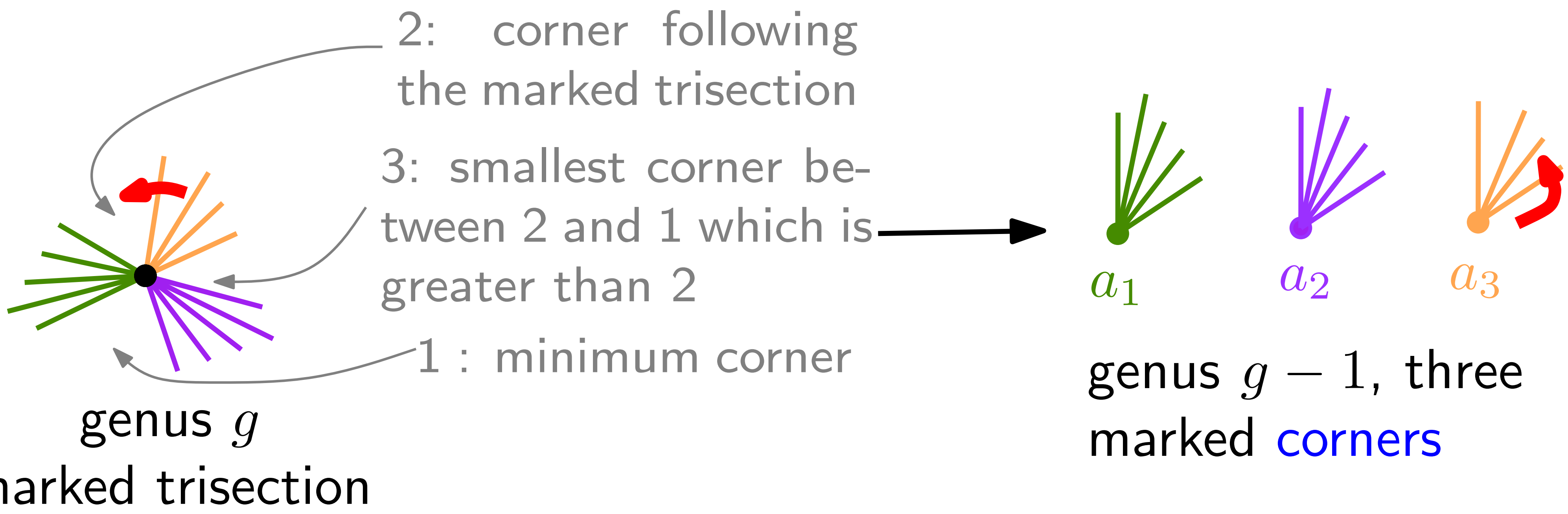

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- This is not always the case for a_3 :
  - If a_3 is the minimum of its vertex : we are in the image of the previous construction.
  - Else a_3 is incident to a trisection of the map of genus $(g-1)$.

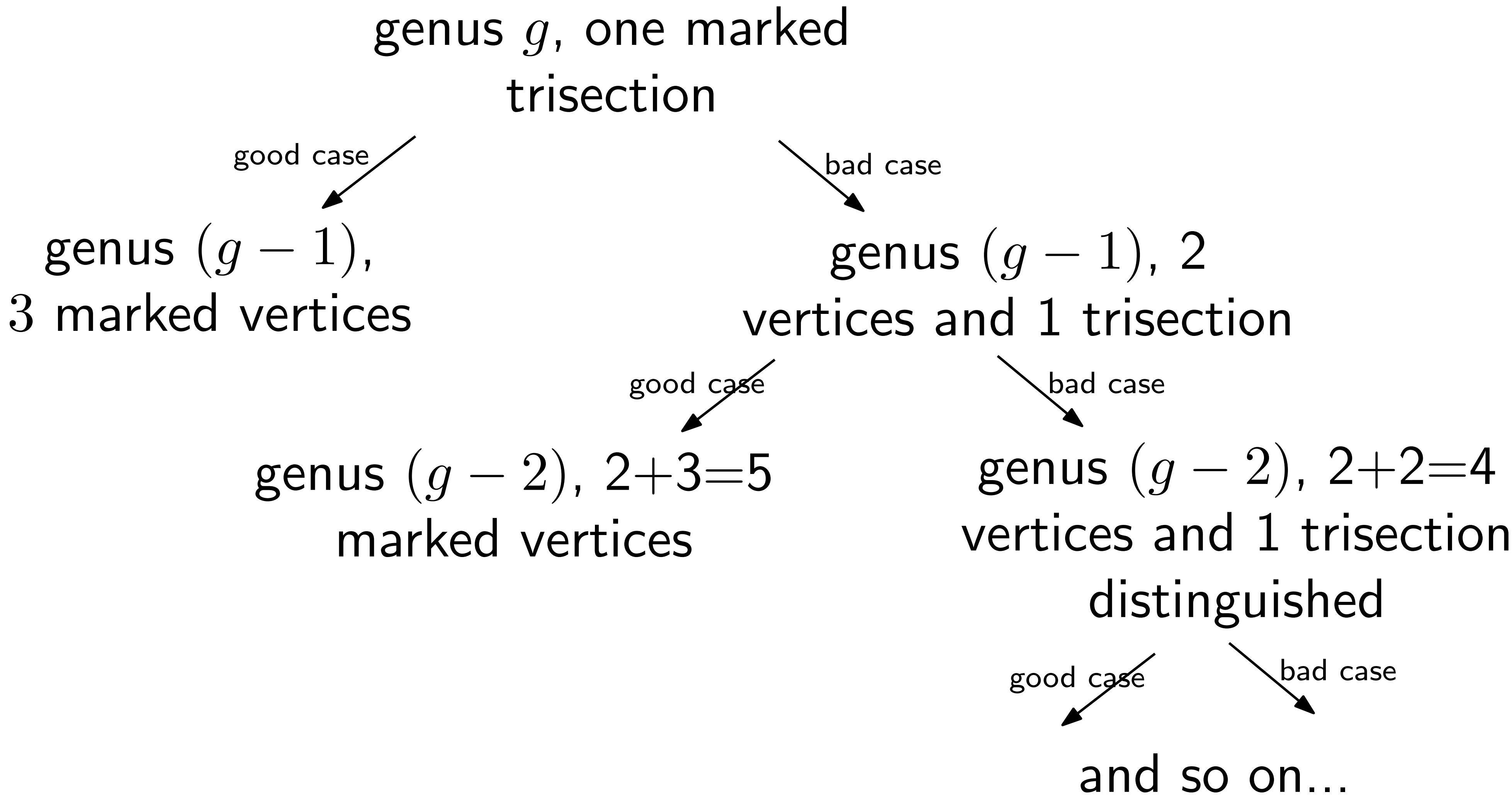
Therefore :

genus g, one marked trisection

good case → genus $(g-1)$, 3 marked vertices

bad case → genus $(g-1)$, 2 vertices and 1 trisection

Therefore :


genus g, one marked trisection
good case
bad case
genus $(g-1)$, 3 marked vertices
genus $(g-1)$, 2 vertices and 1 trisection
good case
bad case
genus $(g-2)$, 2+3=5 marked vertices
genus $(g-2)$, 2+2=4 vertices and 1 trisection distinguished
good case
bad case
and so on...

Our main result:

$$\left.\begin{array}{c}\text{genus } g,\\ \text{one marked trisection}\end{array}\right. \overset{\text{bij.}}{=} \bigcup_{i>0} \left(\begin{array}{l}\text{genus } g-i \text{ and } 2i+1\\ \text{marked vertices.}\end{array}\right)$$

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Everything boils down to plane trees:

$$\epsilon_g(n) = \underbrace{(\text{some polynomial})} \times \mathrm{Cat}(n)$$

$=$ "number" of possibilities for the successive choices of vertices.

$$= \sum_{0=g_0<g_1<\ldots<g_r=g} \prod_{i=1}^{r} \frac{1}{2g_i}\binom{n+1-2g_{i-1}}{2(g_i-g_{i-1})+1}$$

For instance :

$$2 \cdot \epsilon_1(n) = \frac{(n+1)n(n-1)}{6}\mathrm{Cat}(n)$$

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$$4 \cdot \epsilon_2(n) = \tfrac{(n-1)(n-2)(n-3)}{6} \epsilon_1(n) + \tfrac{(n+1)n(n-1)(n-2)(n-3)}{5!} \mathrm{Cat}(n)$$

$$= \tfrac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440} \mathrm{Cat}(n)$$

Extensions

- The formula leads to a differential equation which enables to recover the known closed formulas for the generating functions (Harer-Zagier, Itzykson-Zuber).

- Works the same for bipartite unicellular maps.

- The Marcus-Schaeffer bijection relates general maps on surfaces to labelled unicellular maps. The composition of the two bijections leads to a description of general maps of given genus in terms of labelled trees with distinguished vertices. This gives information about the continuum limit of maps on surfaces (Brownian map of genus g).

Thank you!

Formules d'Harer et Zagier :

Récurrence :

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1)+(2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2)$$

Version sommatoire :

$$\sum_{g\geq 0} \epsilon_g(n) y^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{i\geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$