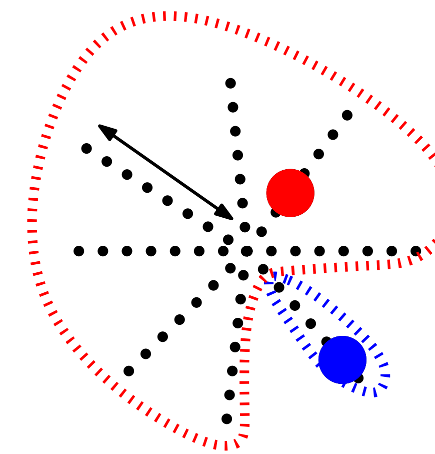
Voronoï Tessellations in the CRT and Continuum Random Maps of Finite Excess



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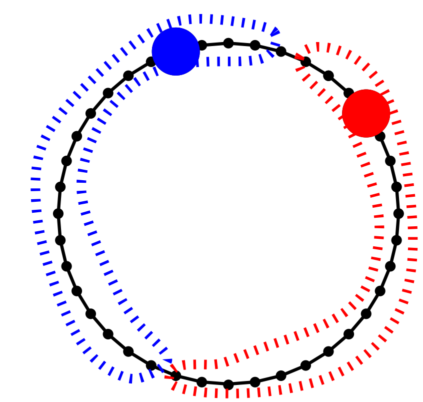

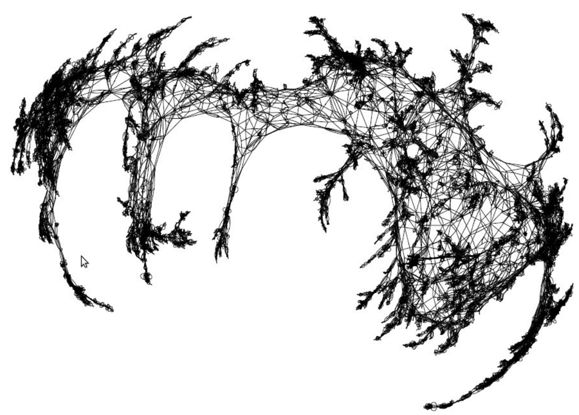

Work supported by the grant ERC – Stg 716083 – “CombiTop”

SODA 2018, New Orleans.

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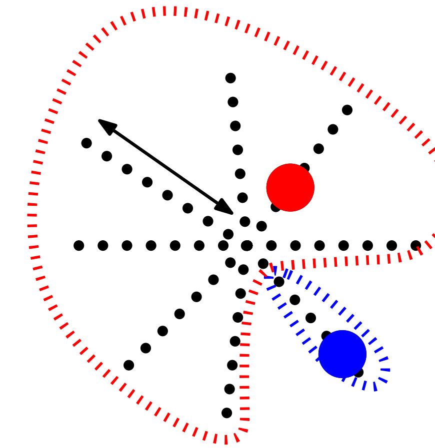

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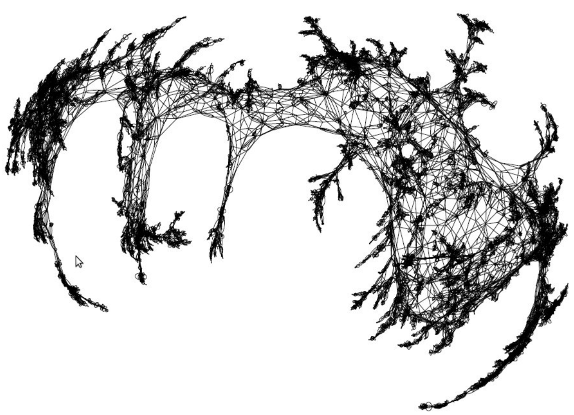

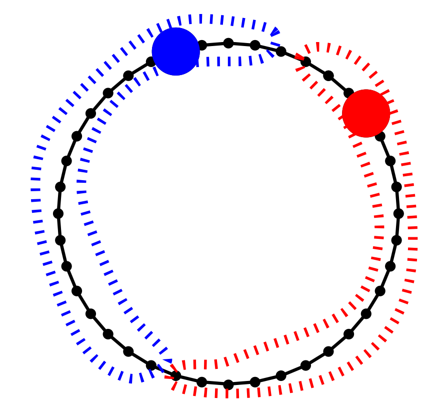

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The Voronoï vector – main definition of the talk!

- Let G_n be your favorite random graph with n vertices ($n \to \infty$)

  Pick k points $v_1, v_2, \ldots, v_k$ uniformly at random (k fixed) and call

  $$V_i = \{x \in V(G), d(x, v_i) = \min_j d(x, v_j)\}$$ (in case of equality, assign to a random V_i among possible choices)

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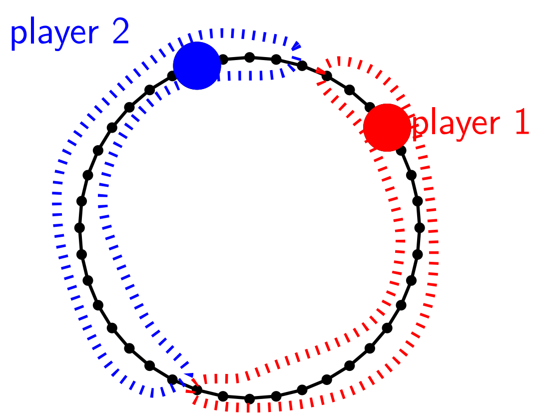
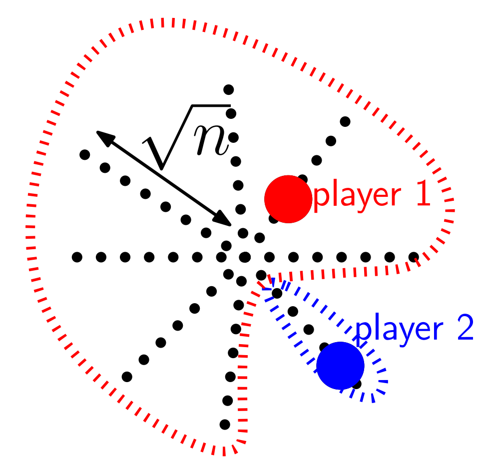
- Question: what is the limit law of the "Voronoï vector" $(\frac{|V_1|}{n}, \frac{|V_2|}{n}, \ldots, \frac{|V_k|}{n})$?

  Examples with $k = 2$



Cycle: deterministic $(\frac{1}{2}, \frac{1}{2})$

$\delta_{\frac{1}{2},\frac{1}{2}}$



"$\sqrt{n} \times \sqrt{n}$-star": winner takes (almost) all

$\frac{1}{2}\delta_{0,1} + \frac{1}{2}\delta_{1,0}$

Conjecture and results

- **Conjecture** [C., published in 2017]

  For a random embedded graph of genus $g \geq 0$ and any $k \geq 2$, the limit law is uniform on the k-simplex. OPEN EVEN FOR PLANAR GRAPHS.

  In particular for $k = 2$ points, each of them gets a $U[0, 1]$ fraction of the mass.

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Random maps of finite excess

Fix $(g; \ell; n_1, \ldots, n_\ell)$ with $g \geq 0$, $\ell \geq 1$, and with $n_i \geq 1$.

Consider a uniform random map (=embedded graph) M with n edges ($n \to \infty$) such that:

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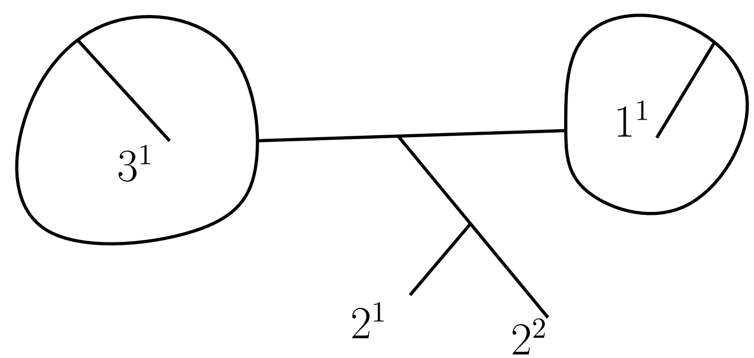
W.h.p. such a map is formed by a cubic skeleton, with edges subdivided in paths of length $O(\sqrt{n})$, and trees attached:

Example:
(0;3;1,2,1)



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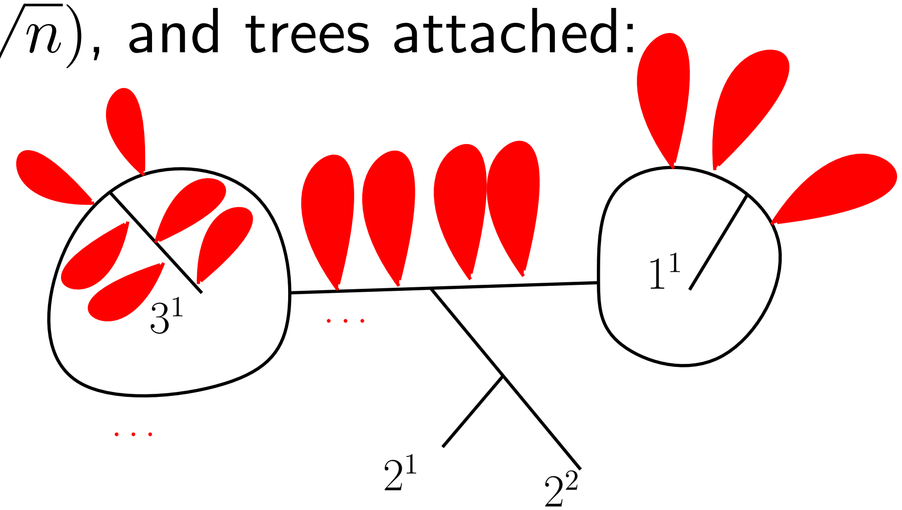

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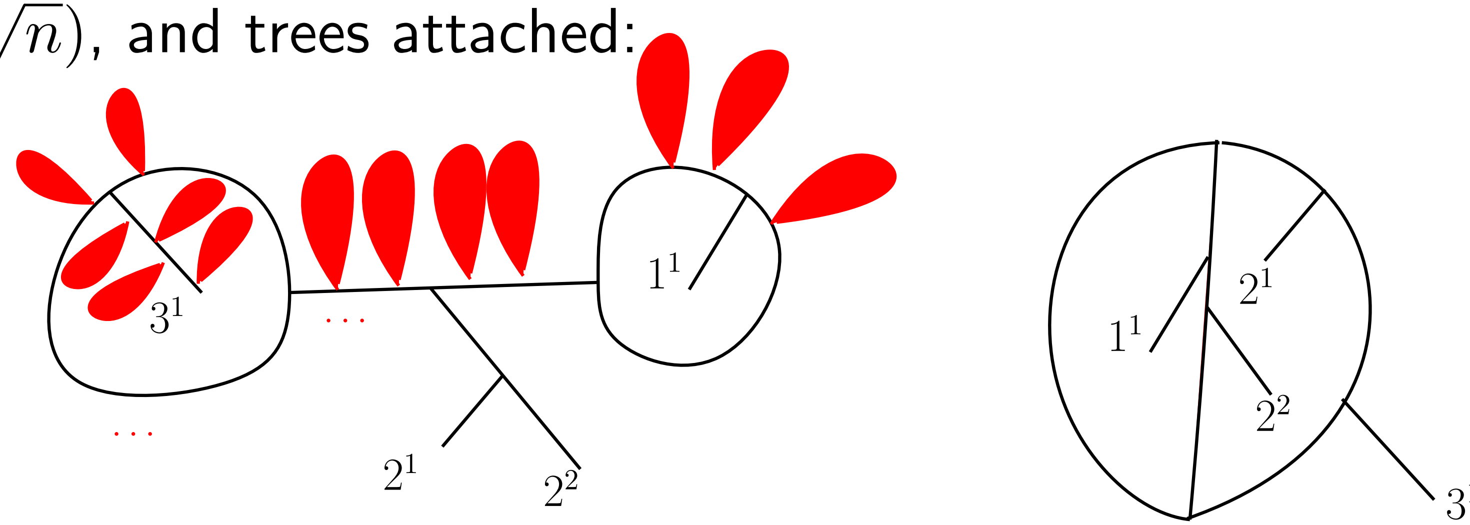

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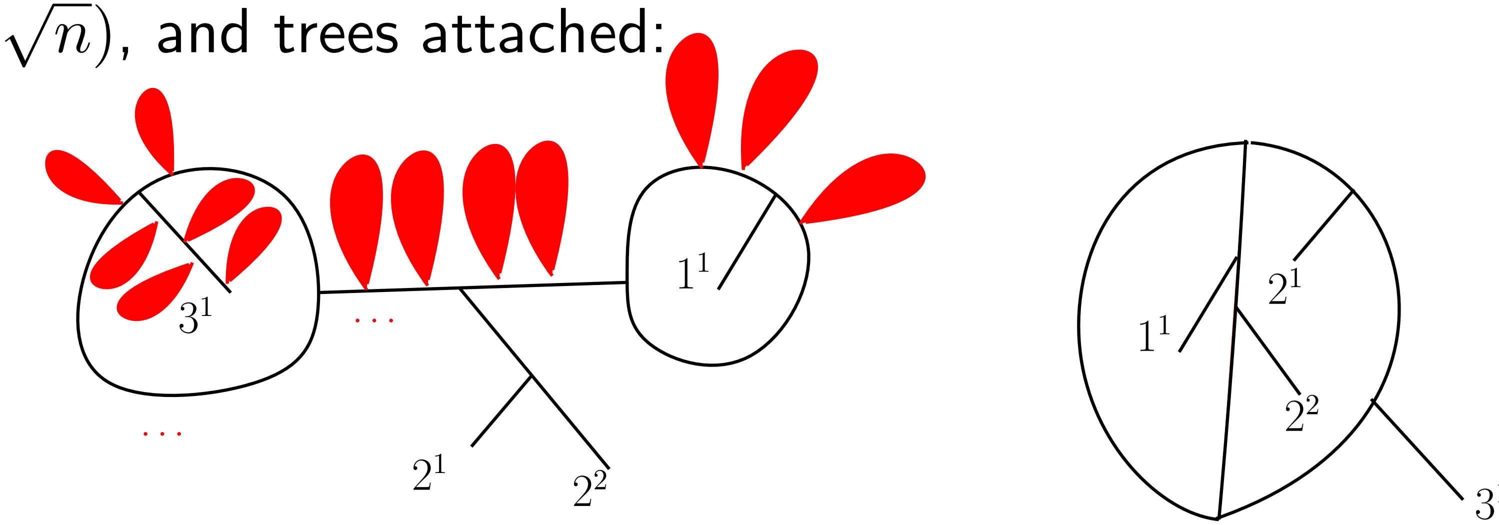

The number of skeletons is finite and all are equaly likely.

Note: $(0;1;k)$= uniform plane tree with k marked points!

Our most general result: Voronoï vs. Interval vectors

$M \sim (g;\ell;n_1,\ldots,n_\ell)$ with $g \geq 0$, $\ell \geq 1$, and with $n_i \geq 1$.

In the map M look at the **two vectors** of length $k = \sum_i n_i$

$$\vec{v} := \left(\frac{|V_1^1|}{n},\ldots,\frac{|V_1^{n_1}|}{n},\ldots,\frac{|V_k^1|}{n},\ldots,\frac{|V_\ell^{n_\ell}|}{n}\right) \text{ Voronoï vector}$$

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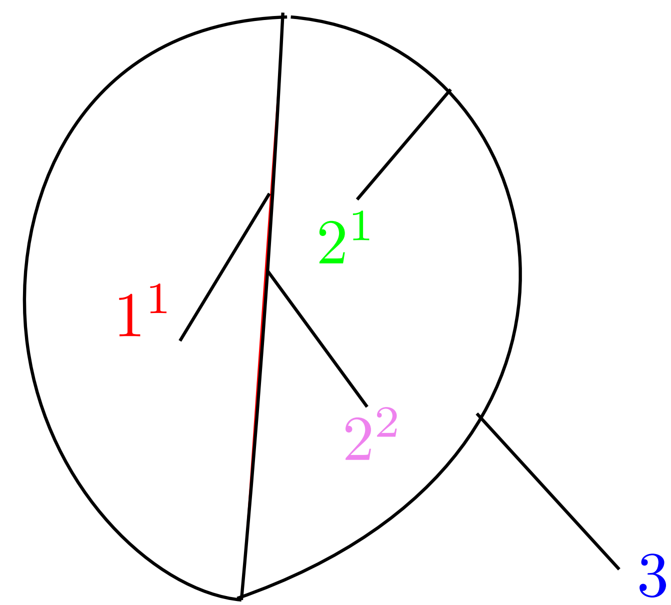

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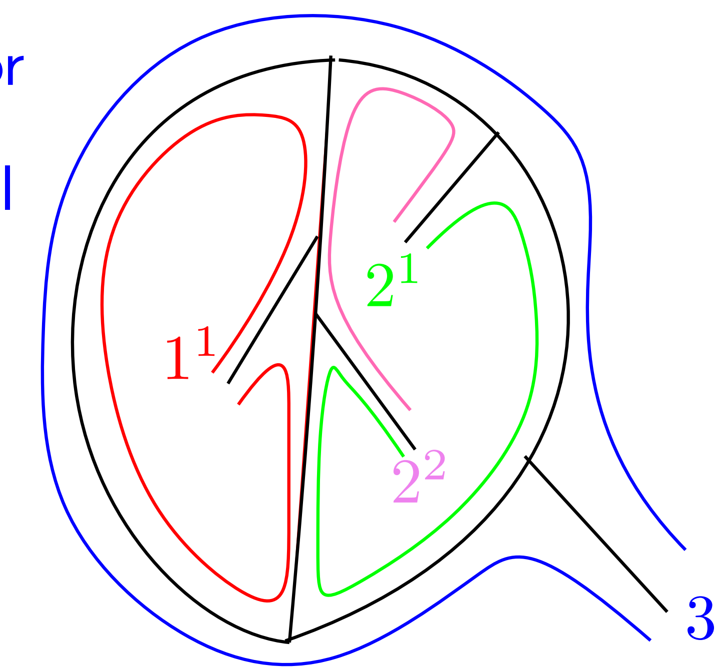

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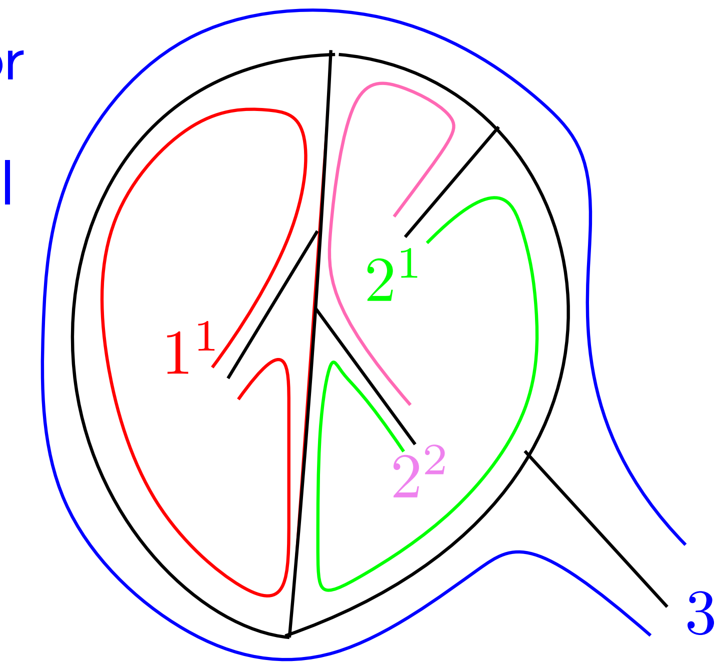

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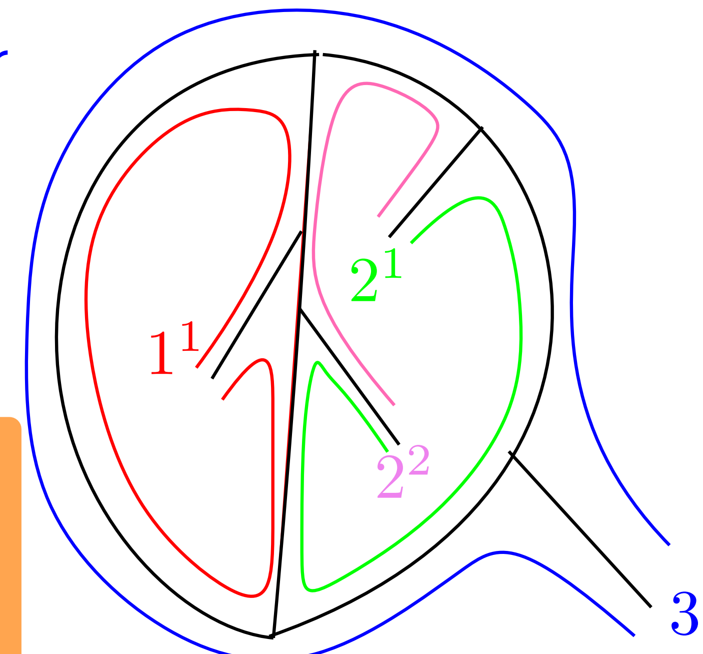

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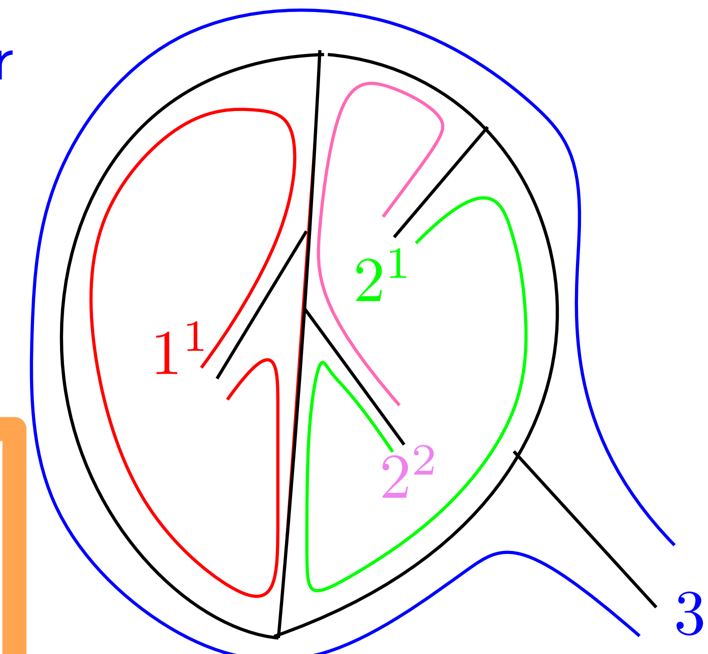

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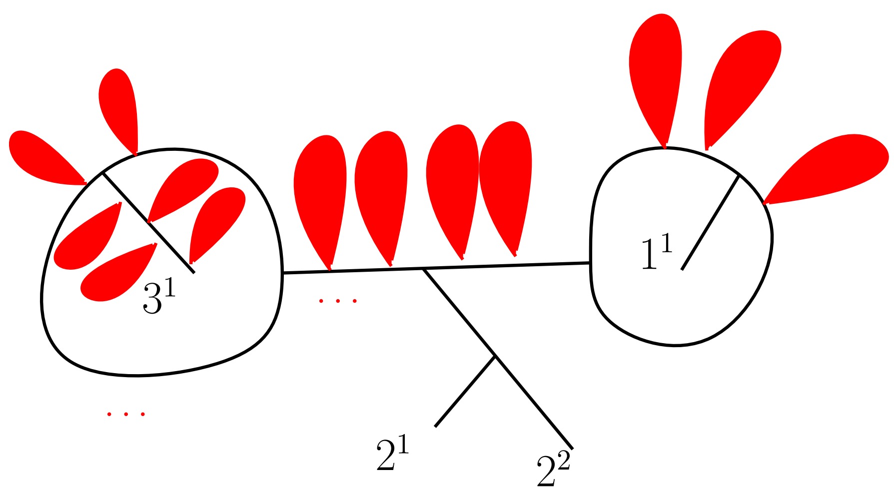
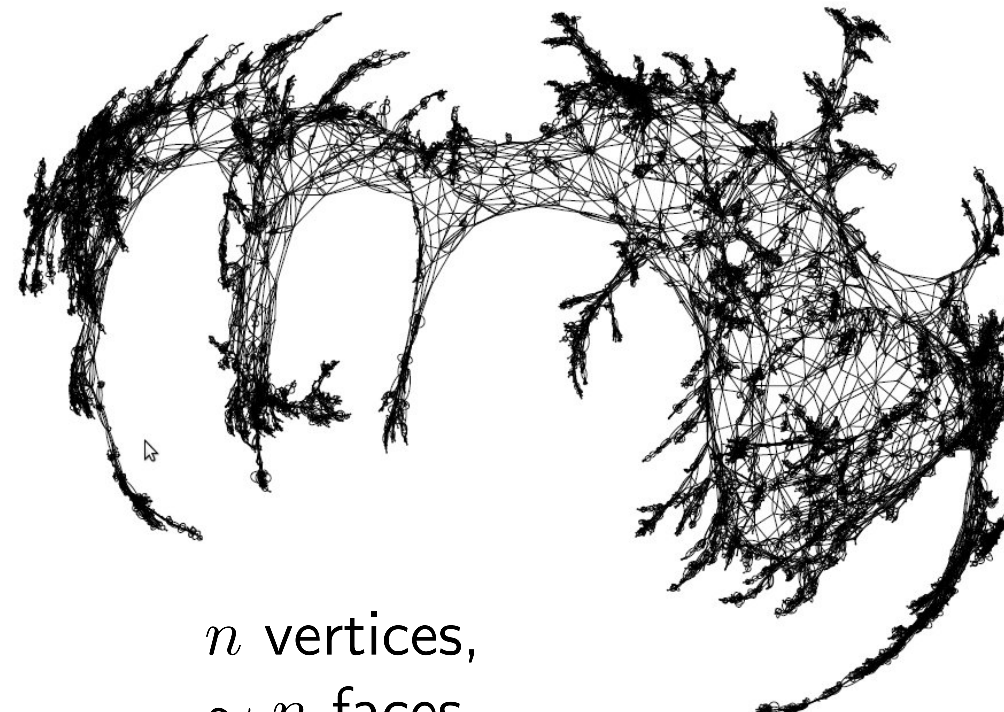
Comments: We DO NOT know how to prove uniformity even for trees without the trick of introducing interval vectors!

The proof is by induction on Euler characteristic

Note

finite excess random maps of genus $g \neq$ general random maps of genus g





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excess $O(1)$

diameter $\Theta(\sqrt{n})$

continuum limit object
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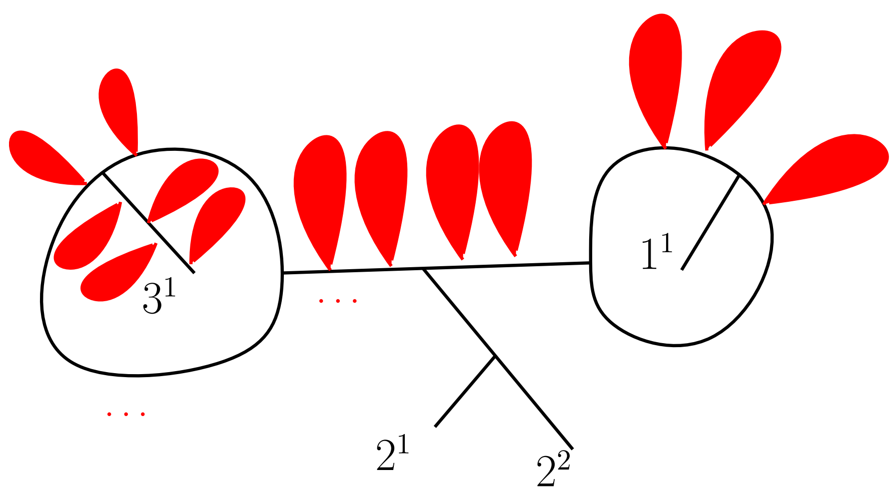

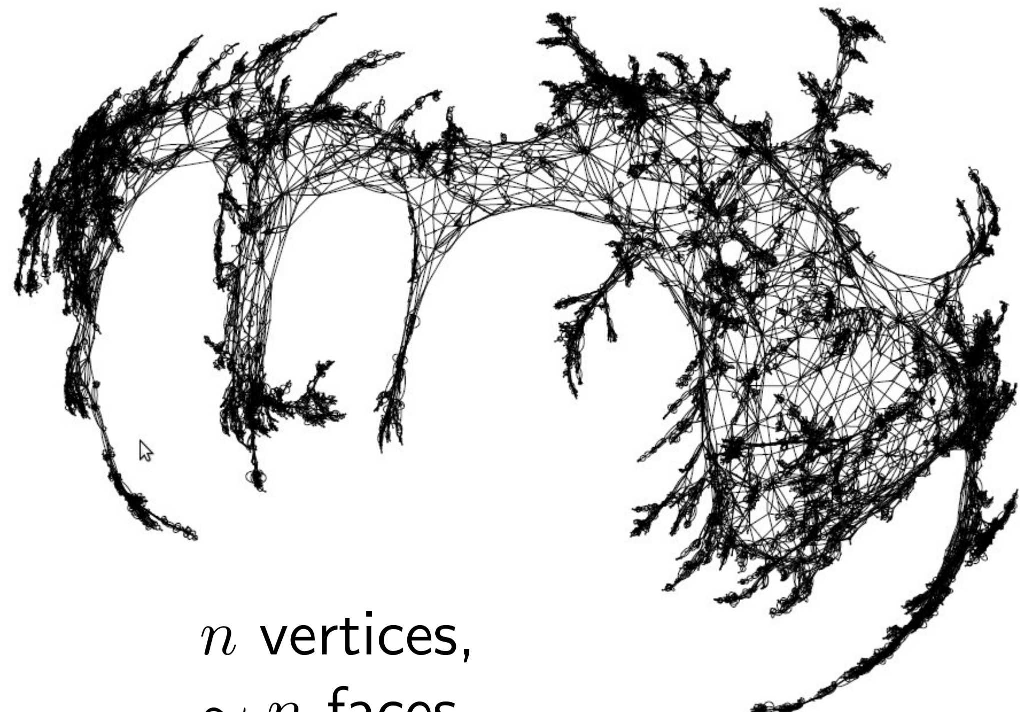

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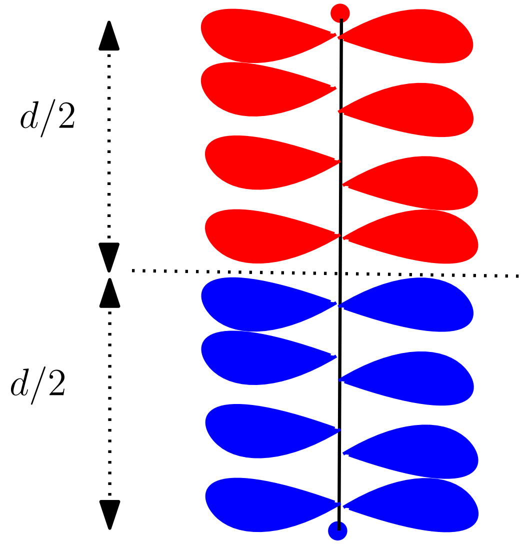
$\longrightarrow$ Why would their Voronoi vectors behave similarly ???

The proof for trees

Start with $k = 2$ (two marked points).



Voronoï game

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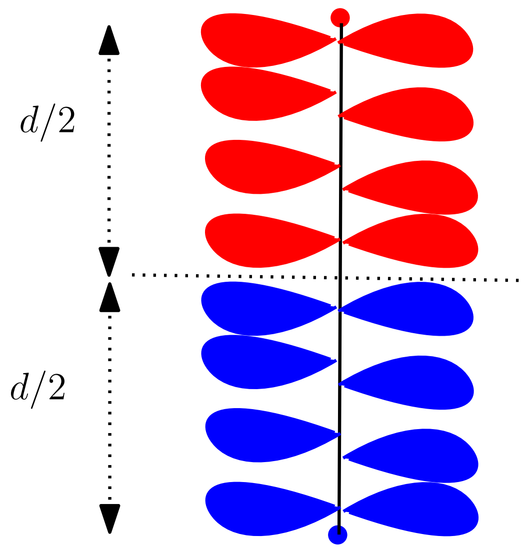

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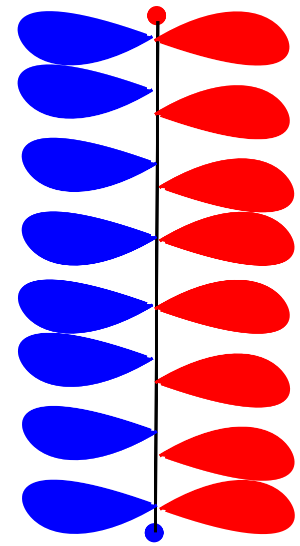

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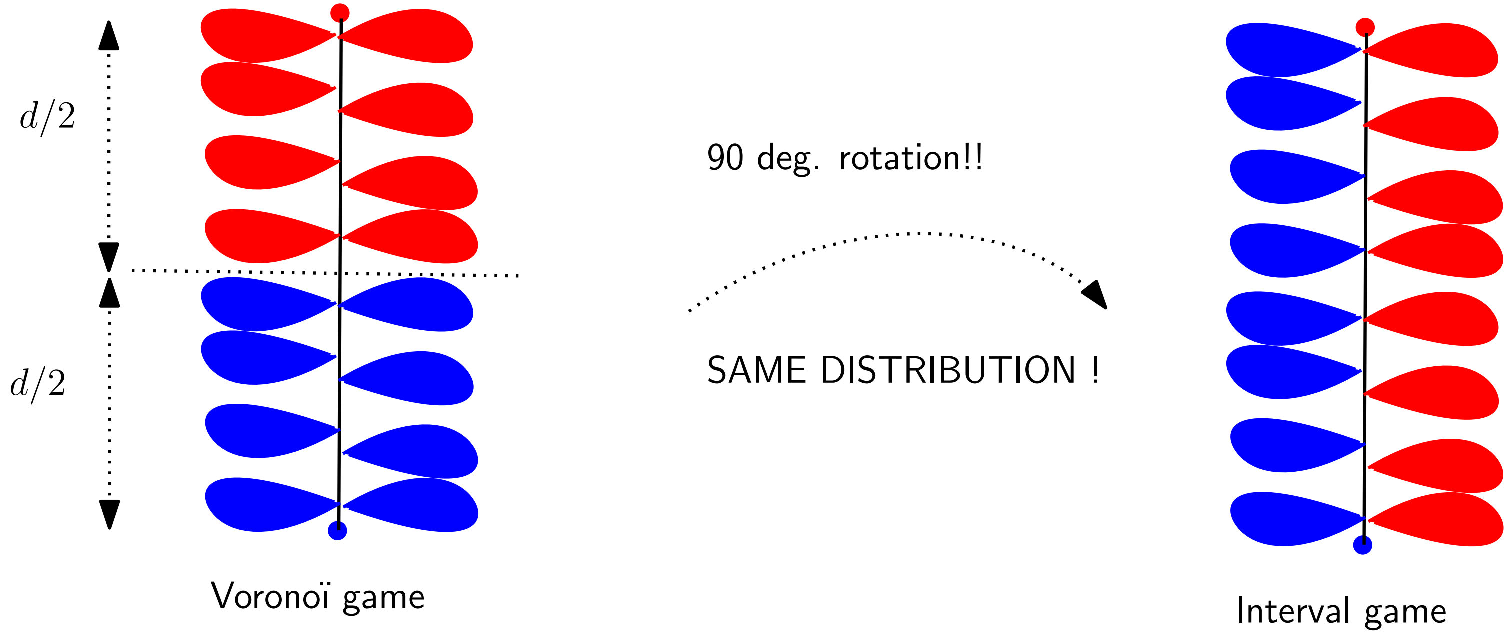

... It took us YEARS to find this trick

The proof for trees, continued $k \geq 2$

Take k players (here $k = 4$) and look at the Voronoi and Interval Games.

Voronoi Game

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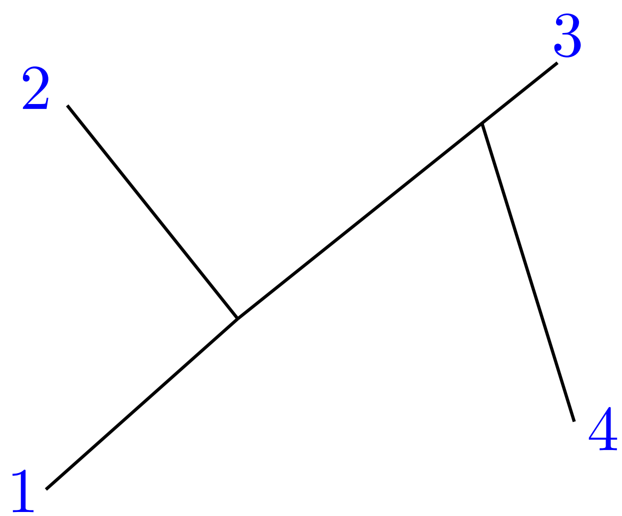

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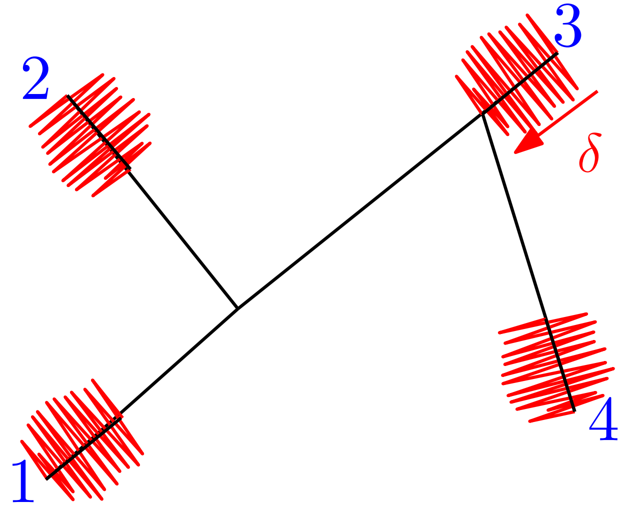

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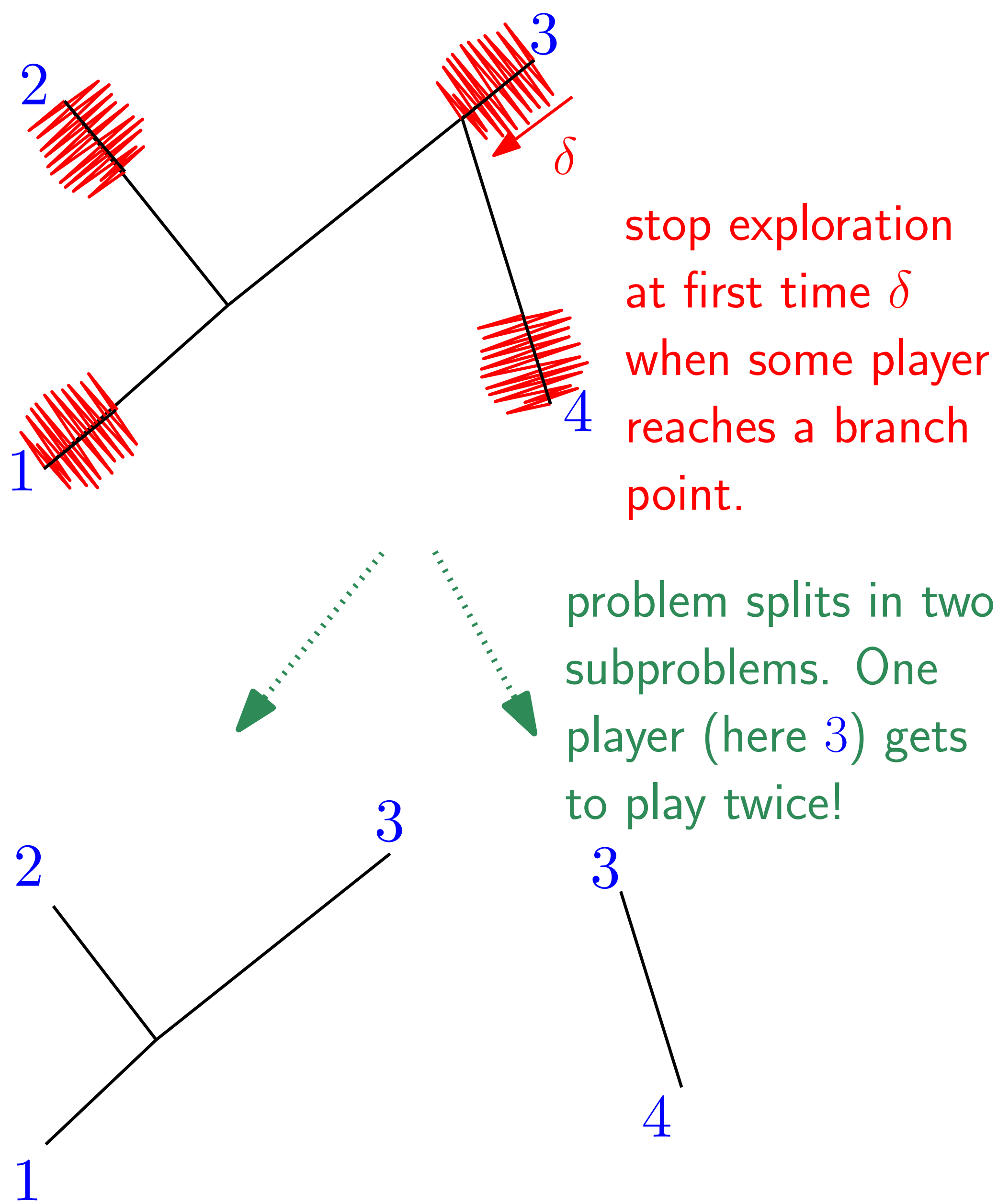

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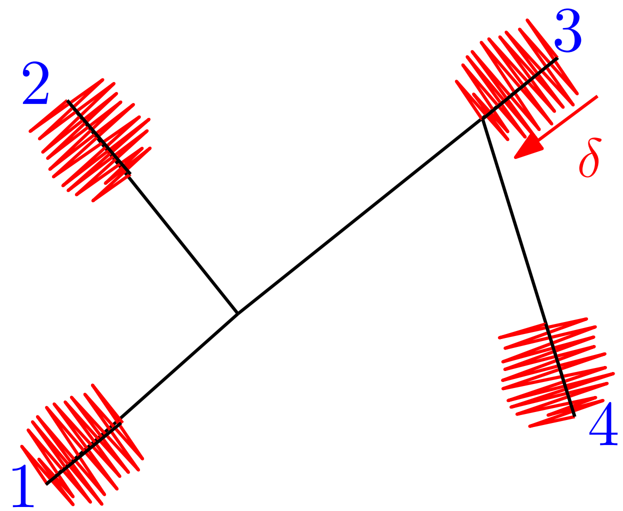

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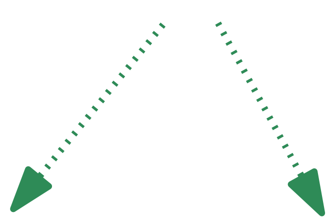
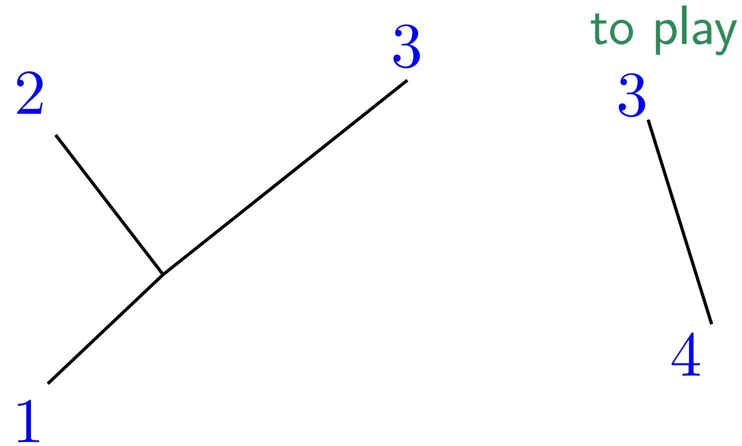
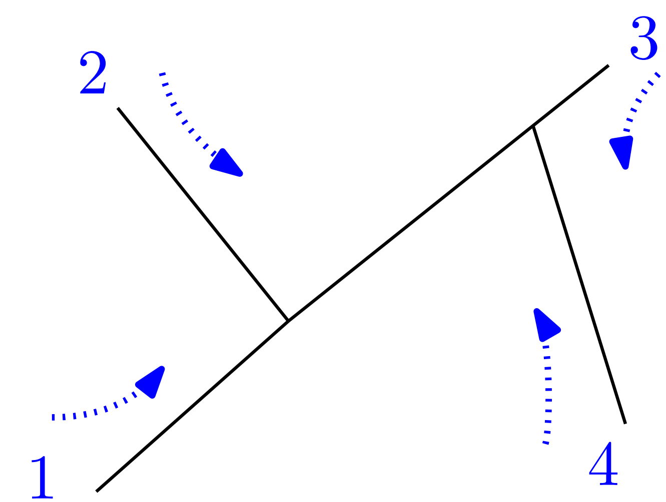


problem splits in two subproblems. One player (here 3) gets to play twice!



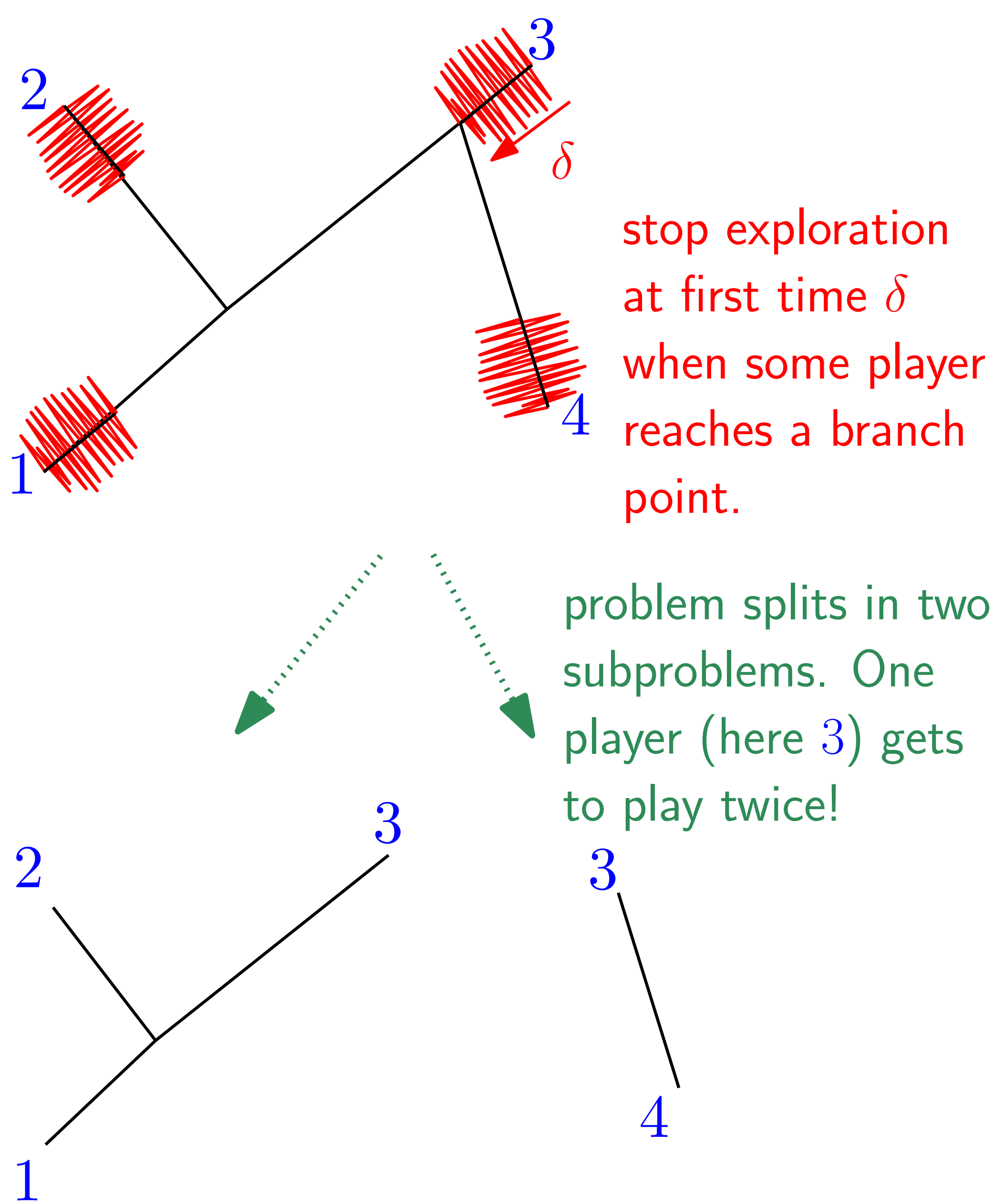
Interval Game



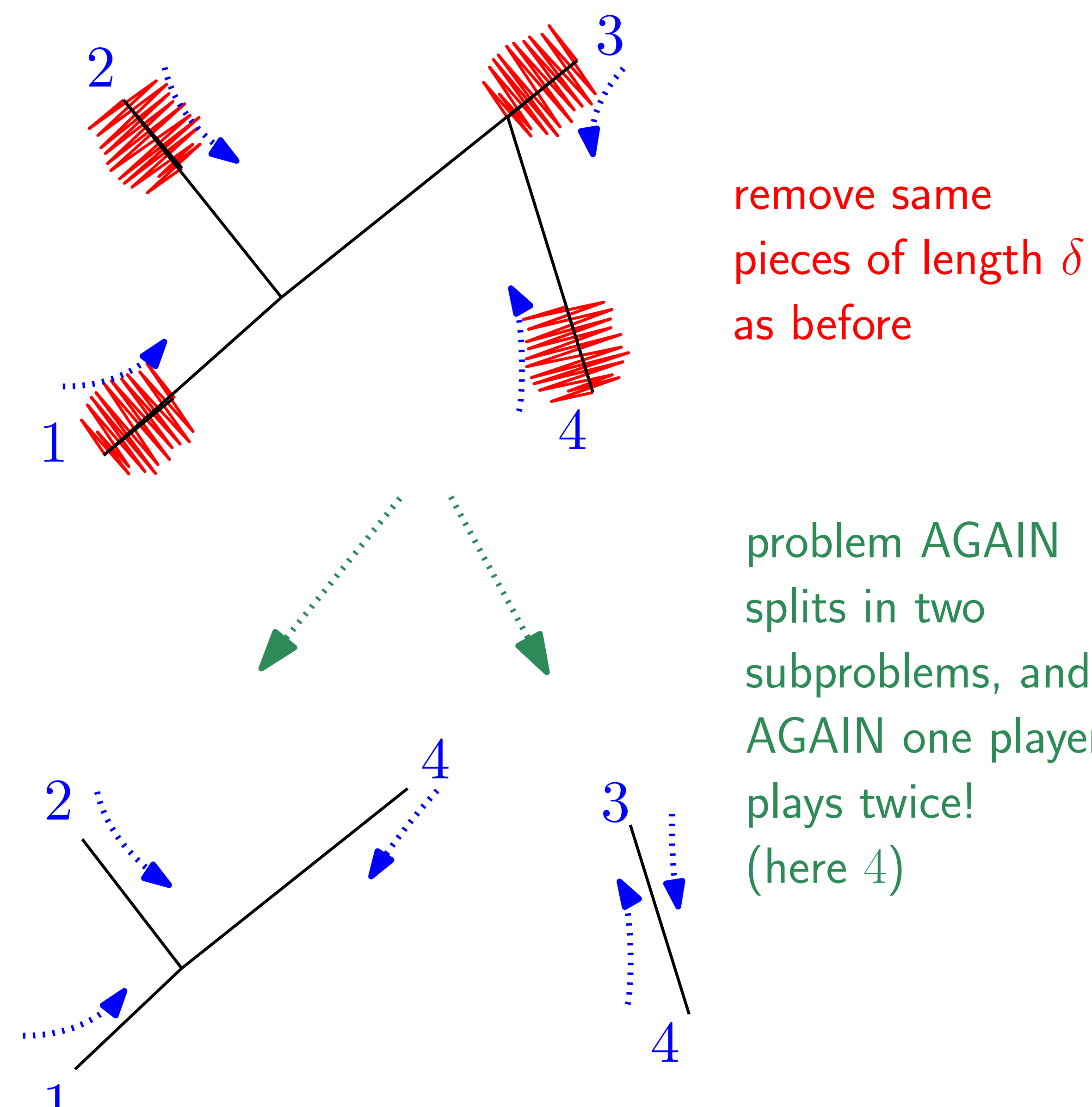
The proof for trees, continued $k \geq 2$

Take k players (here $k = 4$) and look at the Voronoi and Interval Games.

Voronoi Game



Interval Game



$\longrightarrow$ proof complete, by induction!

Conclusion

We only have a "proof from the book" that doesn't explain anything...
But the similarity with the main conjecture is puzzling.

WHY would a model of random graphs or random geometry would have uniform Voronoï tessellations?

THANK YOU