

What about maps in complex reflection groups?

Guillaume Chapuy (CNRS – Université Paris 7)

joint work

Christian Stump (Hannover)

Factorizations of a Coxeter element in complex reflection groups

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Part 1: the objects

Minimal factorizations of a full cycle – Cayley's formula

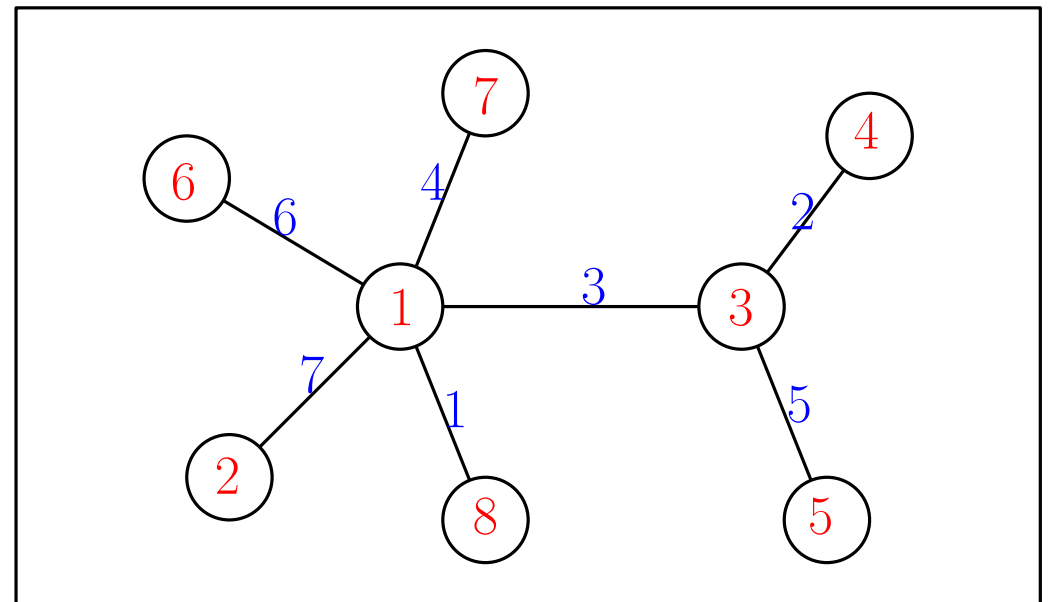
- In the symmetric group S_n we consider factorizations of the full cycle $(1, 2, \dots, n)$ into a product of $(n - 1)$ transpositions
- **Theorem [Cayley's formula]** The number of such factorizations is

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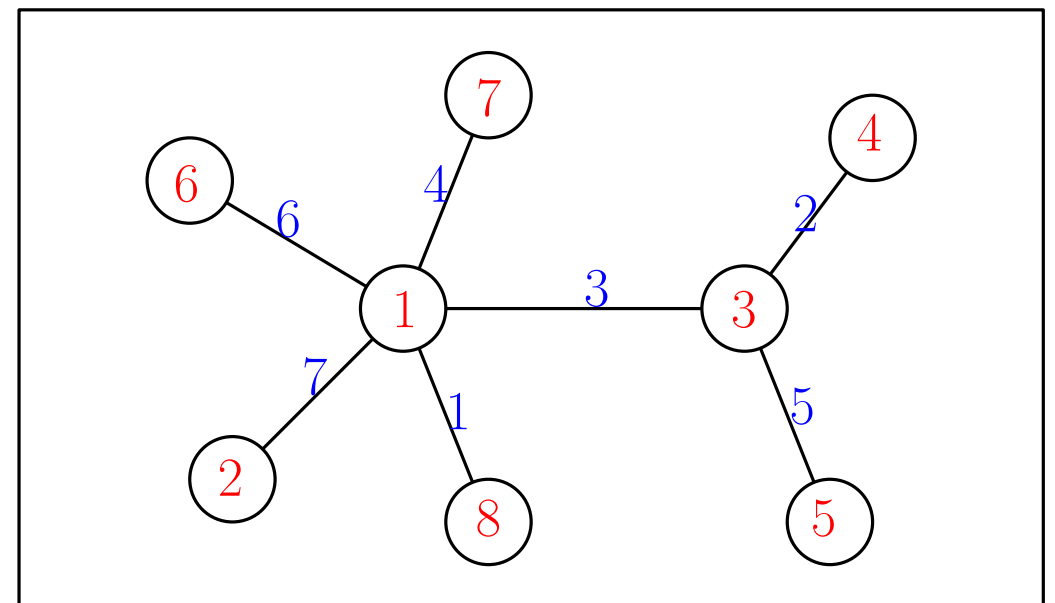
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 \\
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a factorization of an arbitrary full cycle



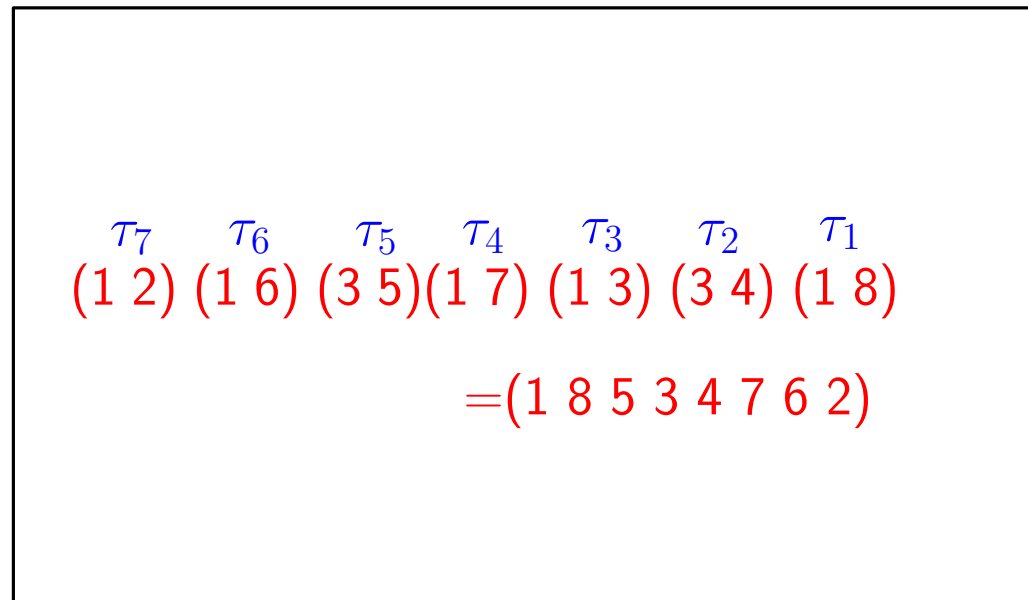
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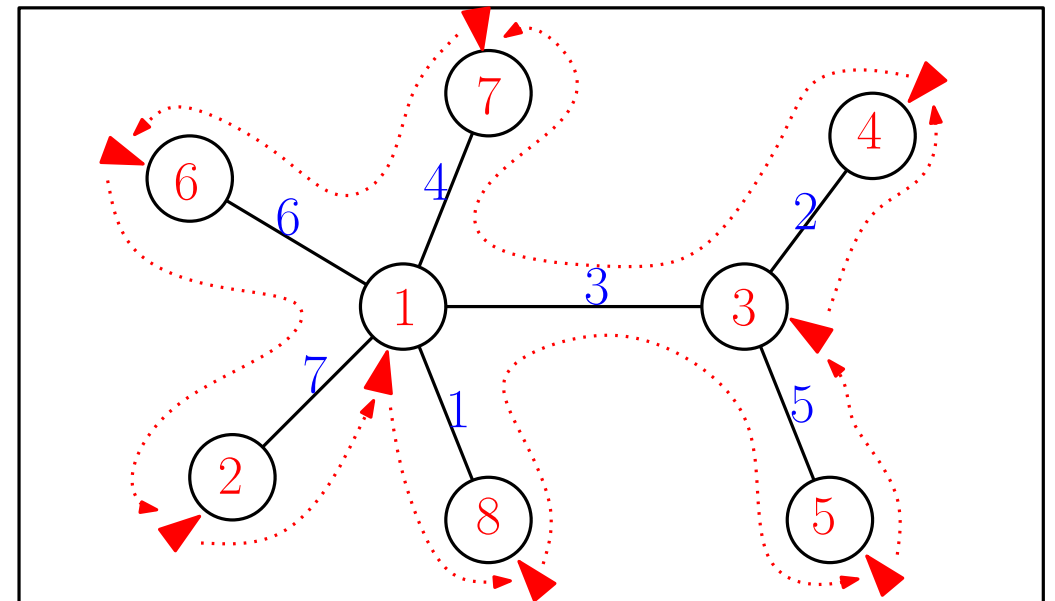
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- From a **topological viewpoint**, we are considering two restrictions:
 - **planar** (\sim factorizations of minimal length)
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$$\sim \frac{1}{n!} (tn)^{n-1} = \frac{t^{n-1}}{(n-1)!} n^{n-2}$$

→ at order 1, this is Cayley's formula.

Reflection groups (I)

- Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A **reflection** is an element $\tau \in \text{GL}(V)$ such that $\ker(\text{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \text{Diag}(1, 1, \dots, 1, \zeta)$ for ζ a root of unity.

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take an $n \times n$ permutation matrix
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product of all entries is an r/p -th root of unity.

Reflection groups (II)

- If $W \subset GL(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If W is irreducible and is generated by $\dim V$ reflections then it is well-generated.
- $S_n \subset GL(\mathbb{C}^n)$ is not irreducible since $V_0 = \{\sum_i x_i = 0\}$ is stable.
- $S_n \subset GL(V_0)$ is irreducible. It has rank $(n - 1)$. It is well-generated, take $s_i = (i \ i + 1)$ for $1 \leq i < n$.

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- If W is irreducible and well-generated there is a notion of Coxeter element that plays the same role as the full cycle for the symmetric group.

In general: it is an element having an eigenvalue ζ a primitive d -th root of unity with d as large as possible.

For real groups, it is the product (in any order) of the $(n - 1)$ generators.

The Coxeter number, h , is the order of the Coxeter element.

Deligne's formula

- **Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let W be an irreducible well-generated complex reflection group of rank n . Then the number of factorizations of a **Coxeter element** into a product of n reflections is

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- Translation for the symmetric group S_m .
 - cox. element = full cycle; its order $h = m$
 - reflection = transposition
 - rank $n = m - 1$
$$\rightarrow \frac{(m-1)!}{m!} m^{m-1} = m^{m-2} \quad \text{Cayley's formula!}$$

Our result – “higher genus” factorizations in w.g.c.r.g.

- **Theorem [C.-Stump]** Let W be an irreducible well-generated complex reflection group of rank n . Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\{\tau_1\tau_2 \dots \tau_\ell = c \text{ where } \tau_i \text{ are reflections}\}$$

Then the generating function is nice:

$$F(t) = \sum_{\ell \geq 0} \frac{t^\ell}{\ell!} h_\ell = \frac{1}{|W|} \left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t} \right)^n.$$

- Parameters: $\frac{h'}{2} = \frac{\#\text{reflections}}{n}$ and $\frac{h''}{2} = \frac{\#\text{reflection hyperplanes}}{n}$

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- For real groups $h' = h'' = h$ (e.g. Shapiro-Shapiro-Vainshtein for \mathbb{S}_m).

Part 2: group characters

Counting factorizations in groups (I)

- Let $\mathcal{R} = \{\text{reflections}\}$ and $c = \text{Coxeter element}$.

$$\text{Let } h_\ell = \#\{\tau_1\tau_2 \dots \tau_\ell = c \text{ where } \tau_i \in \mathcal{R}\}$$

- **Lemma** [the Frobenius formula] Let $\chi_\lambda, \lambda \in \Lambda$ be the list of all irreducible characters of W . Then one has:

$$h_\ell = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \left(\frac{\chi_\lambda(R)}{\dim \lambda} \right)^\ell \chi_\lambda(c^{-1}). \quad \text{where}$$
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- **Sketch of a proof:** Consider the group algebra $\mathbb{C}[W]$.

$$\text{Then } h_\ell = \text{coeff. of } \mathbf{1} \text{ in } \left(R^\ell c^{-1} \right) \text{ where } R = \sum_{\tau \in \mathcal{R}} \tau$$

$$= \frac{1}{|W|} \text{Tr} \left(R^\ell c^{-1} \right) \quad \text{since if } \sigma \in W, \text{ then } \text{Tr}_{\mathbb{C}[W]} \sigma = \begin{cases} |W| & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1 \end{cases}$$

Now use: - the (classical) decomposition of $\mathbb{C}[W]$ as $\mathbb{C}[W] = \bigoplus_{\lambda \in \Lambda} (\dim V^\lambda) V^\lambda$

- the fact that R is central and therefore acts as a scalar on each V^λ .

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W , our generating function is a **finite sum**:

$$F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t\right)$$

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- ask your computer to **factor** it... it works!

$$F_{E_8}(t) = \frac{1}{|E_8|} (e^{15t} - e^{-15t})^8.$$

Part 3: Classification ...and case-by-case proof

Classification and proof strategy

- **Theorem**[Sheppard, Todd, 54] Let W be an irreducible complex reflection group. Then W is (isomorphic to) either:
 - the symmetric group $S_n \subset GL(V_0)$
 - $G(r, p, n)$ for some integer $r \geq 2$, $p, n \geq 1$ with $p|r$.
 - one of 34 exceptional groups
- **Well-generated**: S_n , $G(r, 1, n)$ and $G(r, r, n)$ + 26 exceptional groups.

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MATHS !

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- We start from $F(t) = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_{\lambda}(c^{-1}) \exp\left(\frac{\chi_{\lambda}(R)}{\dim \lambda} \cdot t\right)$

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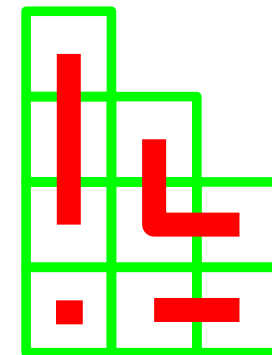
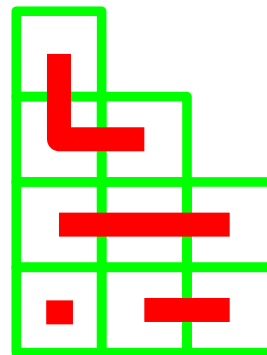
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Murnaghan-Nakayama rule

$\lambda = [3, 3, 2, 1]$ and $\sigma = (1, 3, 4)(2, 8, 9)(5, 7)(6)$

$$\chi_{\lambda}(\sigma) = (-1) + (-1) = (-2)$$



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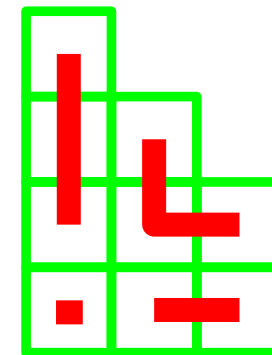
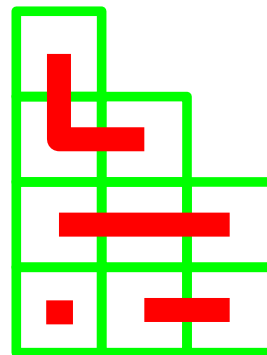
Here $\Lambda = \{\text{partitions of } n\}$ and $c^{-1} = \text{full cycle}$.

- **Crucial fact:** There are **very few** partitions λ such that $\chi_{\lambda}(c^{-1}) \neq 0$

Murnaghan-Nakayama rule

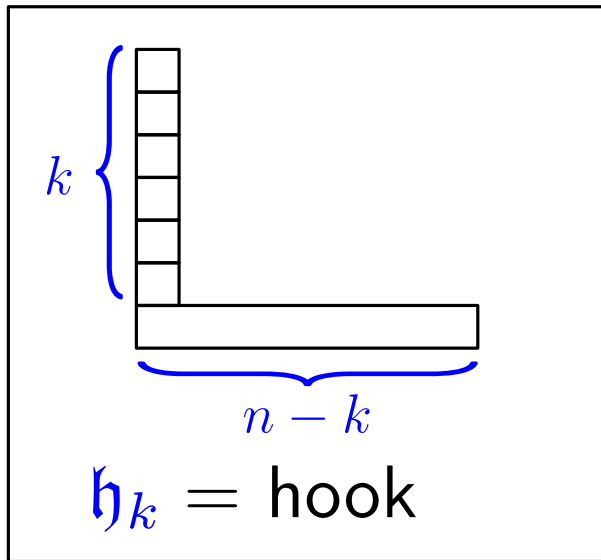
$\lambda = [3, 3, 2, 1]$ and $\sigma = (1, 3, 4)(2, 8, 9)(5, 7)(6)$

$$\chi_{\lambda}(\sigma) = (-1) + (-1) = (-2)$$



Example of S_n (what is so special about the Coxeter element ?) – 2

• We have $F(t) = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_{\lambda}(c^{-1}) \exp\left(\frac{\chi_{\lambda}(R)}{\dim \lambda} \cdot t\right)$

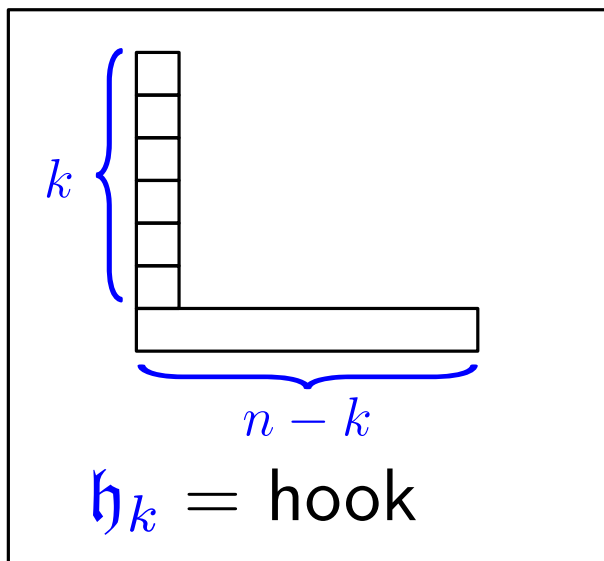


$= \frac{1}{|W|} \sum_{k=0}^{n-1} (\dim \mathfrak{h}_k) \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim \mathfrak{h}_k} \cdot t\right)$

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BIG sum



SMALL sum

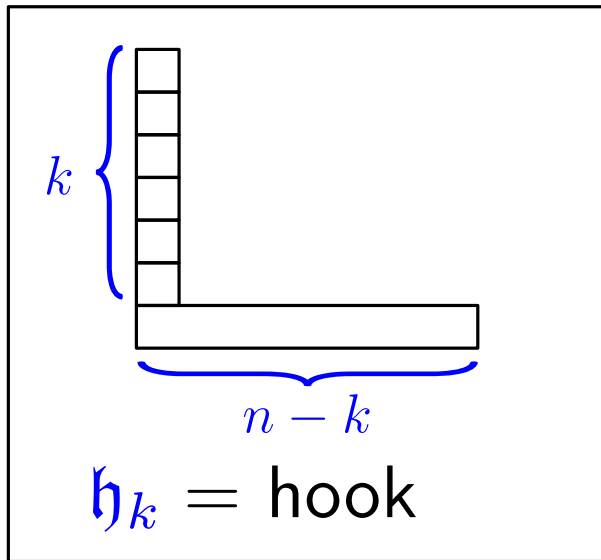
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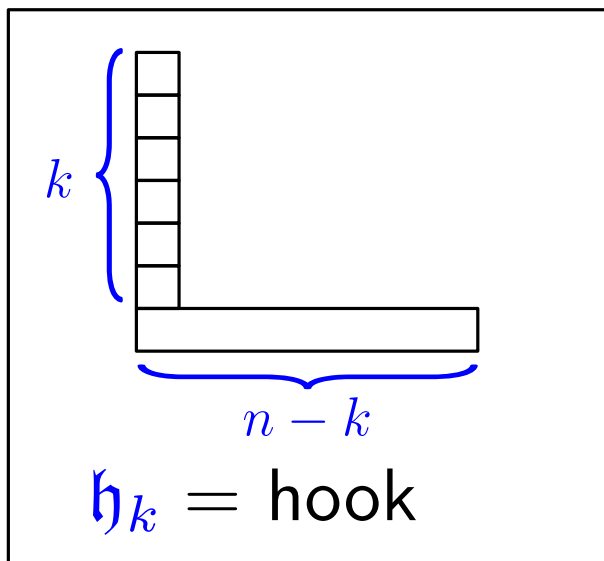
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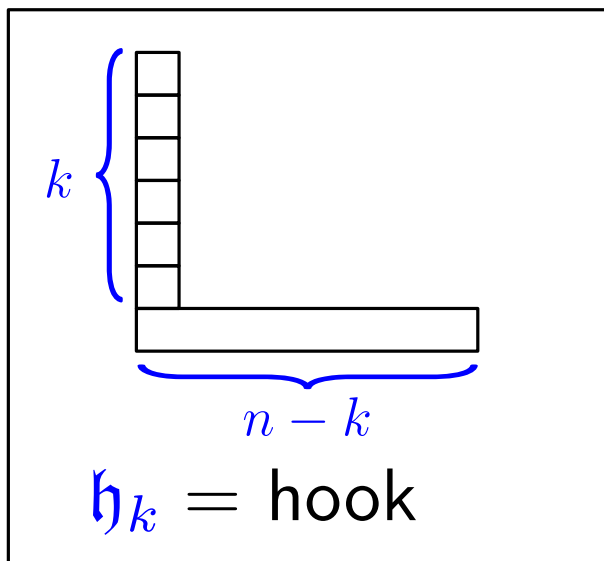
Annotations for the second equation:

- SMALL sum (pointing to the index k)
- # S.Y.T. (pointing to $\dim h_k$)
- Murnaghan Nakayama. (pointing to $\chi_{h_k}(c^{-1})$)
- Use combinatorial rules (e.g. Jucys Murphy or Murnaghan-Nakayama) (pointing to the fraction $\frac{\chi_{h_k}(R)}{\dim h_k}$)

$= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \exp\left(\frac{n(n-2k-2)}{2} \cdot t\right)$

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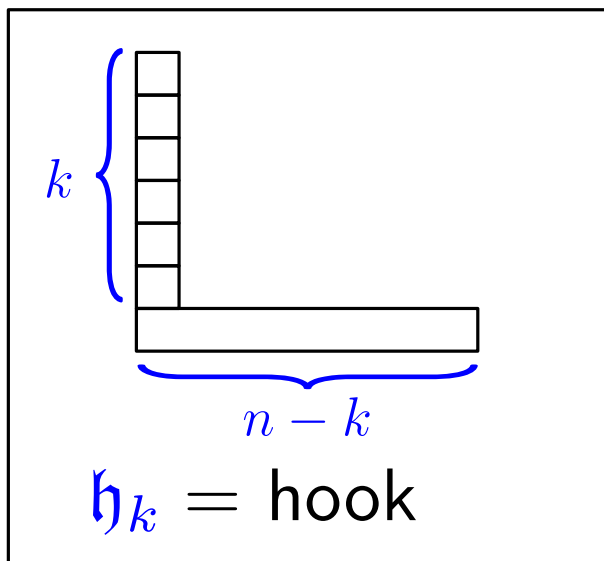
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DONE!

Other infinite families – $G(r, 1, n)$ and $G(r, r, n)$

- We need some combinatorial representation theory for these groups
- $G(r, 1, n) \rightarrow$ standard [MacDonald, Serre...]

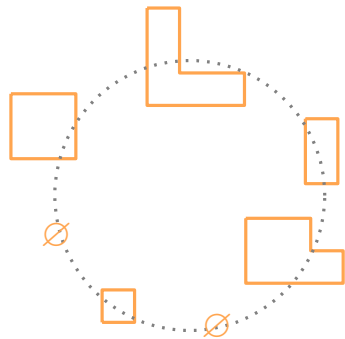
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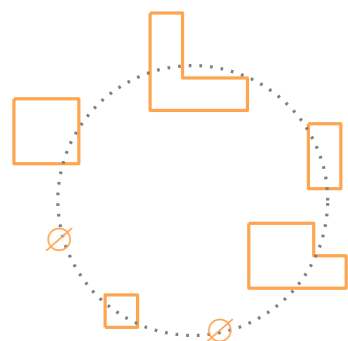
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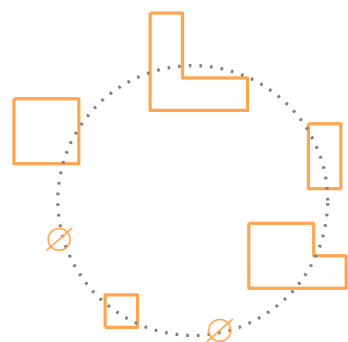
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- **Conclusion:** The formulas are nice but we don’t UNDERSTAND them!

Bimodal conclusion

- Map viewpoint:

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- there exist **many formulas** in map enumeration, that correspond to different **factorization problems in S_n** . Which ones can be generalized to reflection groups ?
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Thank you !