What about maps in complex reflection groups?

Guillaume Chapuy (CNRS – Université Paris 7)

joint work
Christian Stump (Hannover)

Journées Cartes, June 2013.

Factorizations of a Coxeter element in complex reflection groups

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Part 1: the objects

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

- **Theorem** [Cayley's formula] The number of such factorizations is

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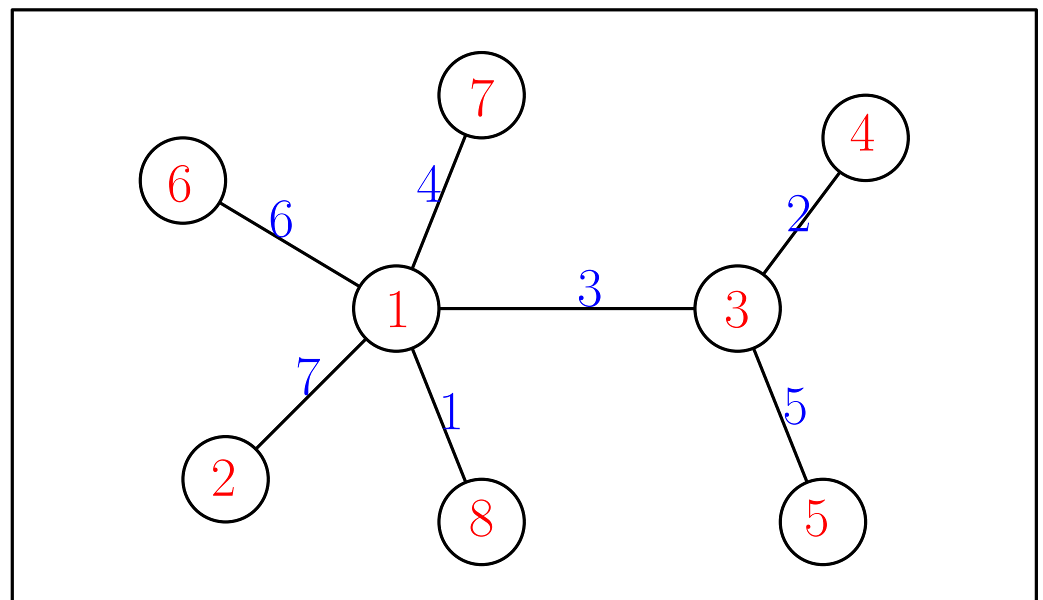

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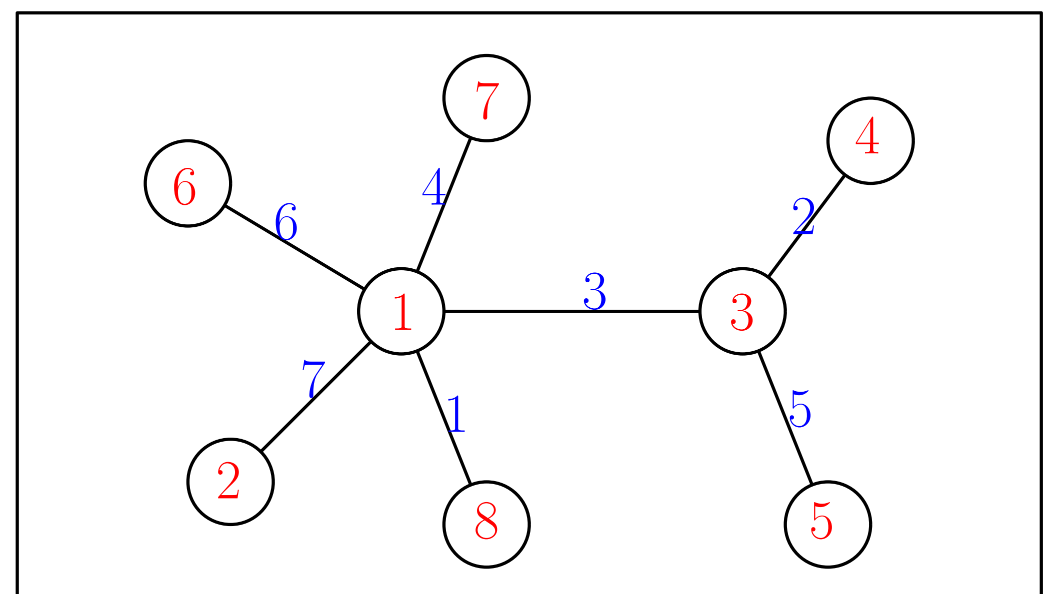
a factorization of an arbitrary full cycle



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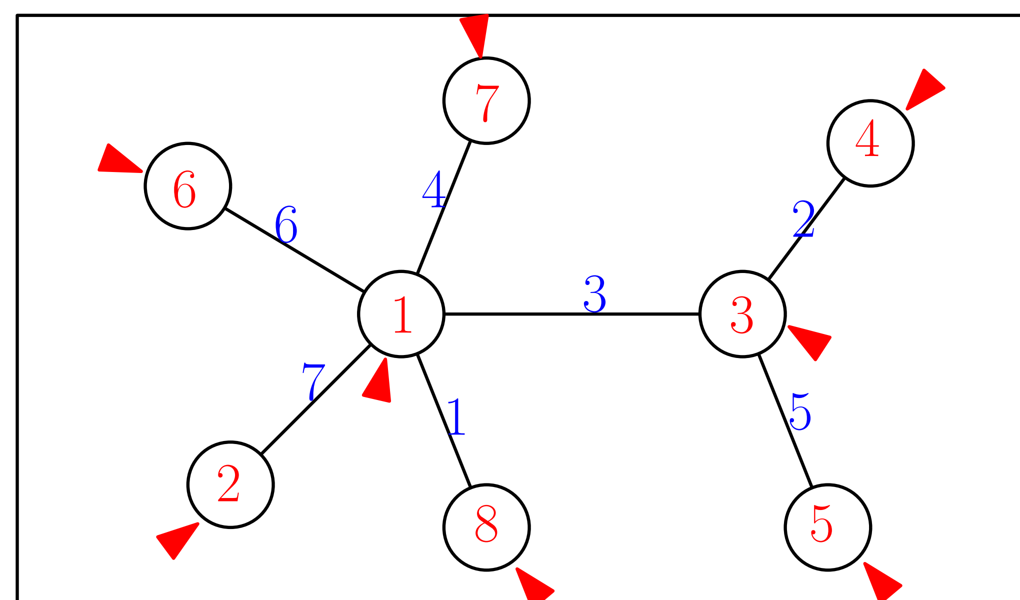

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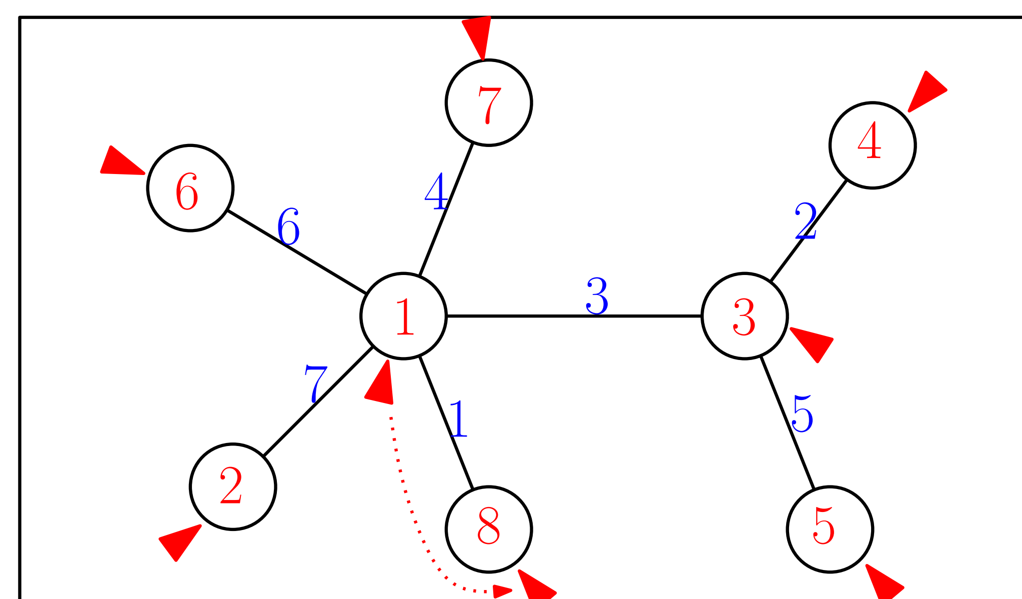

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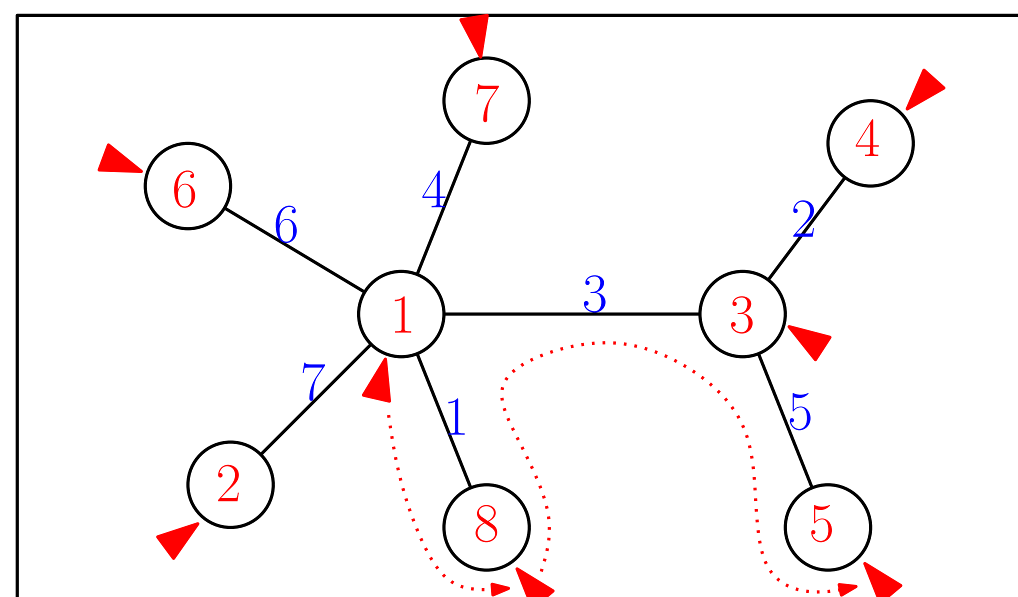

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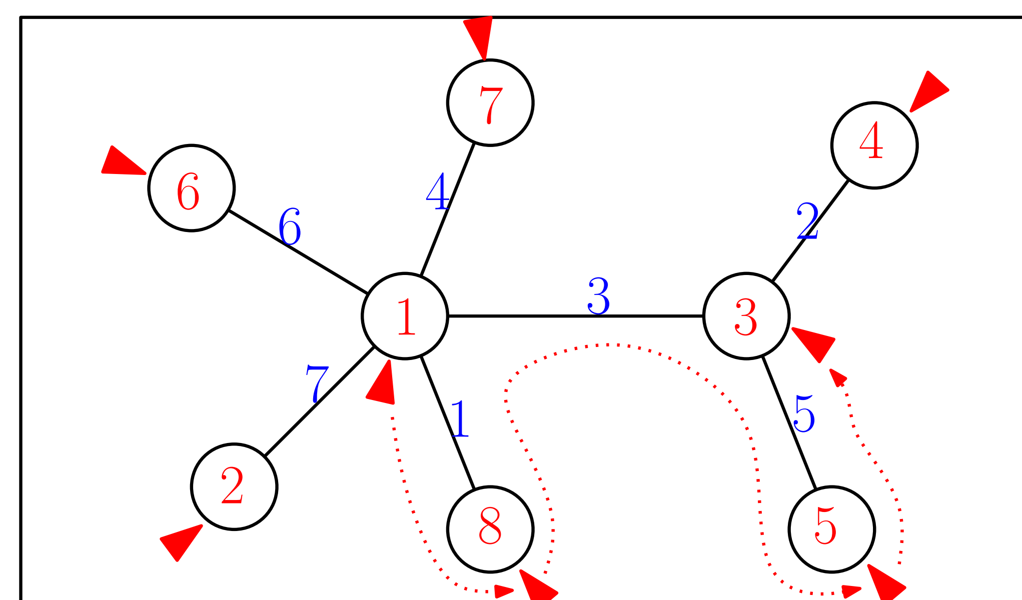

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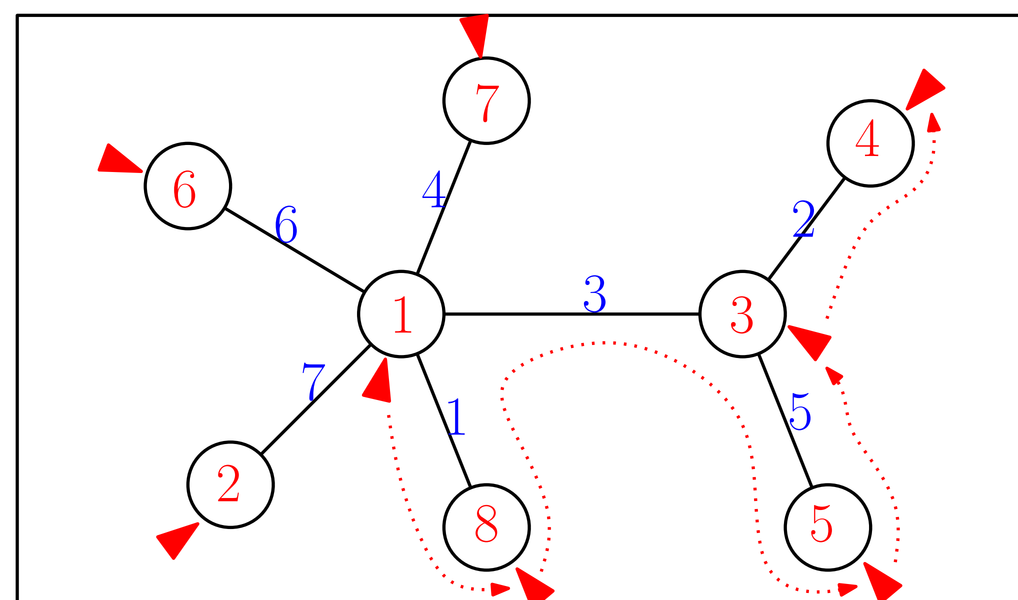

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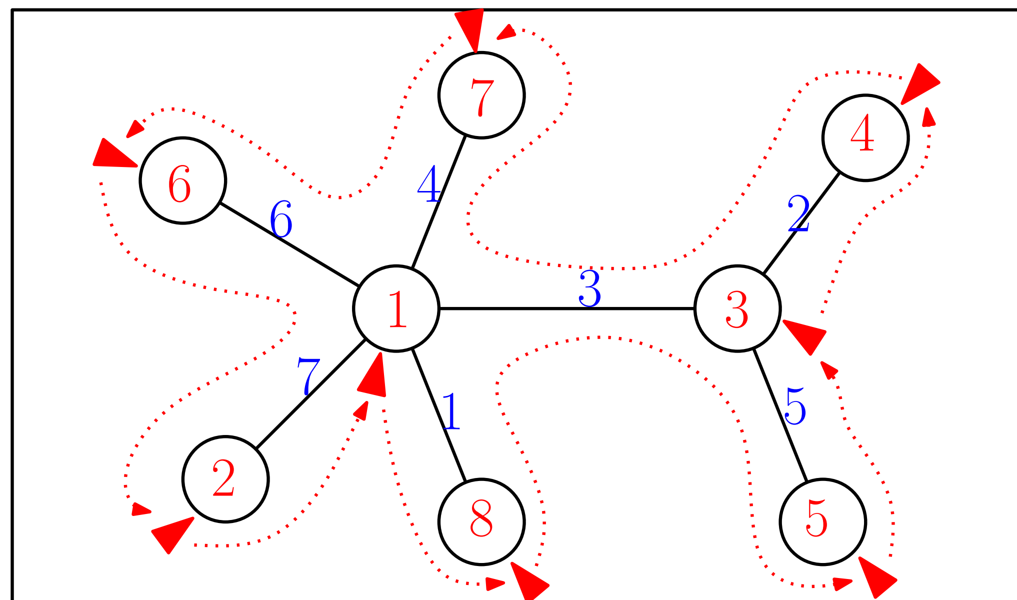

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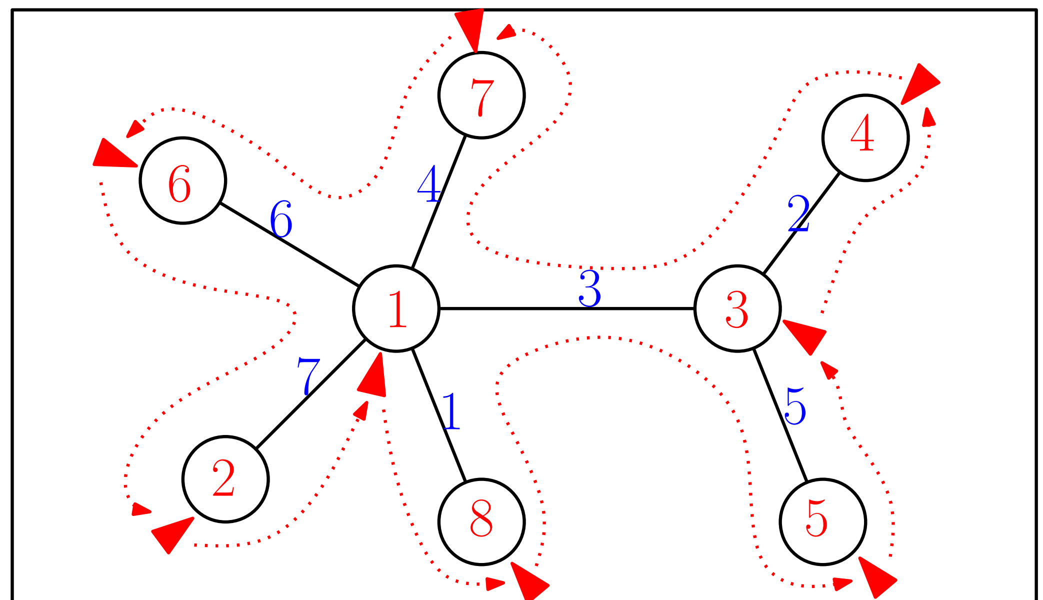

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$$h_{n,g} = \#\{\tau_1\tau_2\ldots\tau_{n-1+2g} = (1,2,\ldots,n)\} = ?$$

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$$F(t) = \sum_{g\geq 0} \frac{t^{n-1+2g}}{(n-1+2g)!} h_{n,g} = \frac{1}{n!}\left(e^{\frac{nt}{2}} - e^{-\frac{nt}{2}}\right)^{n-1}.$$

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$$\sim \frac{1}{n!}(tn)^{n-1} = \frac{t^{n-1}}{(n-1)!} n^{n-2}$$

→ at order 1, this is Cayley's formula.

Reflection groups (I)

• Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

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take an $n \times n$ permutation matrix
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product of all entries is an r/p-th root of unity.

Reflection groups (II)

- If $W \subset \mathrm{GL}(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If W is irreducible and is generated by $\dim V$ reflections then it is well-generated.

- $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ is not irreducible since $V_0 = \{\sum_i x_i = 0\}$ is stable.

- $\mathbb{S}_n \subset \mathrm{GL}(V_0)$ is irreducible. It has rank $(n-1)$. It is well-generated, take $s_i = (i\ i+1)$ for $1 \leq i < n$.

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- If W is irreducible and well-generated there is a notion of Coxeter element that plays the same role as the full cycle for the symmetric group.

In general: it is an element having an eigenvalue ζ a primitive d-th root of unity with d as large as possible.

For real groups, it is the product (in any order) of the $(n-1)$ generators.

The Coxeter number, h, is the order of the Coxeter element.

Deligne's formula

- **Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let W be an irreducible well-generated complex reflection group of rank n. Then the number of factorizations of a Coxeter element into a product of n reflections is

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- Translation for the symmetric group $\mathbb{S}_m$.

  - cox. element = full cycle; its order $h = m$
  - reflection = transposition
  - rank $n = m - 1$

$\rightarrow \frac{(m-1)!}{m!}m^{m-1} = m^{m-2}$ Cayley's formula!

Our result – "higher genus" factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let W be an irreducible well-generated complex reflection group of rank n. Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\big\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \text{ are reflections}\big\}$$

Then the generating function is nice:

$$F(t) = \sum_{\ell\geq 0}\frac{t^\ell}{\ell!}h_\ell = \frac{1}{|W|}\left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t}\right)^n.$$

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- For real groups $h' = h'' = h$ (e.g. Shapiro-Shapiro-Vainshtein for $\mathbb{S}_m$).

Part 2: group characters

Counting factorizations in groups (I)

- Let $\mathcal{R} = \{\text{reflections}\}$ and $c =$ Coxeter element.
  Let $h_\ell = \#\{\tau_1 \tau_2 \ldots \tau_\ell = c \text{ where } \tau_i \in \mathcal{R}\}$

- **Lemma** [the Frobenius formula] Let $\chi_\lambda, \lambda \in \Lambda$ be the list of all irreducible characters of W. Then one has:

$$h_\ell = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \left(\frac{\chi_\lambda(R)}{\dim \lambda} \right)^\ell \chi_\lambda(c^{-1}).$$

where $\chi_\lambda(R) := \sum_{\tau \in \mathcal{R}} \chi_\lambda(\tau)$.

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- **Sketch of a proof:** Consider the group algebra $\mathbb{C}[W]$.

Then $h_\ell =$ coeff. of 1 in $\left(R^\ell c^{-1}\right)$ where $R = \sum_{\tau\in\mathcal{R}}\tau$

$$= \frac{1}{|W|}\mathrm{Tr}\left(R^\ell c^{-1}\right)$$

since if $\sigma \in W$, then $\mathrm{Tr}_{\mathbb{C}[W]}\sigma = \begin{cases} |W| \text{ if } \sigma = 1 \\ 0 \text{ if } \sigma \neq 1 \end{cases}$

Now use: - the (classical) decomposition of $\mathbb{C}[W]$ as $C[W] = \bigoplus_{\lambda\in\Lambda}(\dim V^\lambda)V^\lambda$

- the fact that R is central and therefore acts as a scalar on each V^λ.

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Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

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- ask your computer to factor it... it works!

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Part 3: Classification
...and case-by-case proof

Classification and proof strategy

- **Theorem**[Sheppard, Todd, 54] Let W be an irreducible complex reflection group. Then W is (isomorphic to) either:

  - the symmetric group $\mathbb{S}_n \subset \mathrm{GL}(V_0)$
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  - one of 34 exceptional groups

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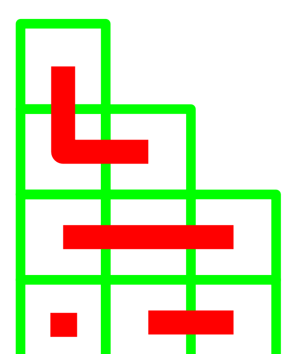
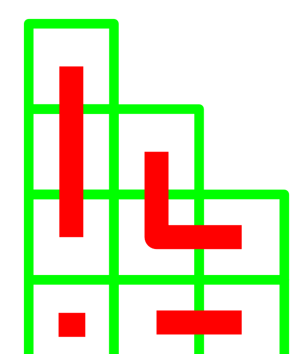
$\lambda = [3,3,2,1]$ and $\sigma = (1,3,4)(2,8,9)(5,7)(6)$

$$\chi_\lambda(\sigma) = \quad (-1) \quad + \quad (-1) \quad = (-2)$$

ribbon strip





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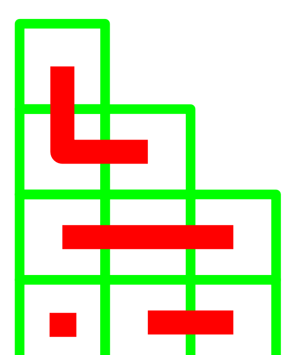

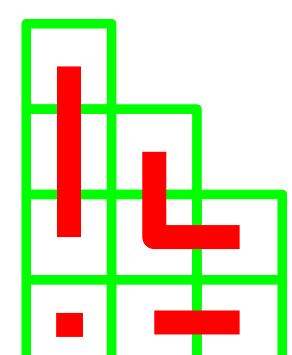

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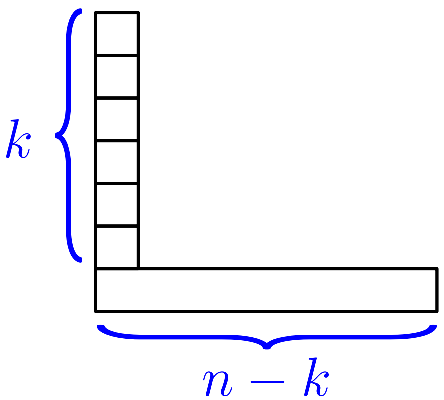

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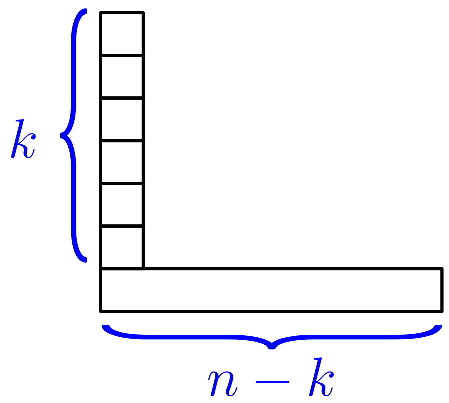

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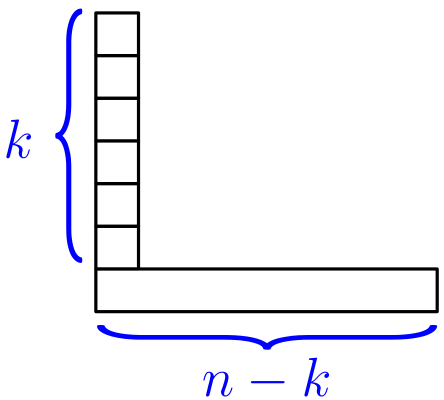

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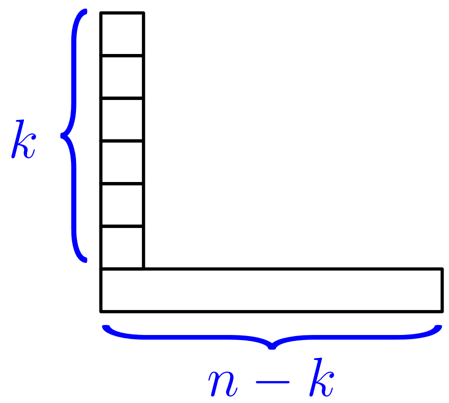

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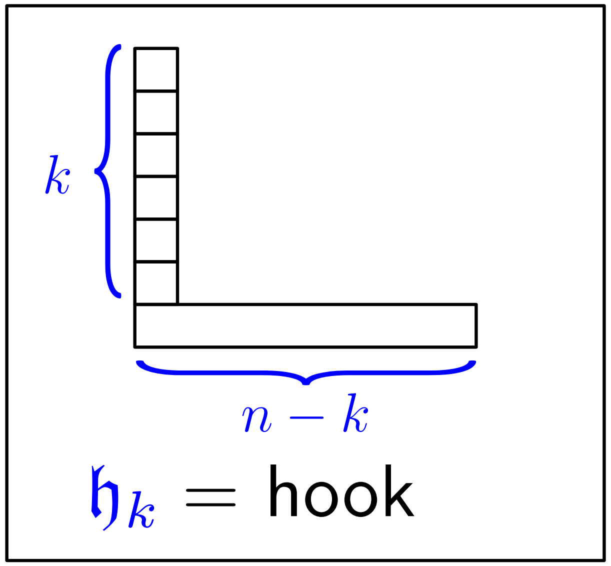

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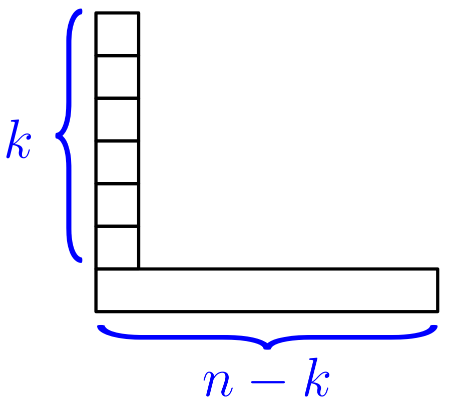

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- We need some combinatorial representation theory for these groups

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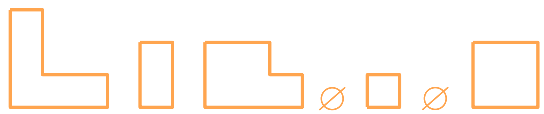

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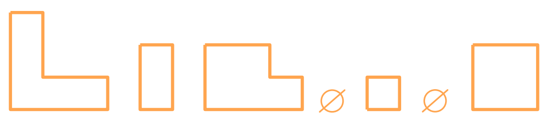

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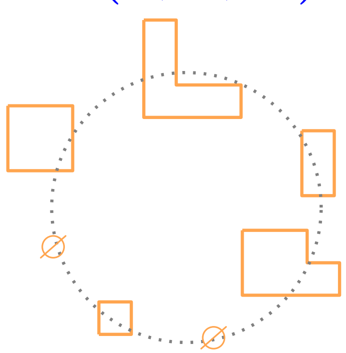

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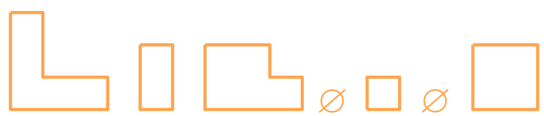

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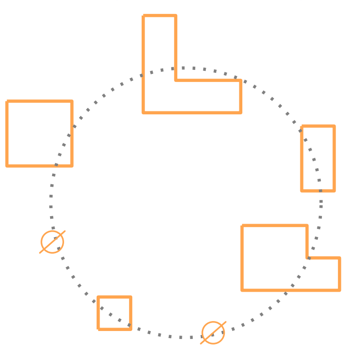

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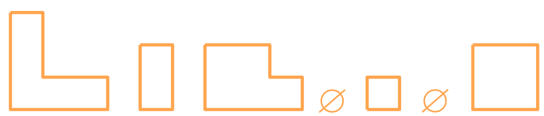

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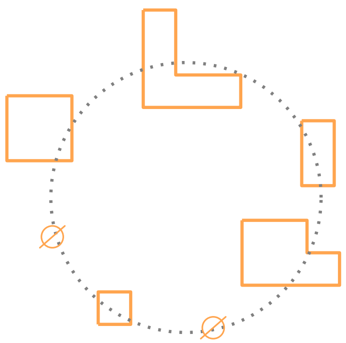

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