On the diameter of random planar graphs

Guillaume Chapuy, CNRS & LIAFA, Paris

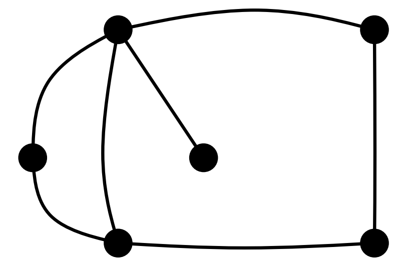
joint work with

Éric Fusy, Paris,
Omer Giménez, ex-Barcelona,
Marc Noy, Barcelona.

Probability and Graphs, Eurandom, Eindhoven, 2014.

Planar graphs and maps

- Planar graph = (connected) graph on $V = \{1, 2, \ldots, n\}$ that can be drawn in the plane without edge crossing.



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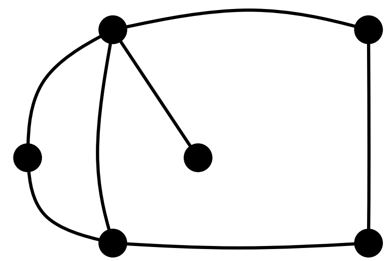
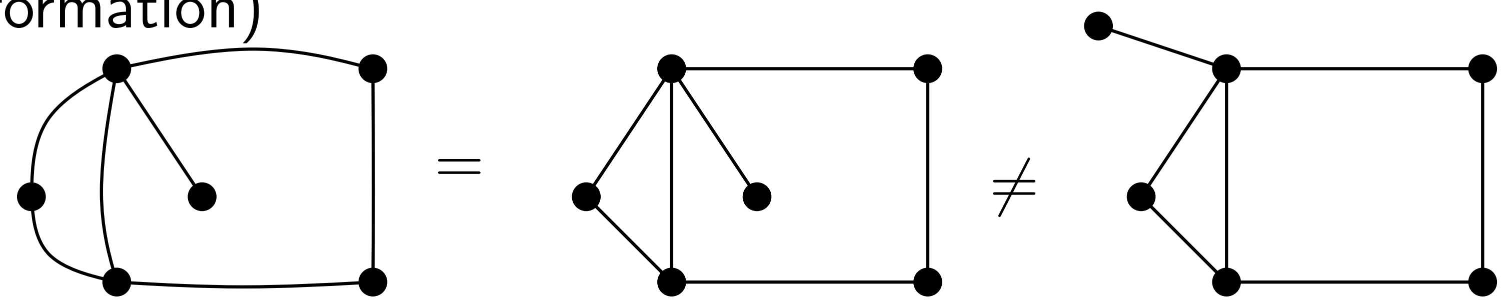


- Planar map = planar graph + planar drawing of this graph (up to continuous deformation)



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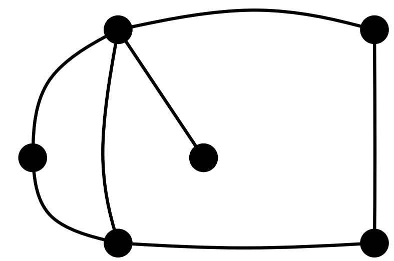

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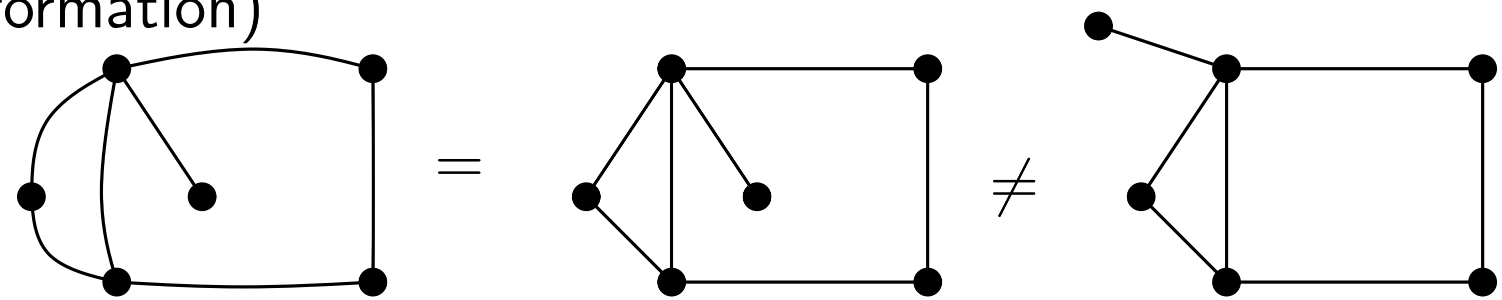

same graph
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- Note: the number of embeddings depends on the graph...

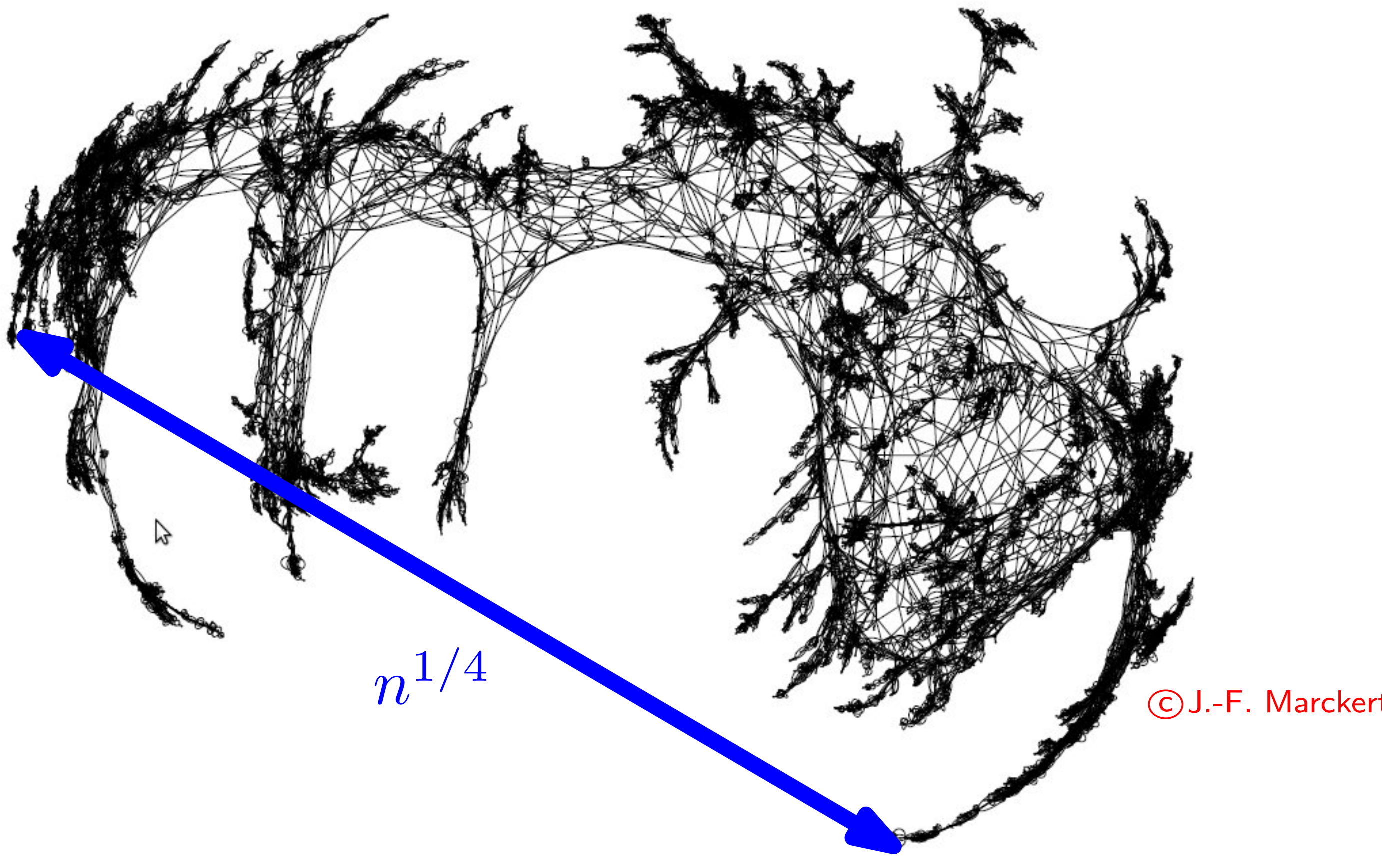
Uniform random planar map $\neq$ Uniform random planar graph!

Some known results for maps (stated approximately)

- **Thm** [Chassaing-Schaeffer '04], [Marckert, Miermont '06], [Ambjörn-Budd '13]

  In a uniform random map M_n of size n, distances are of order $n^{1/4}$.

  For example one has $\dfrac{\mathrm{Diam}(M_n)}{n^{1/4}} \to$ some real random variable



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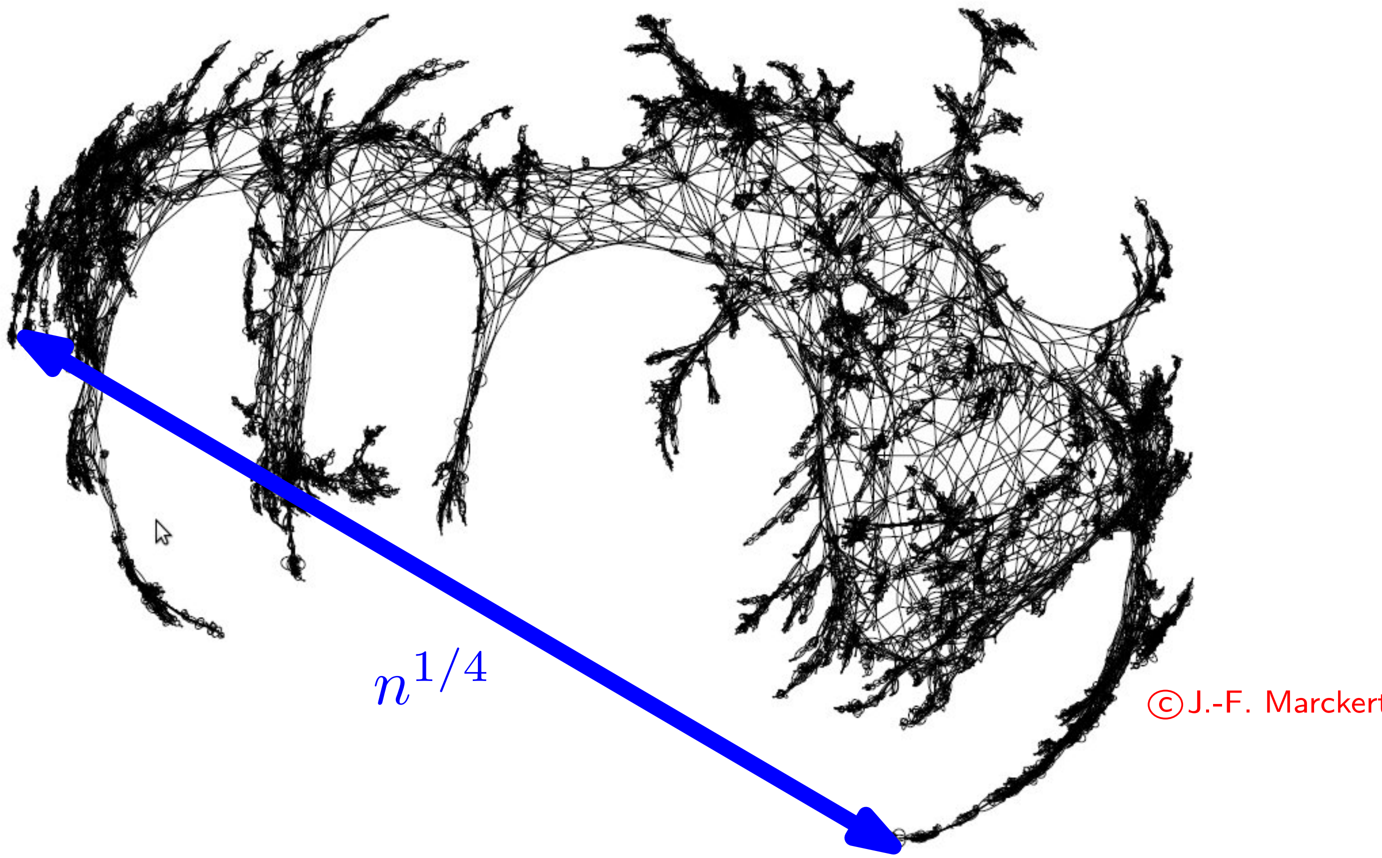

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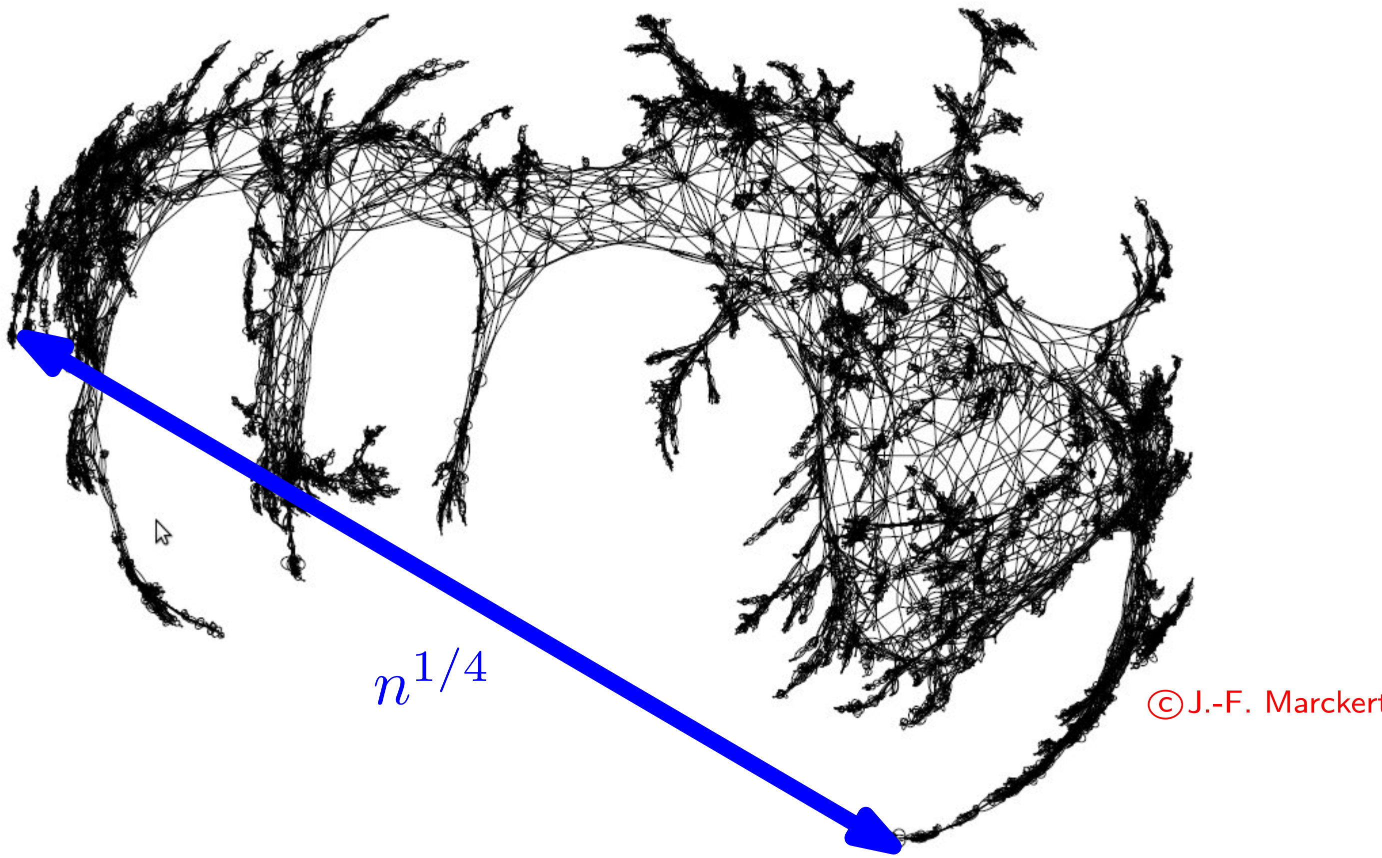

A lot of (very strong) things are known – very active field of research since 2004 [Bouttier, Di Francesco, Guitter, Le Gall, Miermont, Paulin, Addario-Berry, Albenque...]

Our main result: diameter of random planar GRAPHS

- **Thm** [C, Fusy, Giménez, Noy 2010+]

  Let G_n be the uniform random planar graph with n vertices.

  $$\text{Then } \mathrm{Diam}(G_n) = n^{1/4+o(1)} \text{ w.h.p.}$$

  More precisely $\mathbb{P}\Big(\mathrm{Diam}(G_n) \notin \big[n^{1/4-\epsilon}, n^{1/4+\epsilon}\big]\Big) = O(e^{-n^{\Theta(\epsilon)}})$.

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- This is some kind of large deviation result. We also conjecture convergence in law:

$$\frac{\mathrm{Diam}(G_n)}{n^{1/4}} \to \text{some real random variable}$$

- Note: for random trees,

$$\frac{\mathrm{Diam}(T_n)}{n^{1/2}} \to \text{some real random variable}$$

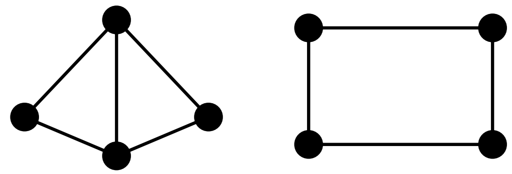
$$\mathbb{P}\Big(\mathrm{Diam}(T_n) \notin \big[n^{1/2-\epsilon}, n^{1/2+\epsilon}\big]\Big) = O(e^{-n^{\Theta(\epsilon)}})$$

[Flajolet et al '93]

(0) Connectivity in graphs

A graph is k-connected if one needs to remove at least k vertices to disconnect it.

General



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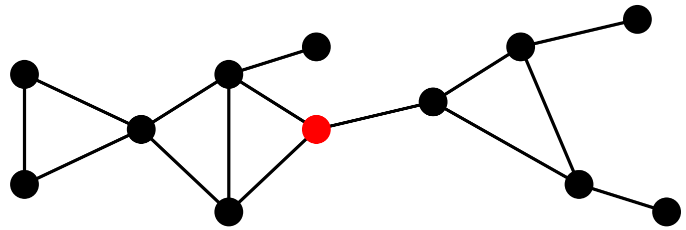

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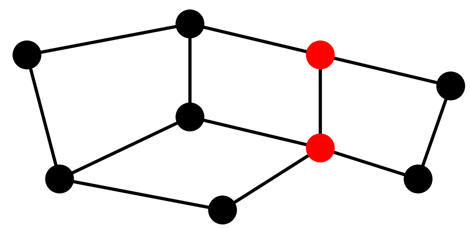

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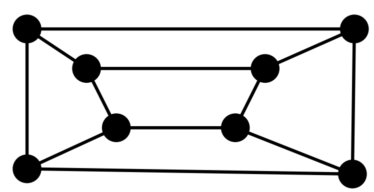

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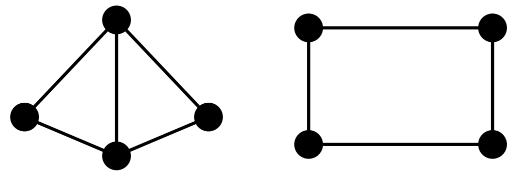

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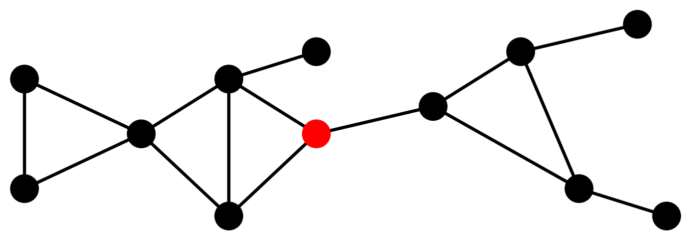

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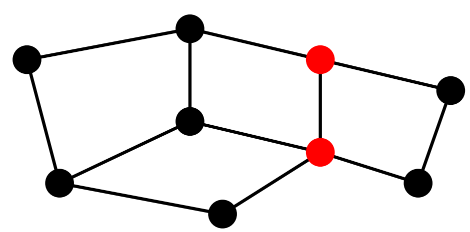

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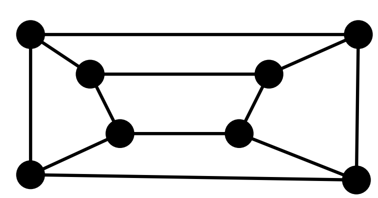

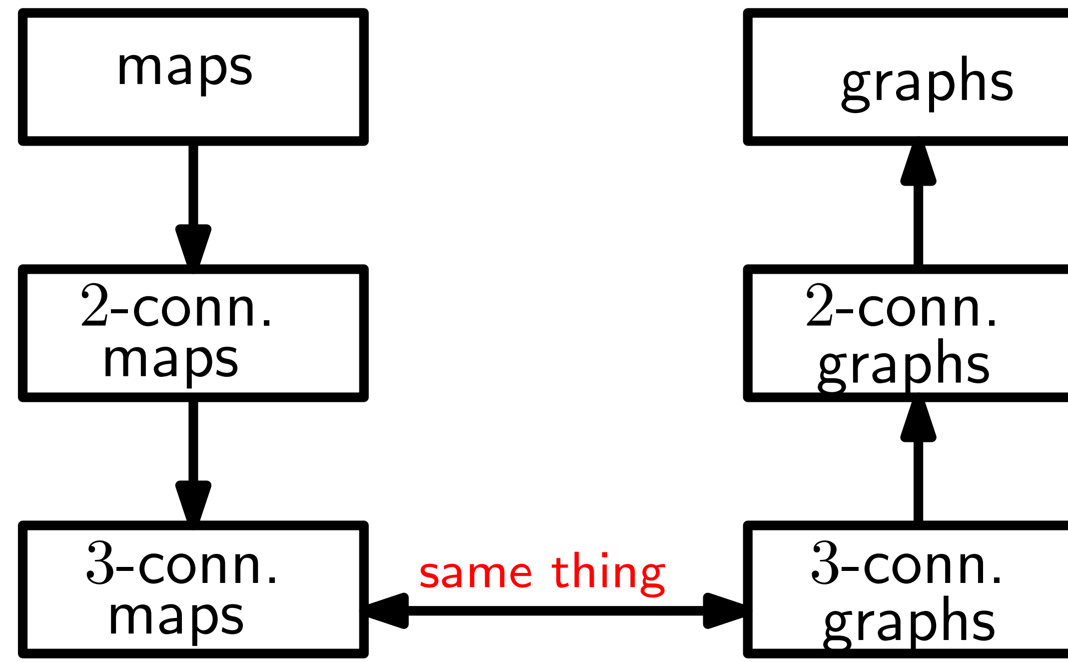

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- a 2-connected graph decomposes into 3-connected components

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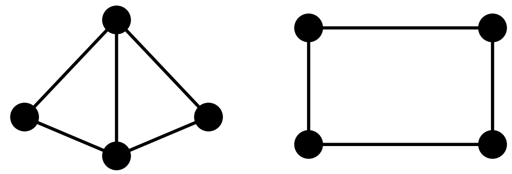

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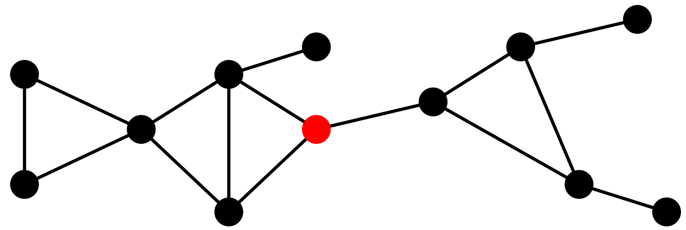

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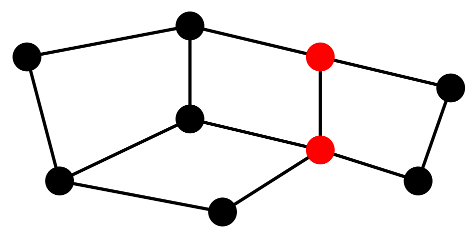

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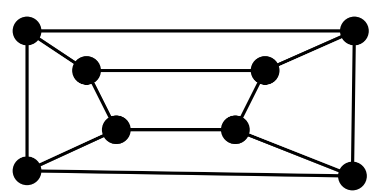

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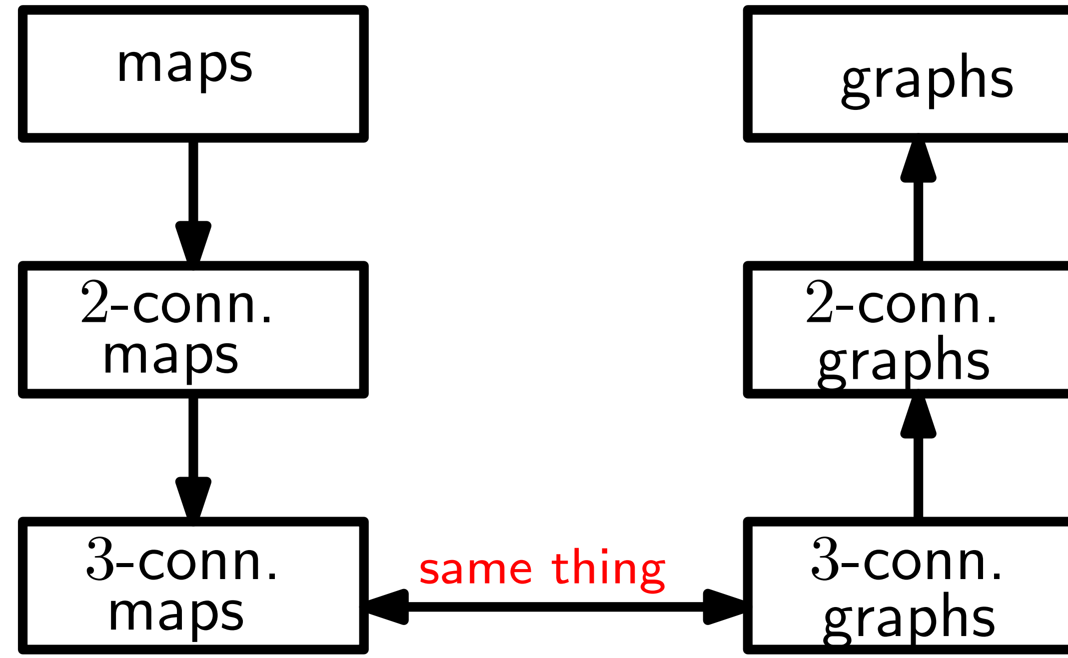

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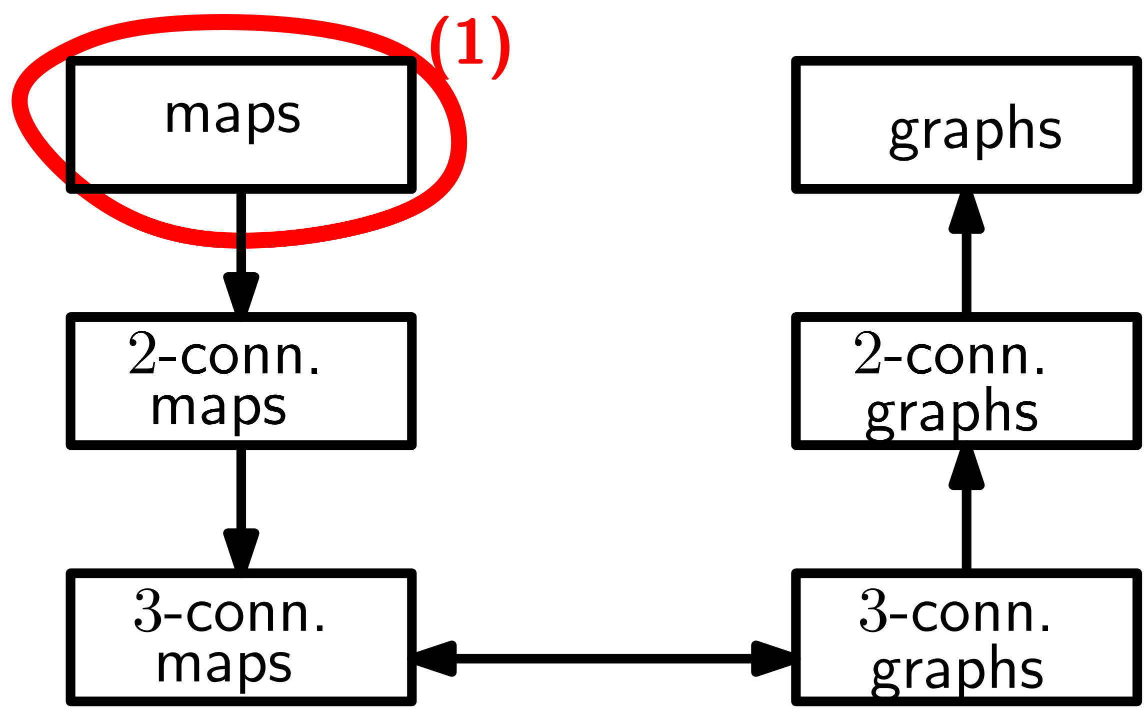
[Tutte 60s], [Bender,Gao,Wormald'02], [Giménez, Noy'05] followed this path carrying counting results along the scheme $\rightarrow$ exact counting of planar graphs!
Here we follow the same path and carry deviations statements for the diameter.


(1)
maps
graphs
2-conn.
maps
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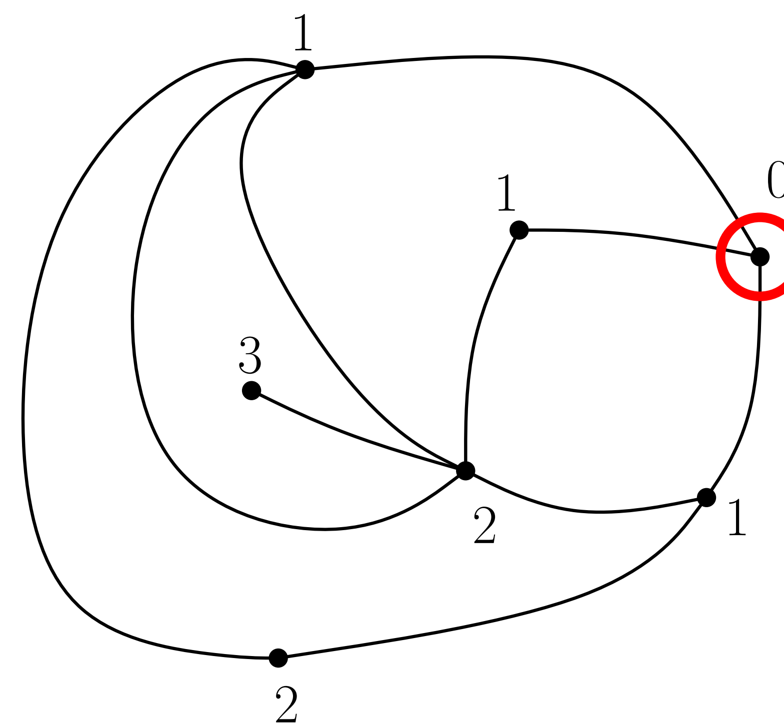
(1) Maps: the Cori-Vauquelin-Schaeffer bijection (1981-1999-2008+)

- To simplify the exposition we consider a quadrangular planar map (faces have degree 4)

  1. Label vertices by their graph-distance to some root vertex



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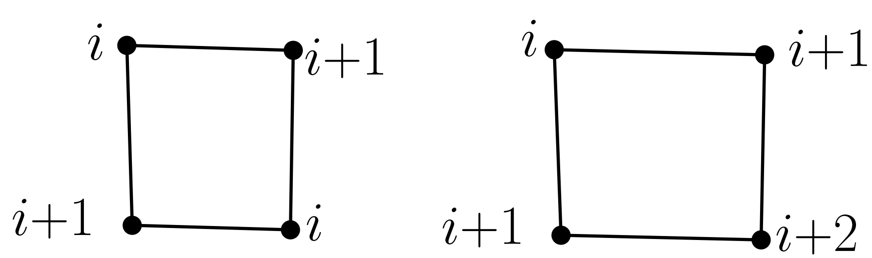

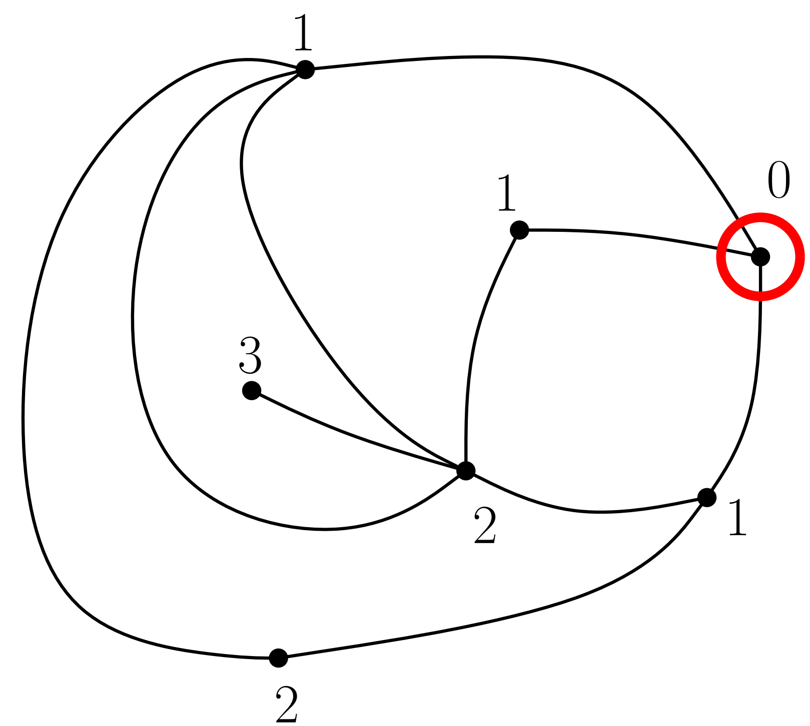

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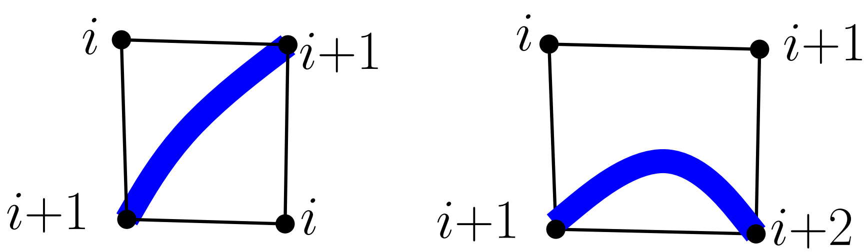

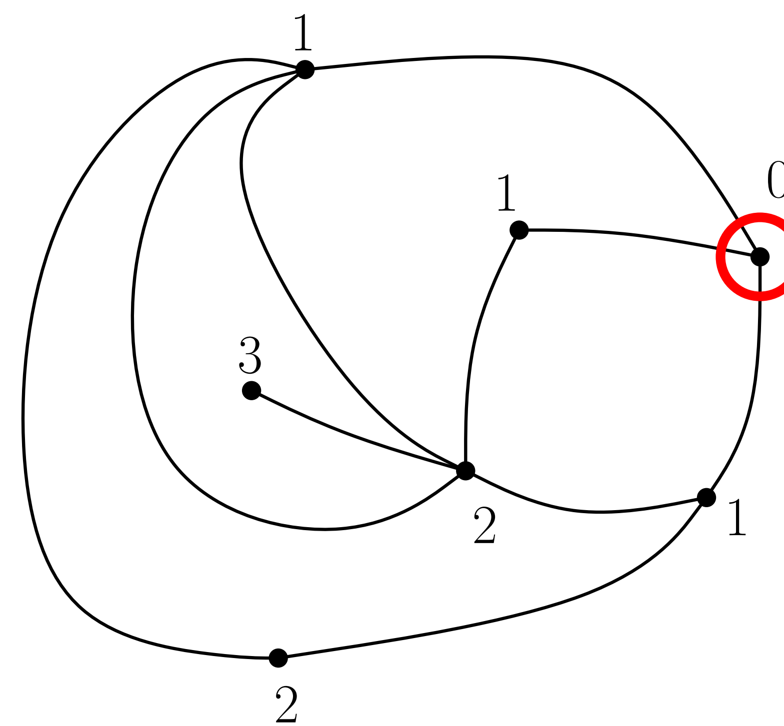

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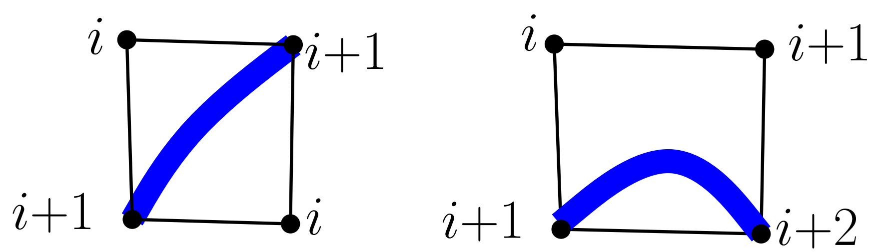

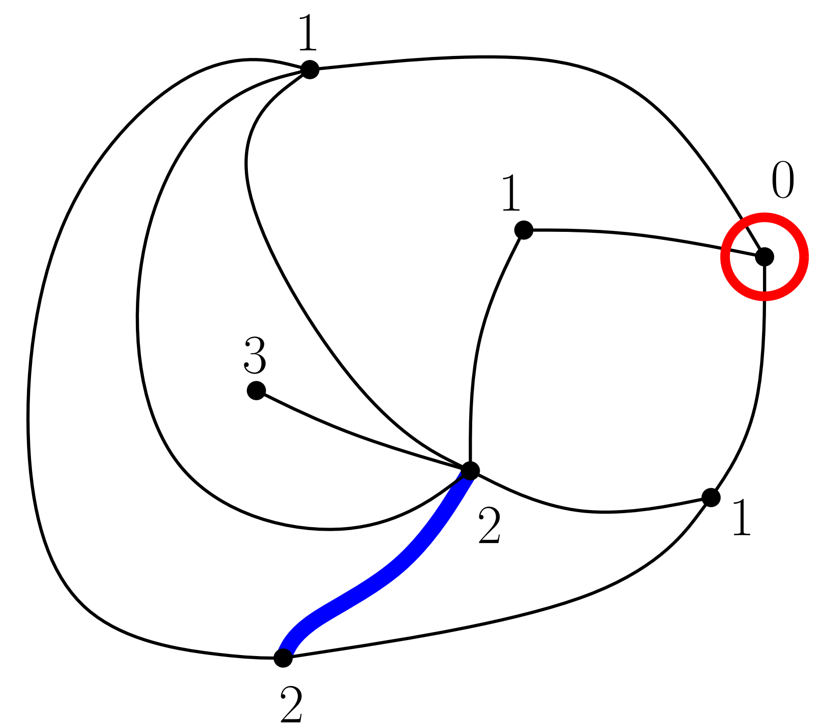

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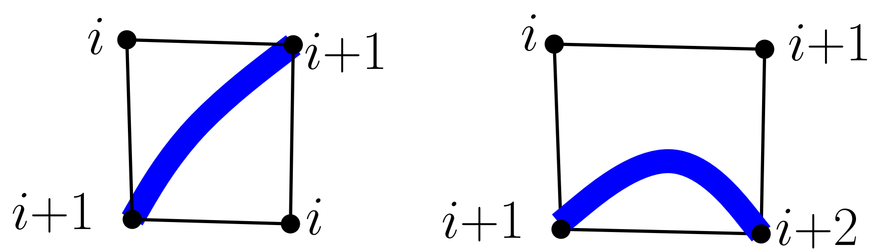

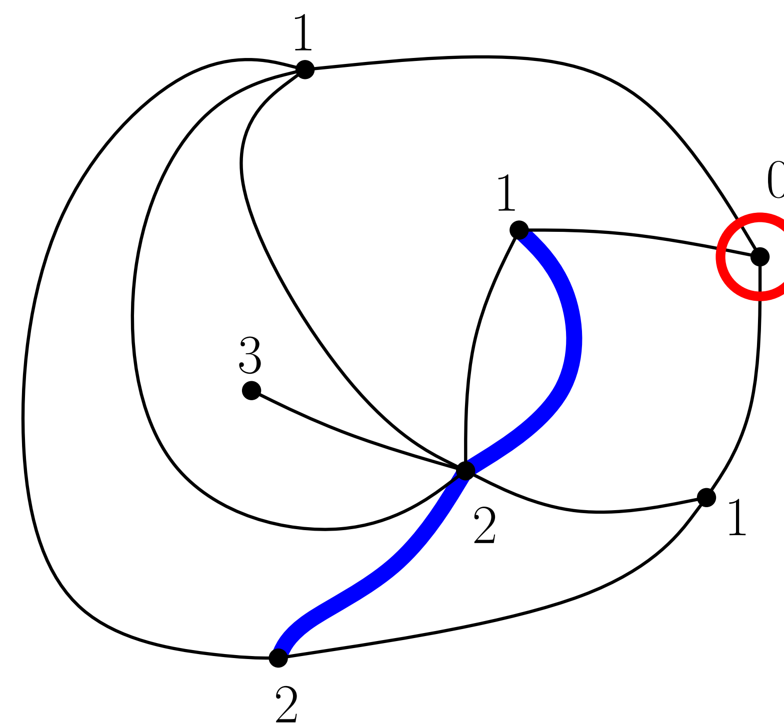

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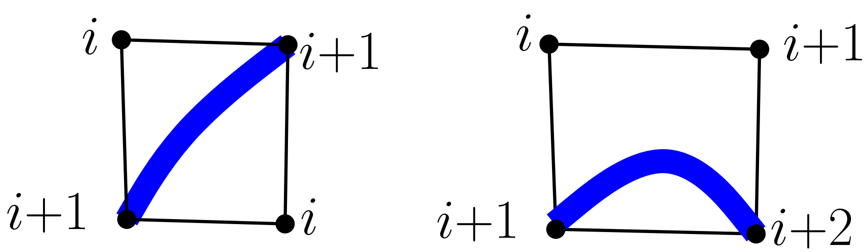

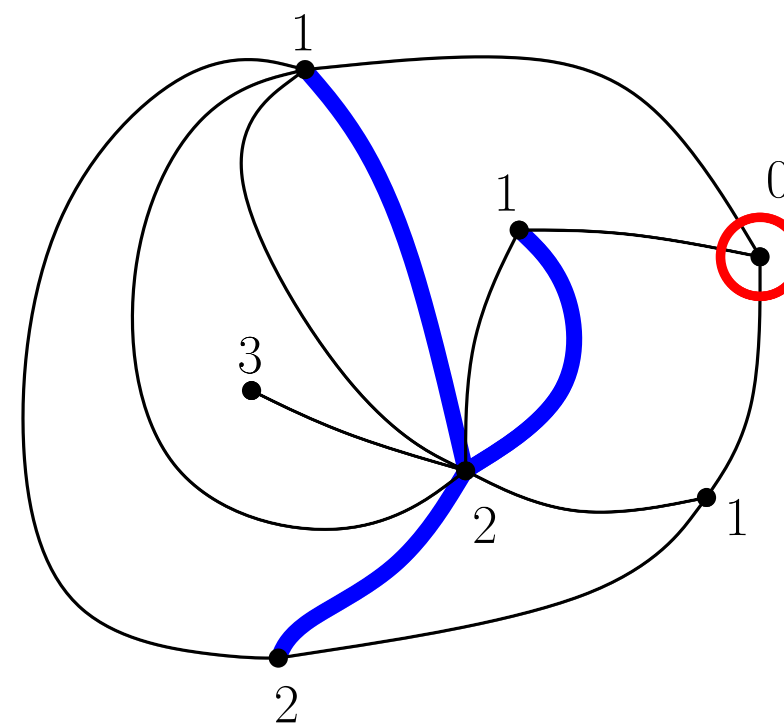

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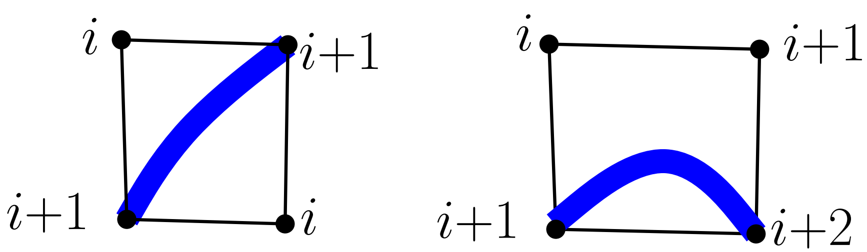

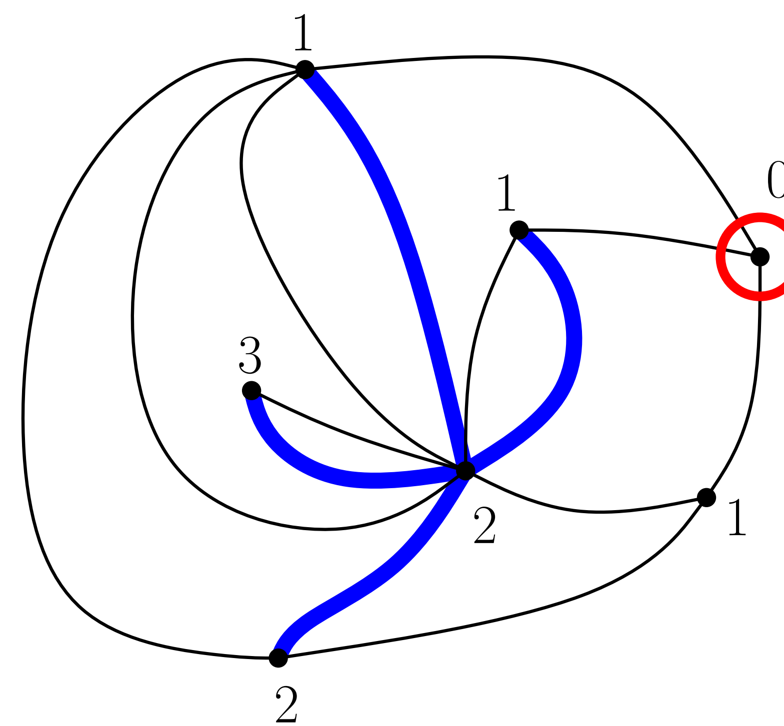

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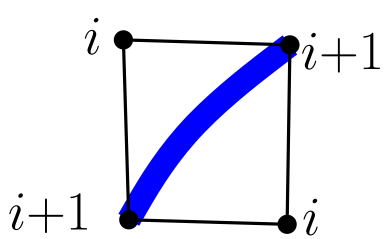

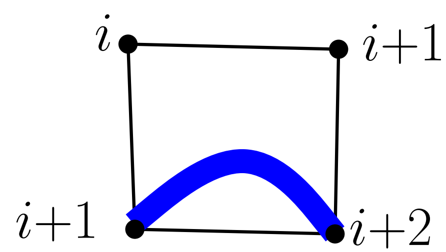

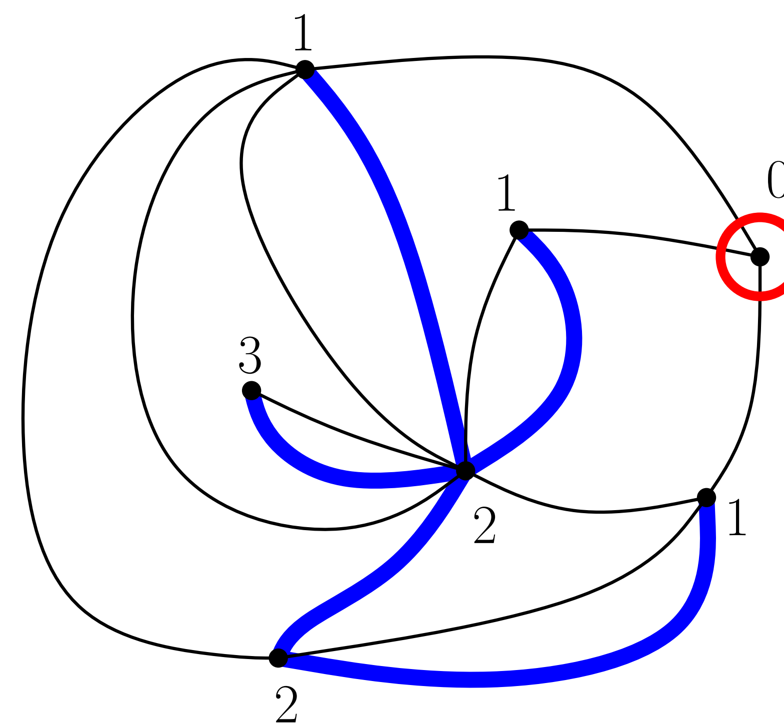

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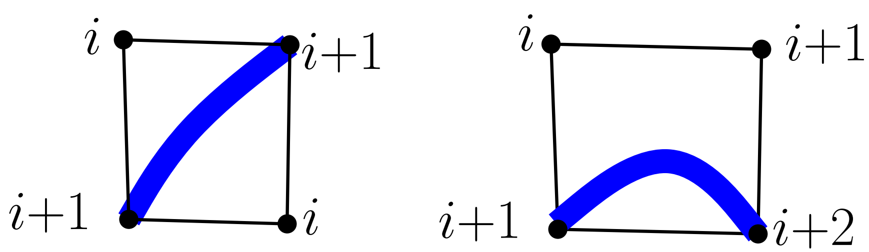

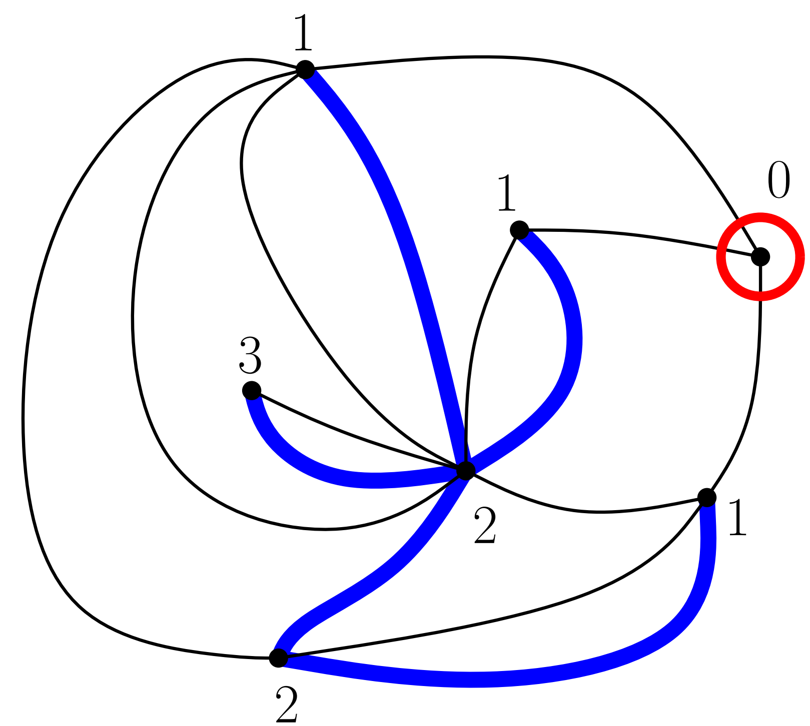

Fact: the blue map is a tree.

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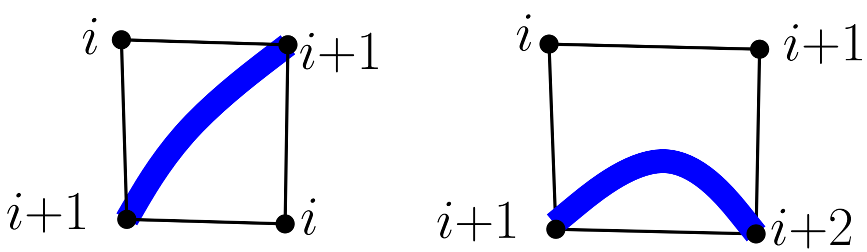

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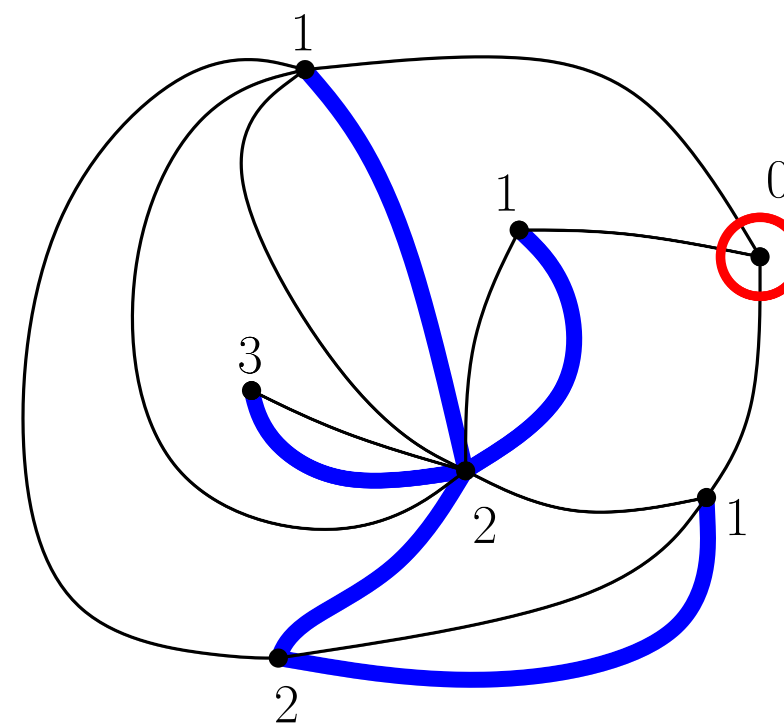

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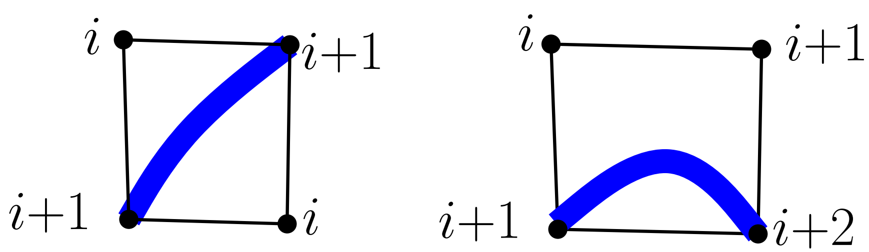

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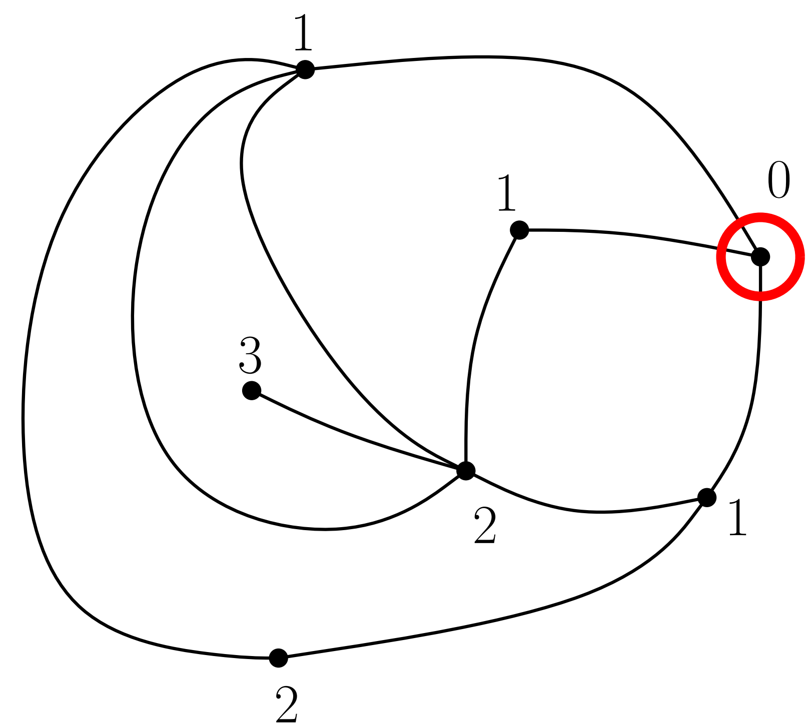

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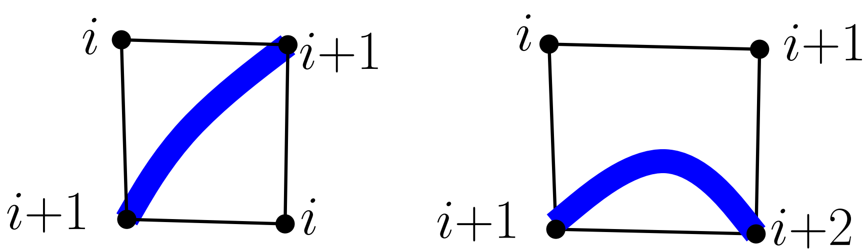

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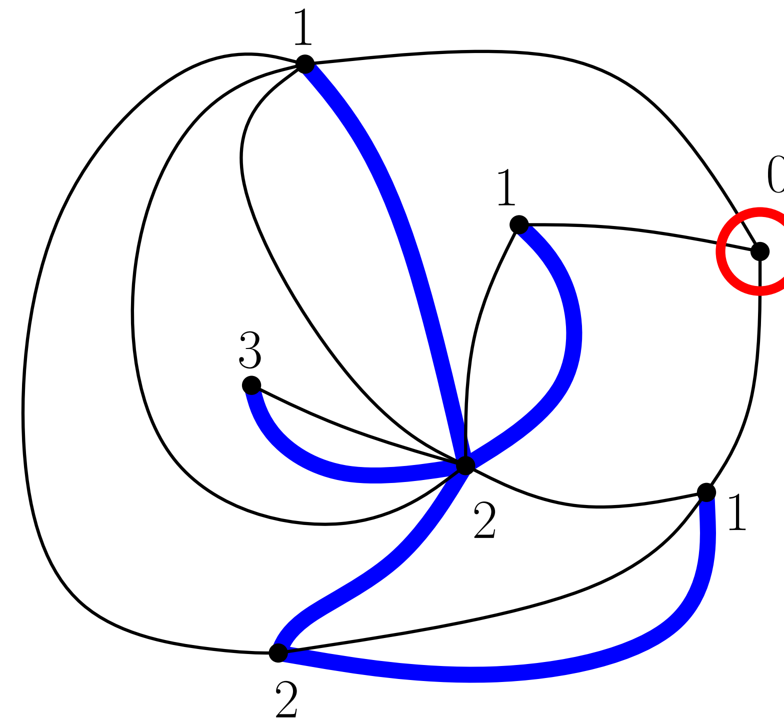

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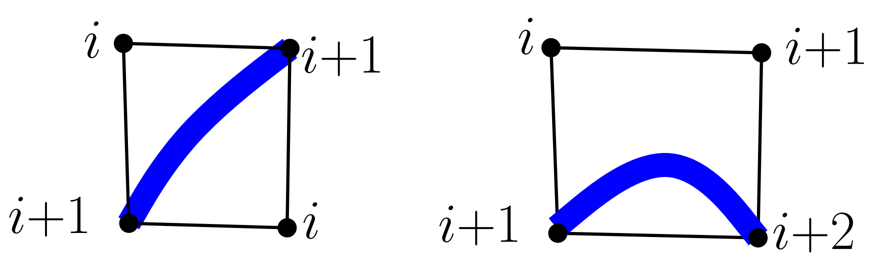

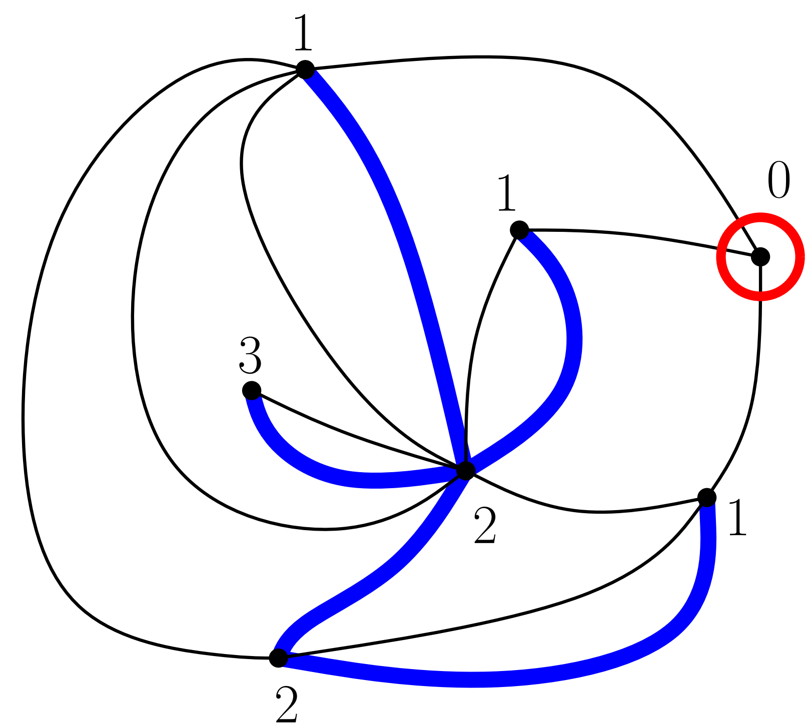

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If one remembers the labels the construction is bijective!

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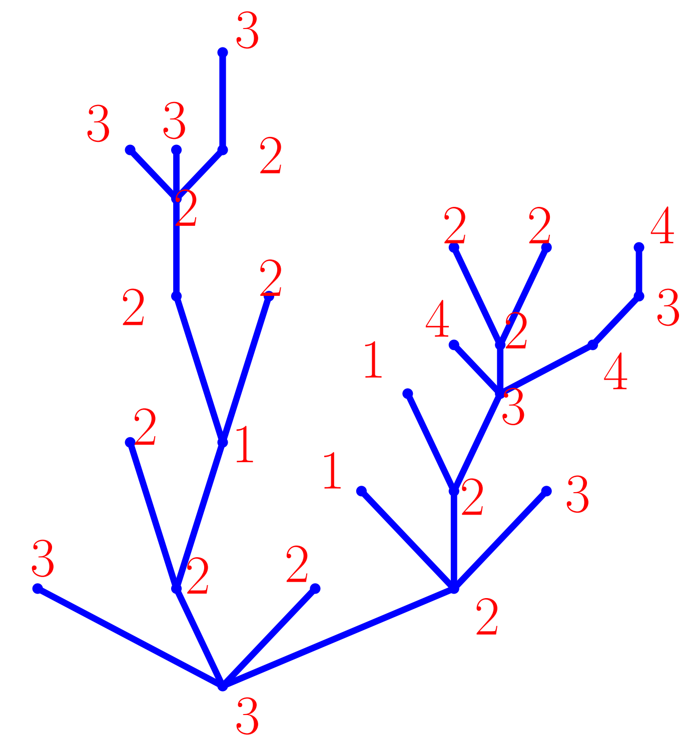
- A well-labelled tree is a plane tree together with a mapping $l : V \to \mathbb{Z}_{>0}$ such that
  - if $v \sim v'$ then $|l(v) - l(v')| \leq 1$
  - $\min_v l(v) = 1$



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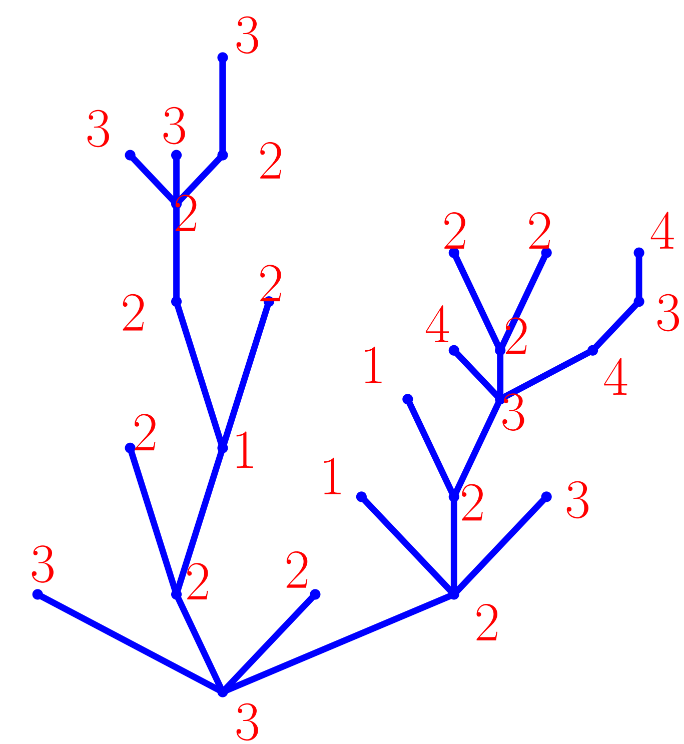
- **Thm** [Cori-Vauquelin'81;Schaeffer'99]
  There is a bijection between quadrangular planar maps with a pointed vertex and $n+1$ vertices and well-labelled trees with n vertices. The labels in the tree correspond to distances to the root in the map.



(1) Maps: the Cori-Vauquelin-Schaeffer bijection (1981-1999-2008+)

- A well-labelled tree is a plane tree together with a mapping $l : V \to \mathbb{Z}_{>0}$ such that

  - if $v \sim v'$ then $|l(v) - l(v')| \leq 1$

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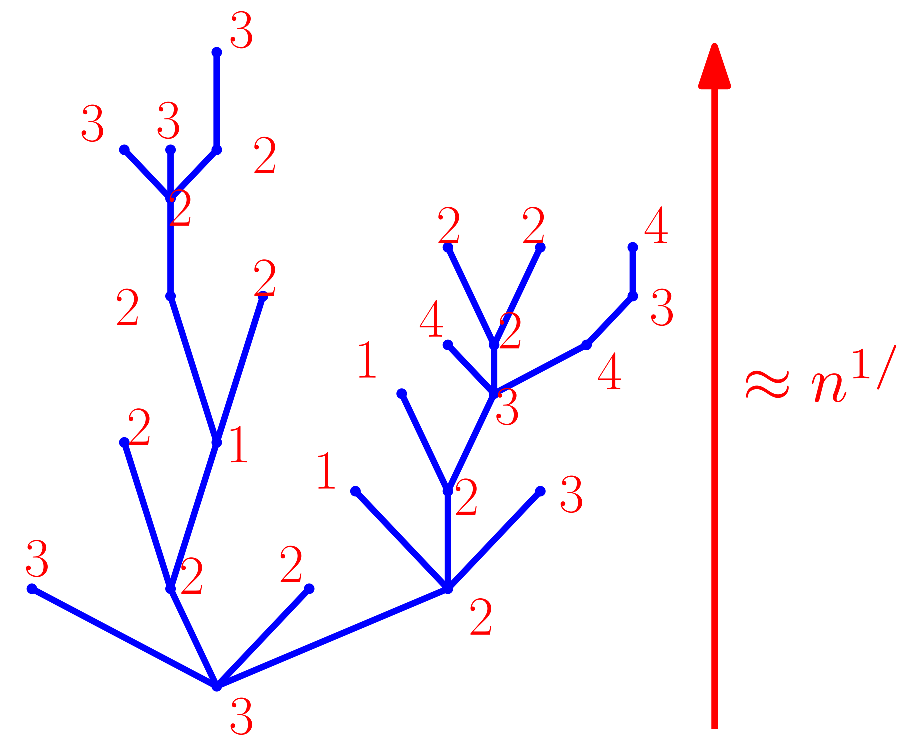

Corollary: $\mathrm{Diam}(M_n) = n^{1/4+o(1)}$

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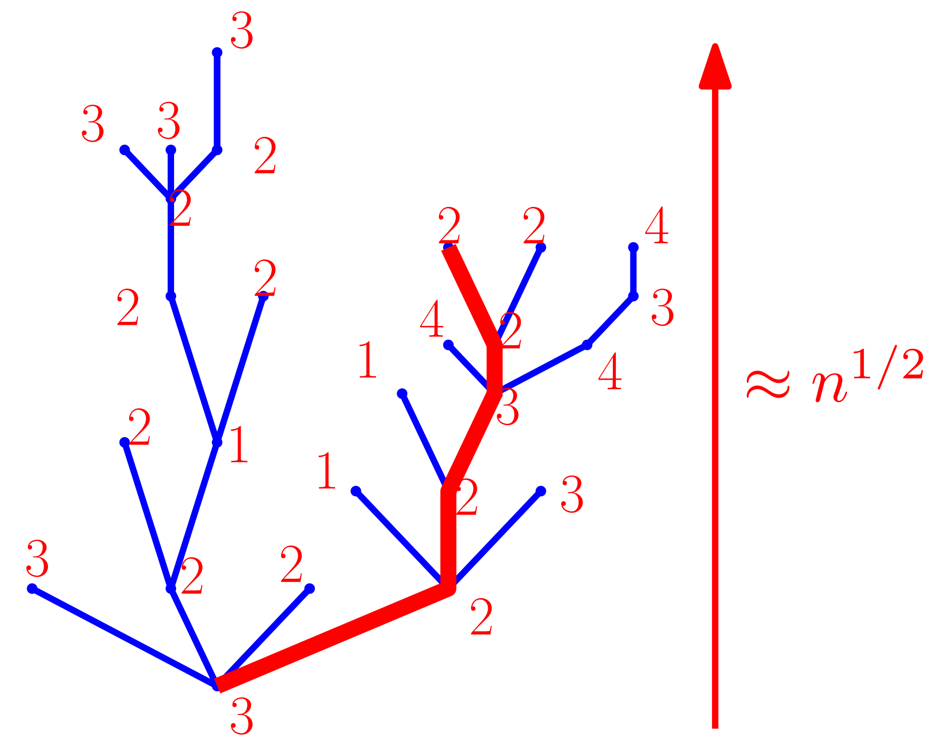

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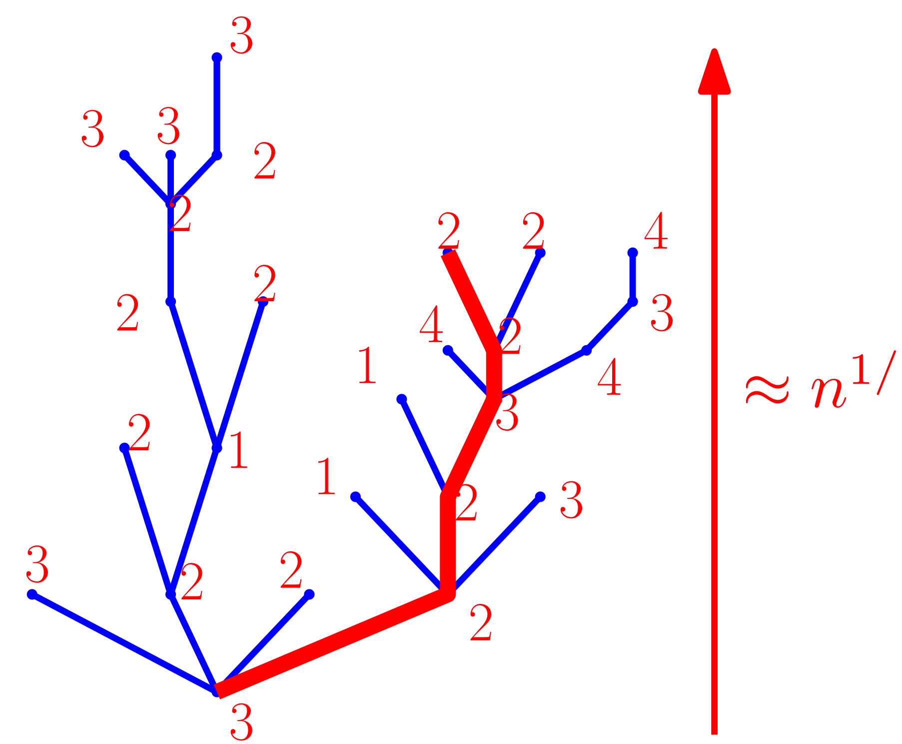

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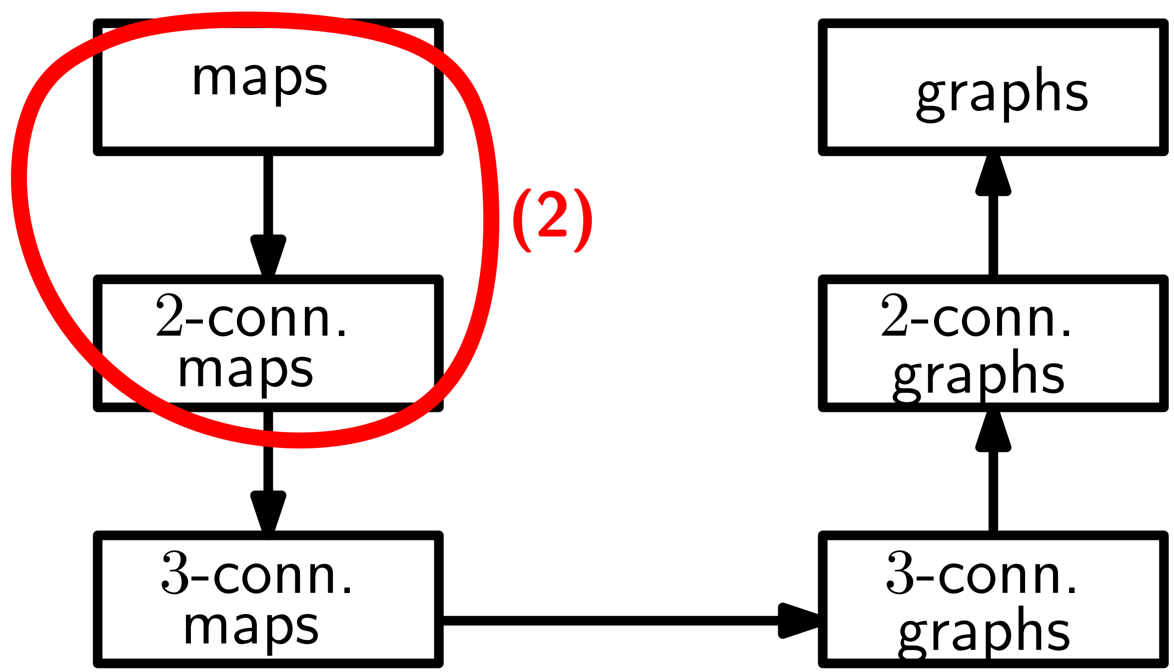
[Chassaing-Schaeffer'04]



maps
2-conn.
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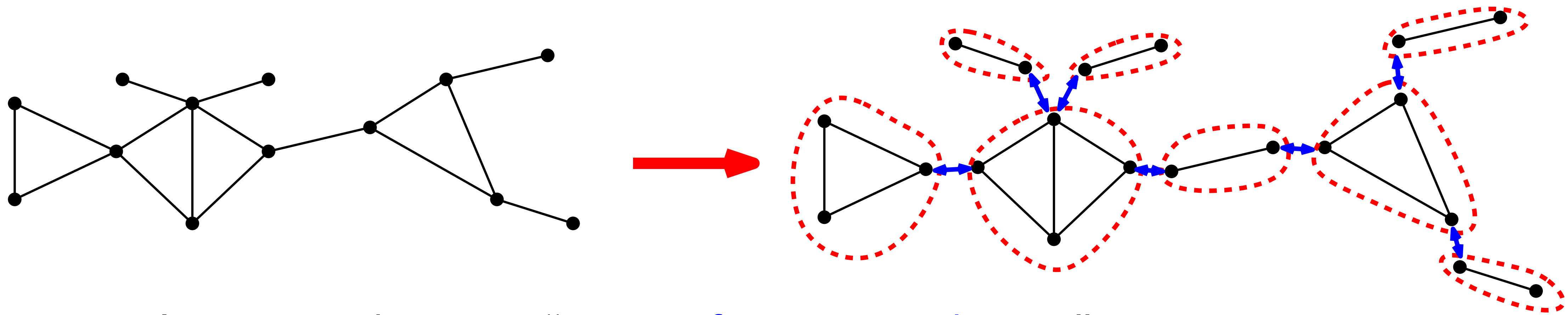
graphs
(2)

(2) Decomposition into 2-connected components



A connected map is “a tree of 2-connected maps”

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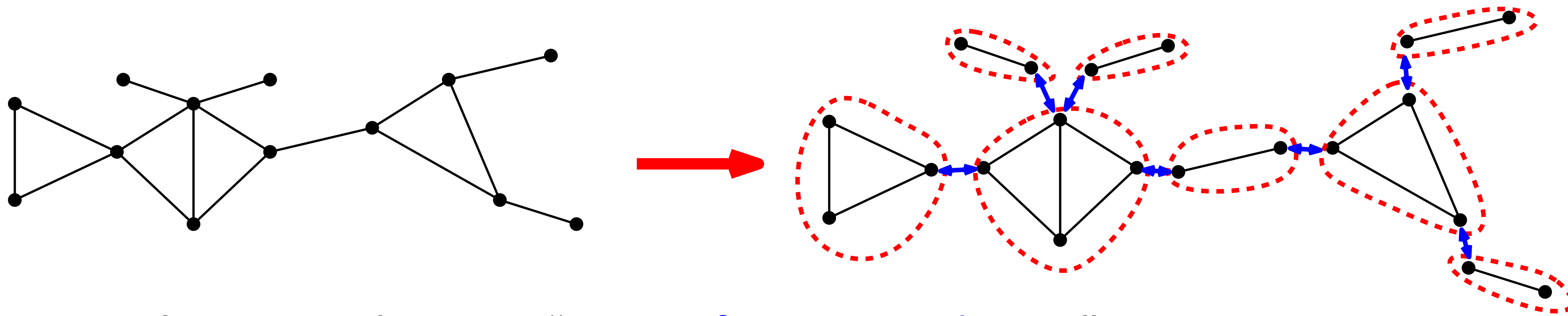

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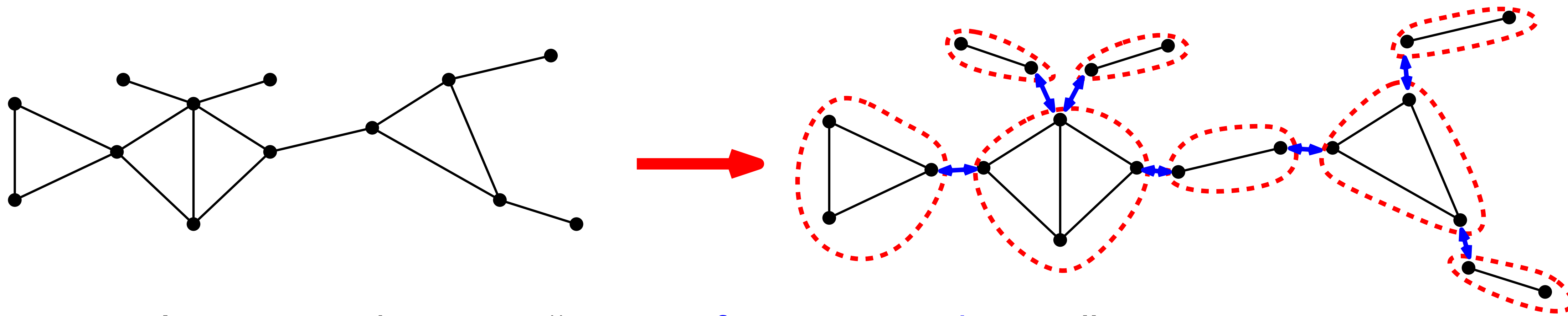

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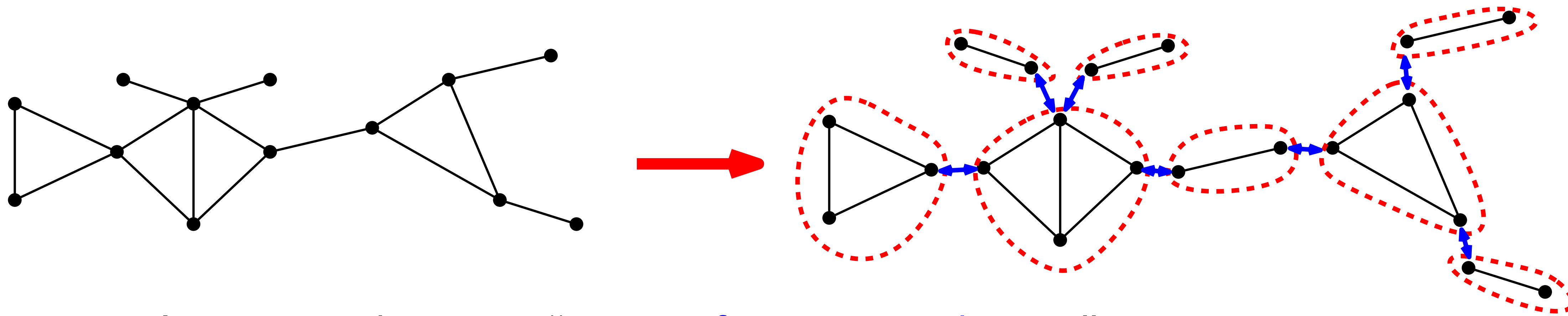

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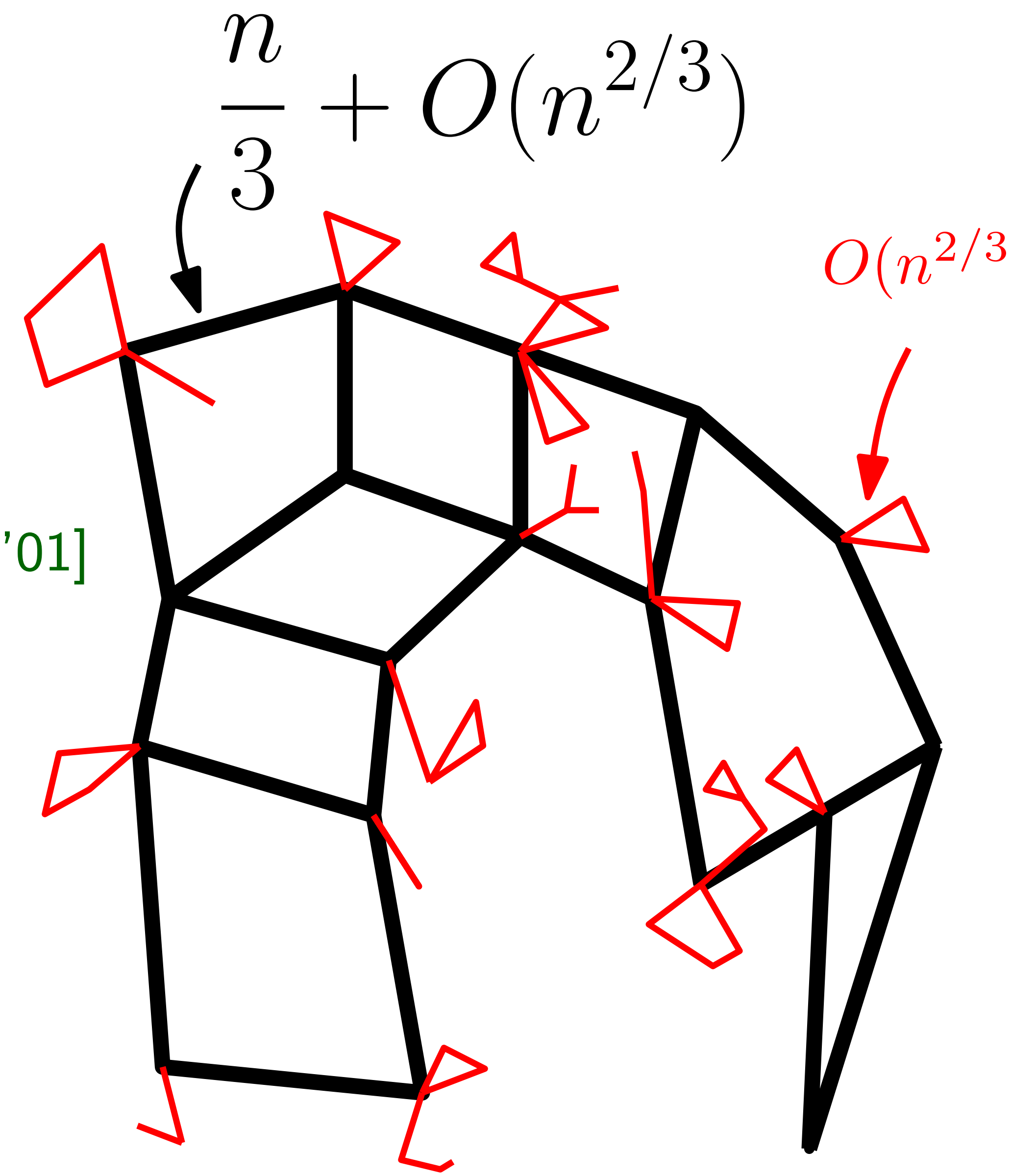
The largest 2-connected component has size $\frac{n}{3} + n^{2/3}A$ where A converges to an explicit law.

The second-largest component has size $O(n^{2/3})$.

[Gao, Wormald'99] [Banderier, Flajolet, Schaeffer, Soria '01]



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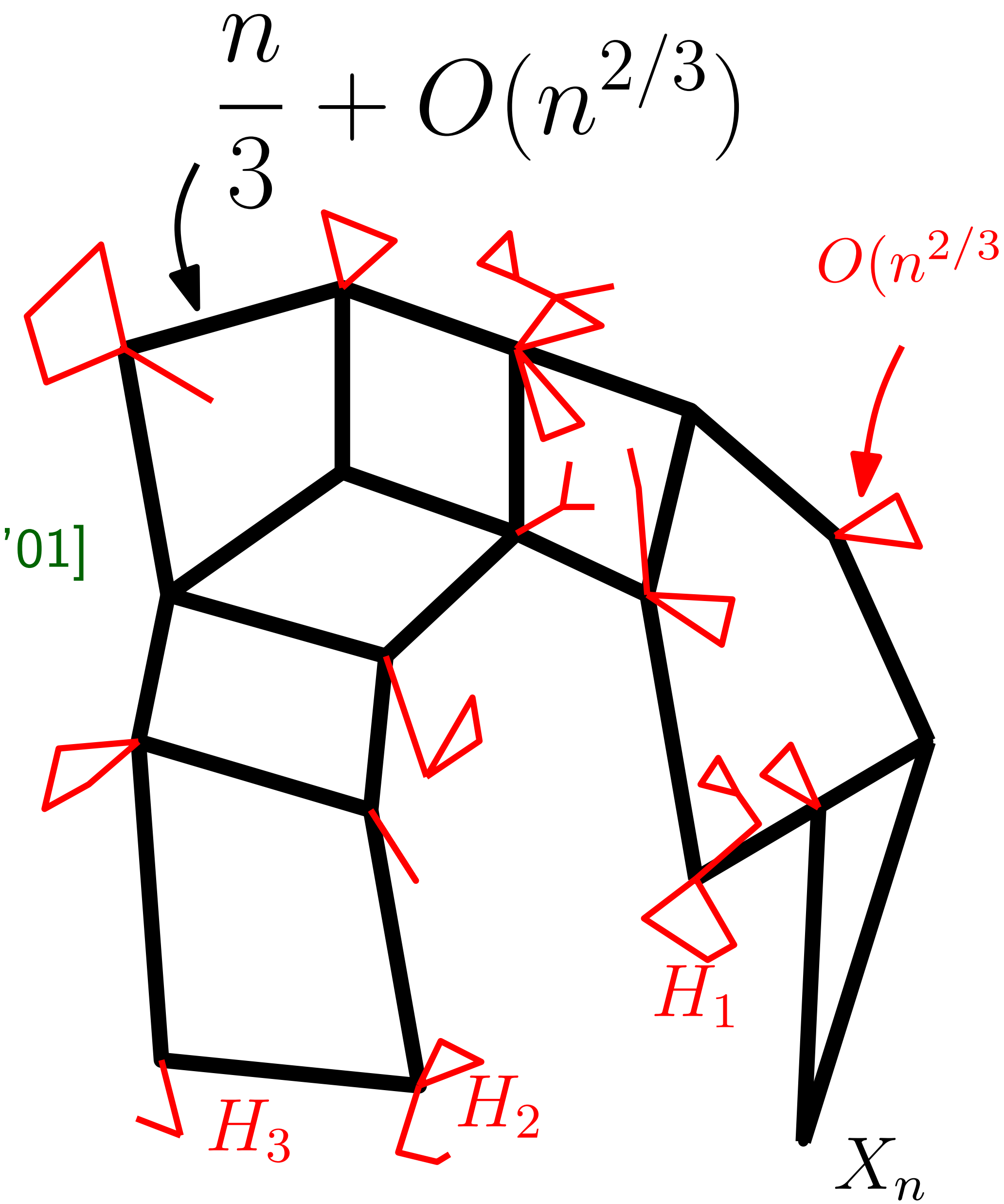

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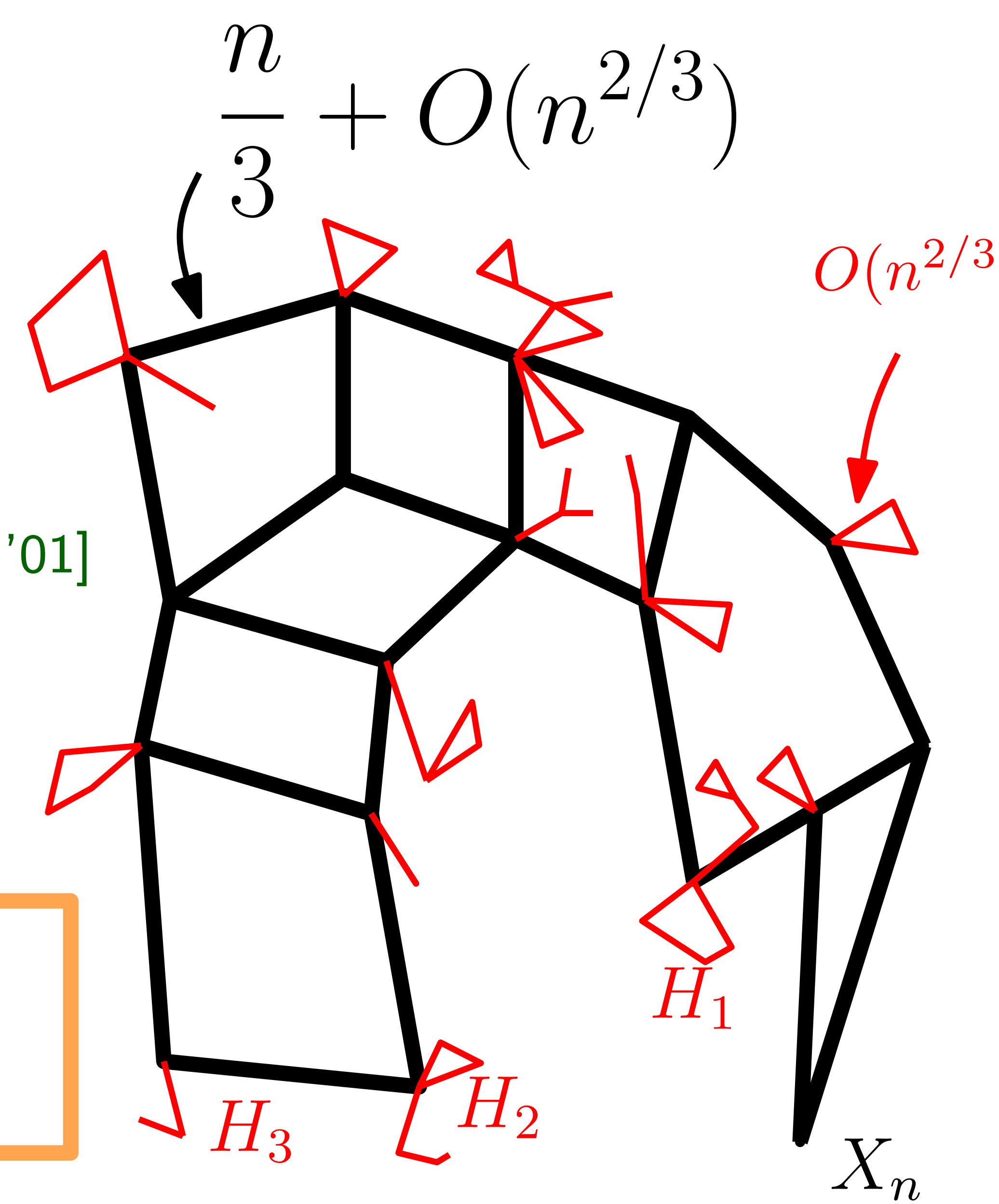
random 2-conn. map of size $n/3$



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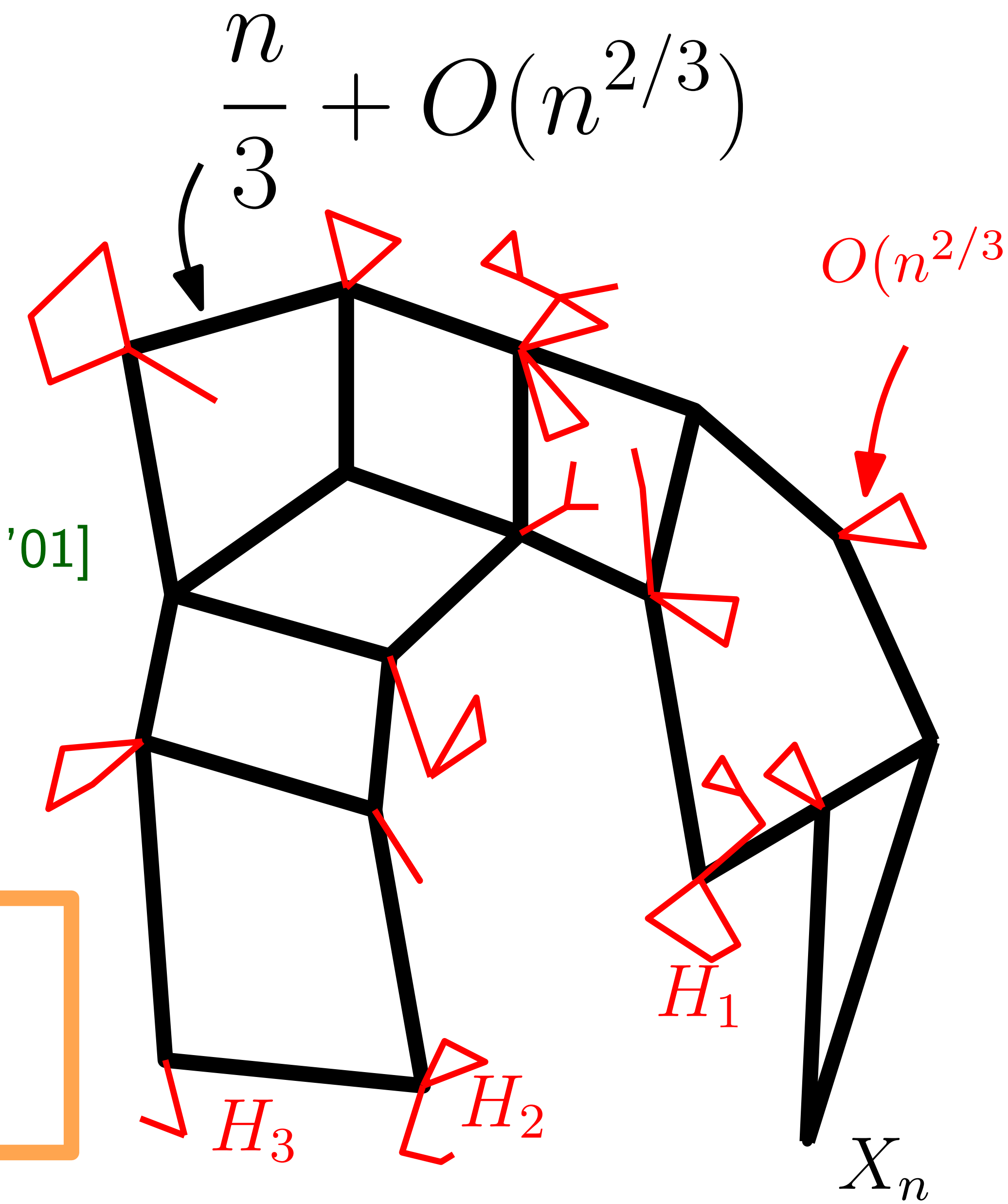

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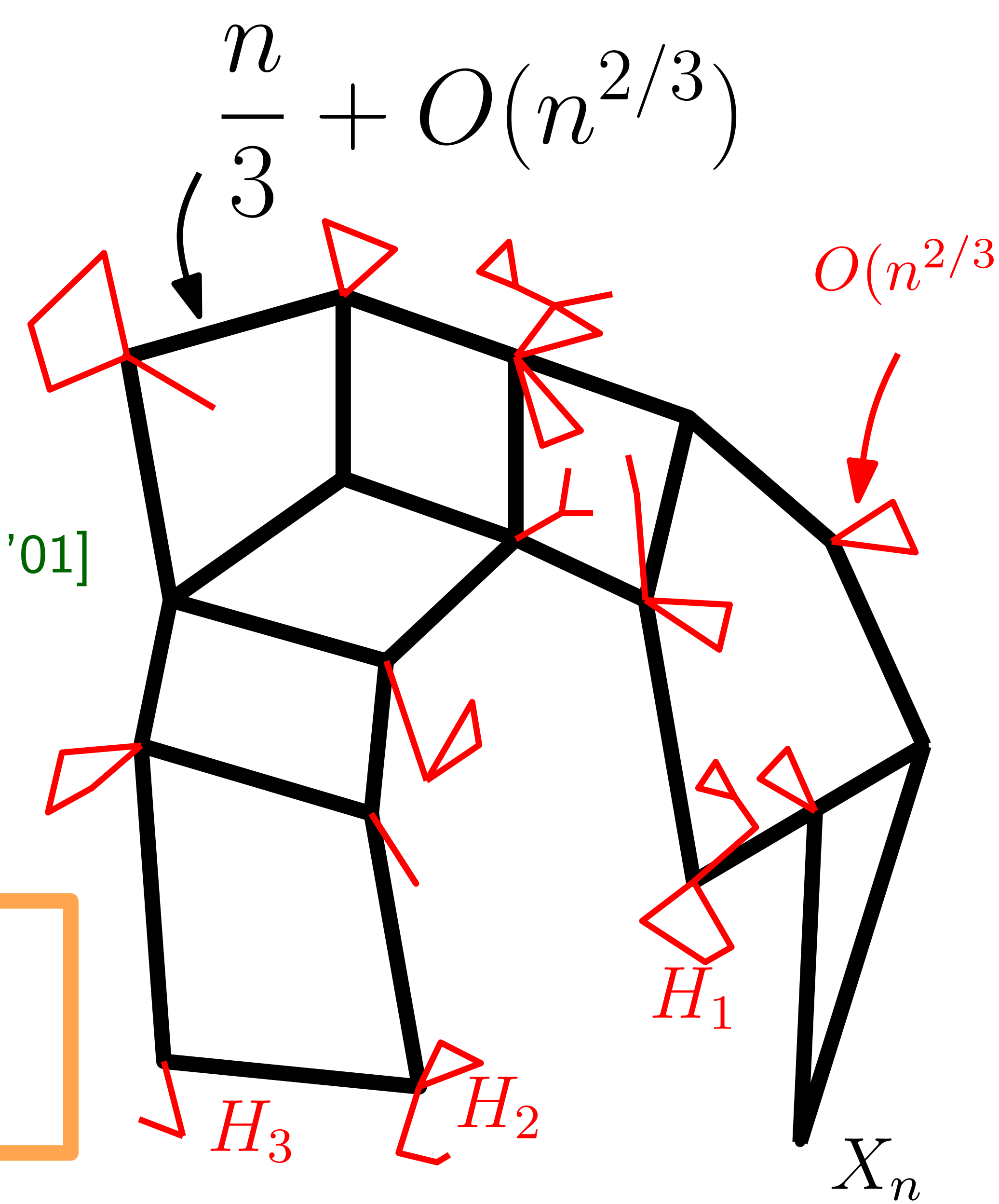

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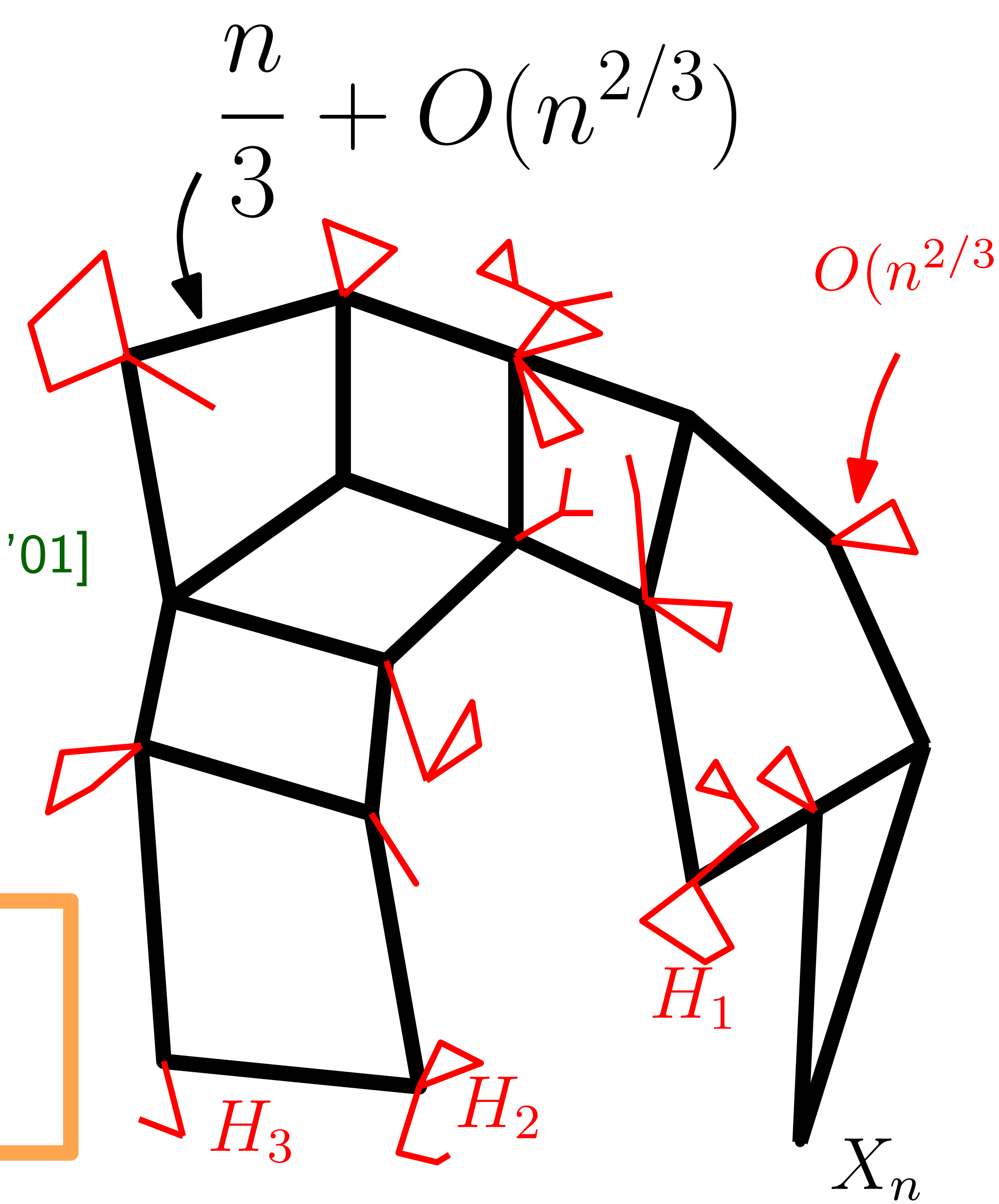

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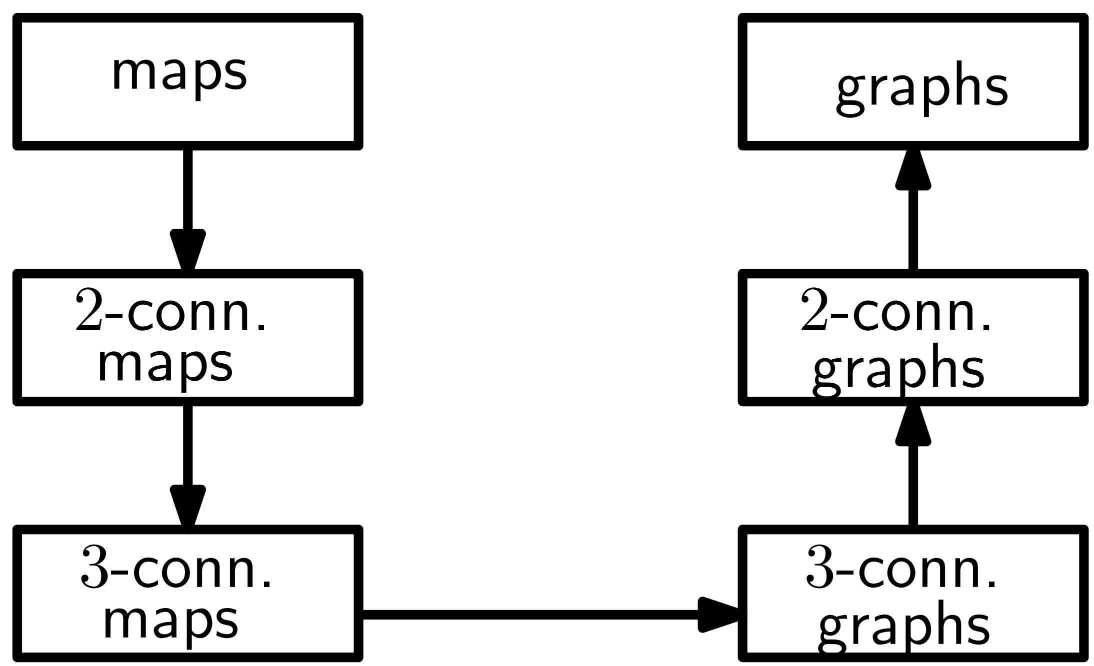
and X_n is essentially a random 2-conn. map of size $n/3$.



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3-conn. maps
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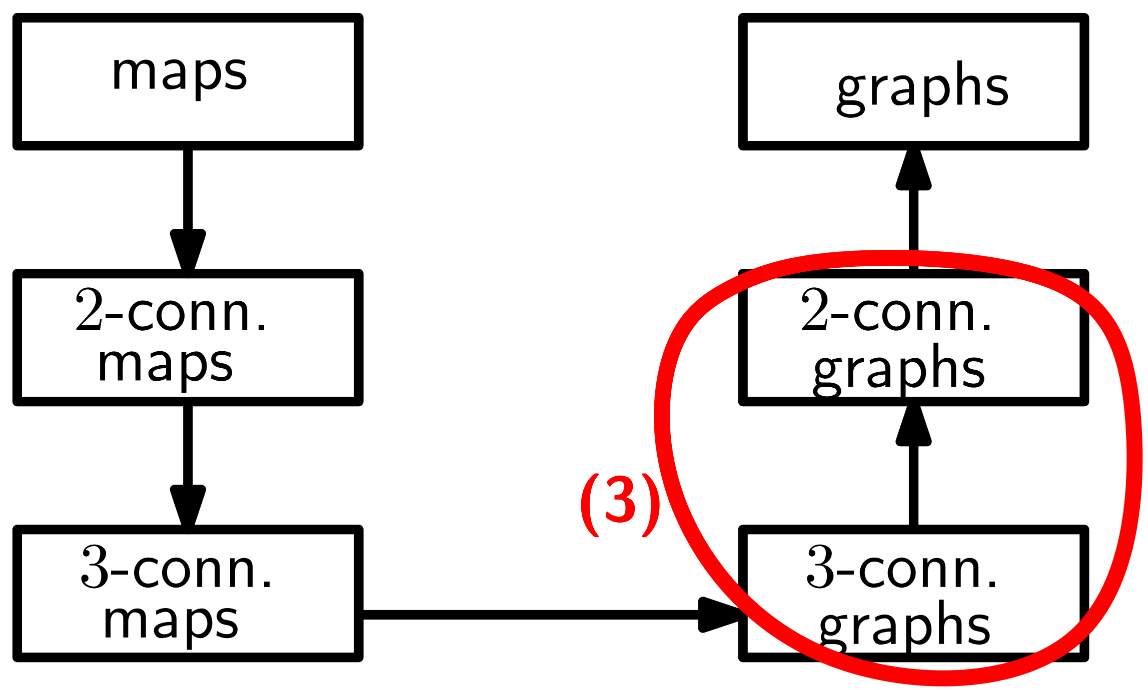
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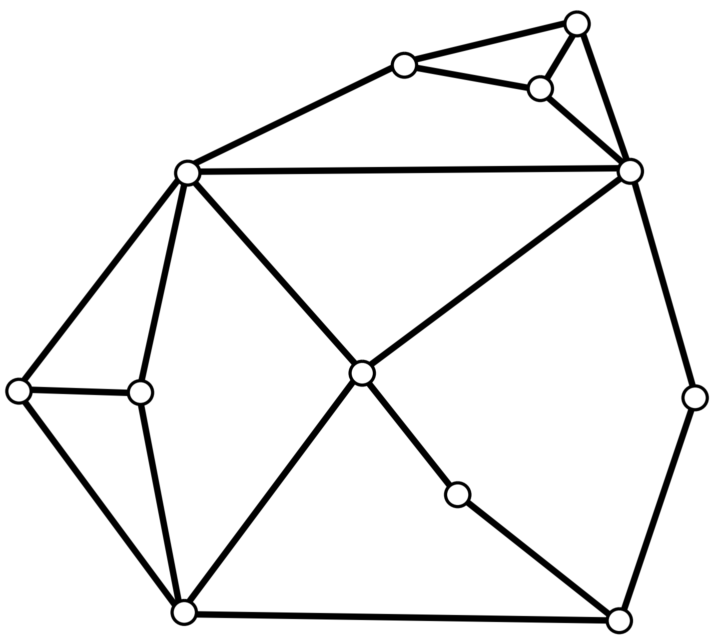
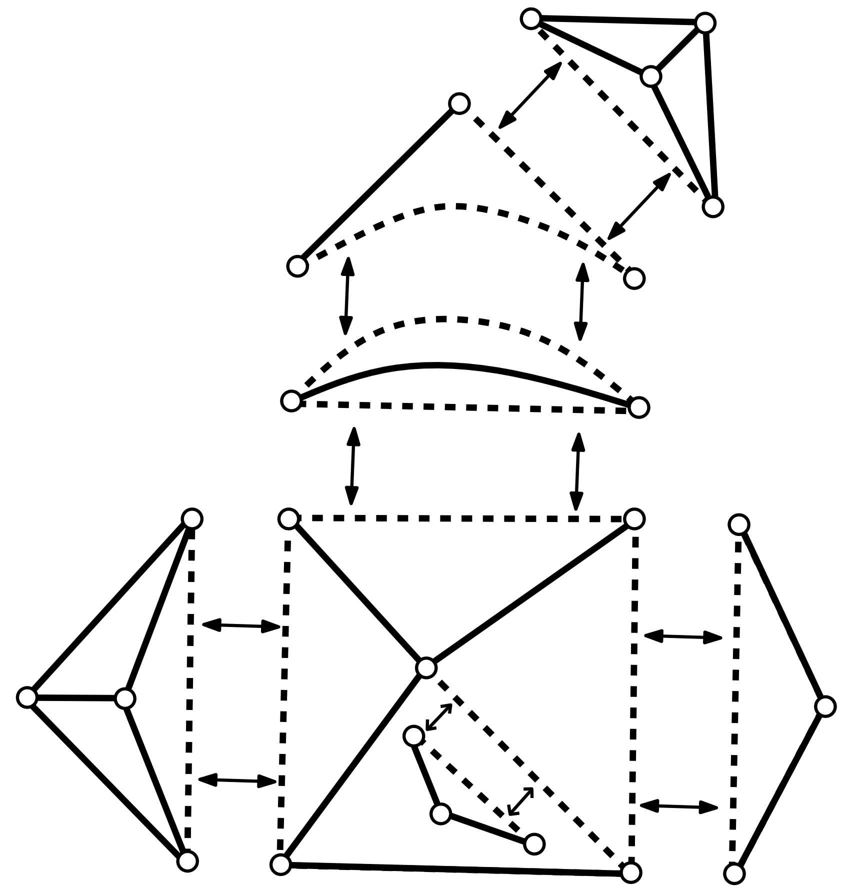
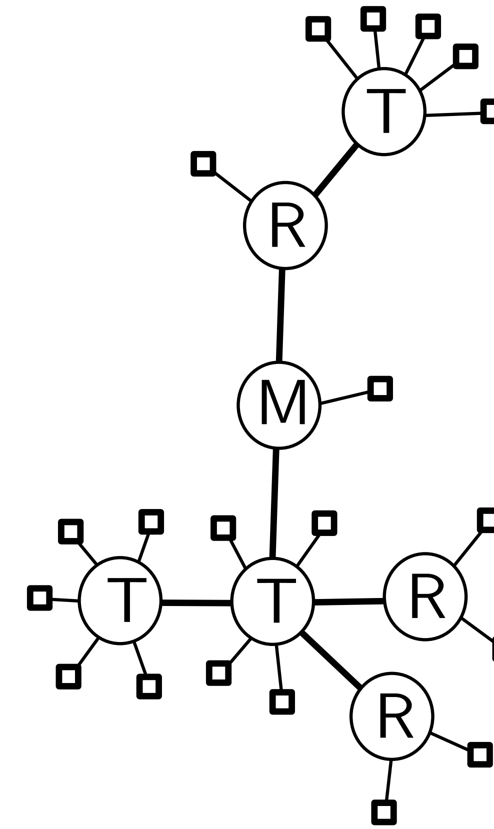
(3)

(3) Decomposition into 3-connected components



2-connected





Again one can write everything in terms of generating functions.

→ deduce the g.f. of 3-conn. maps from the one of 2-connected maps. [Tutte 60's].

→ deduce the g.f. of 2-conn. graphs from the one of 3-connected graphs [Bender, Gao, Wormald'02].

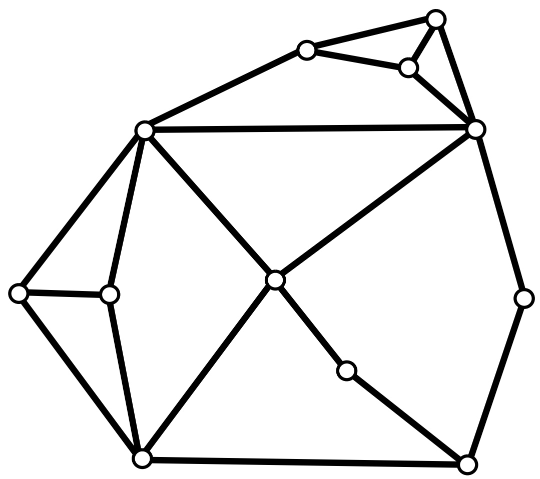
T = 3-connected component

R = series composition

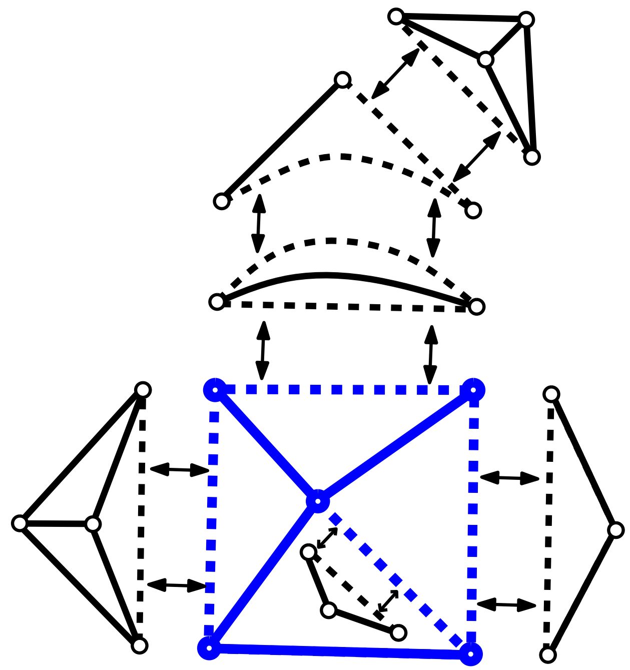
M = parallel composition

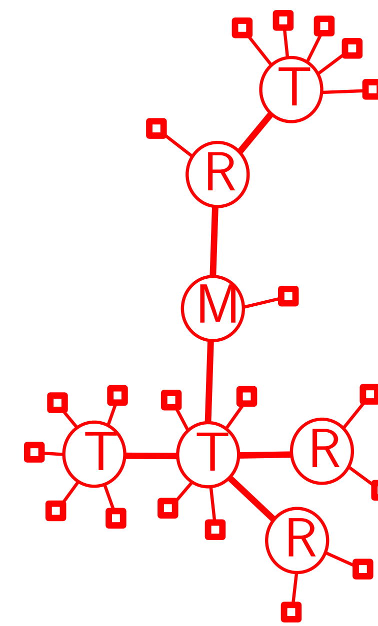
(3) Decomposition into 3-connected components

Prop A random 2-connected planar graph with n edges has diameter $n^{1/4+o(1)}$ with high probability.



a 2-conn. graph B_n





RMT tree

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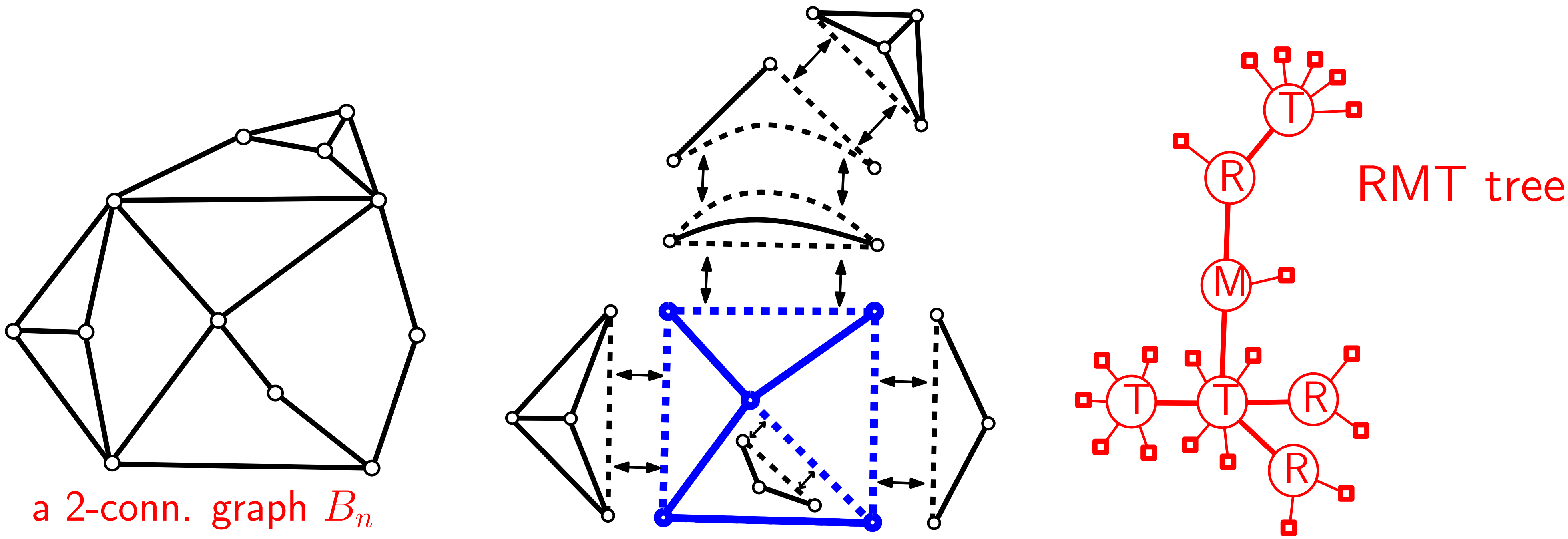

Same idea:

- there exists a T-component Y_n of linear size w.h.p.

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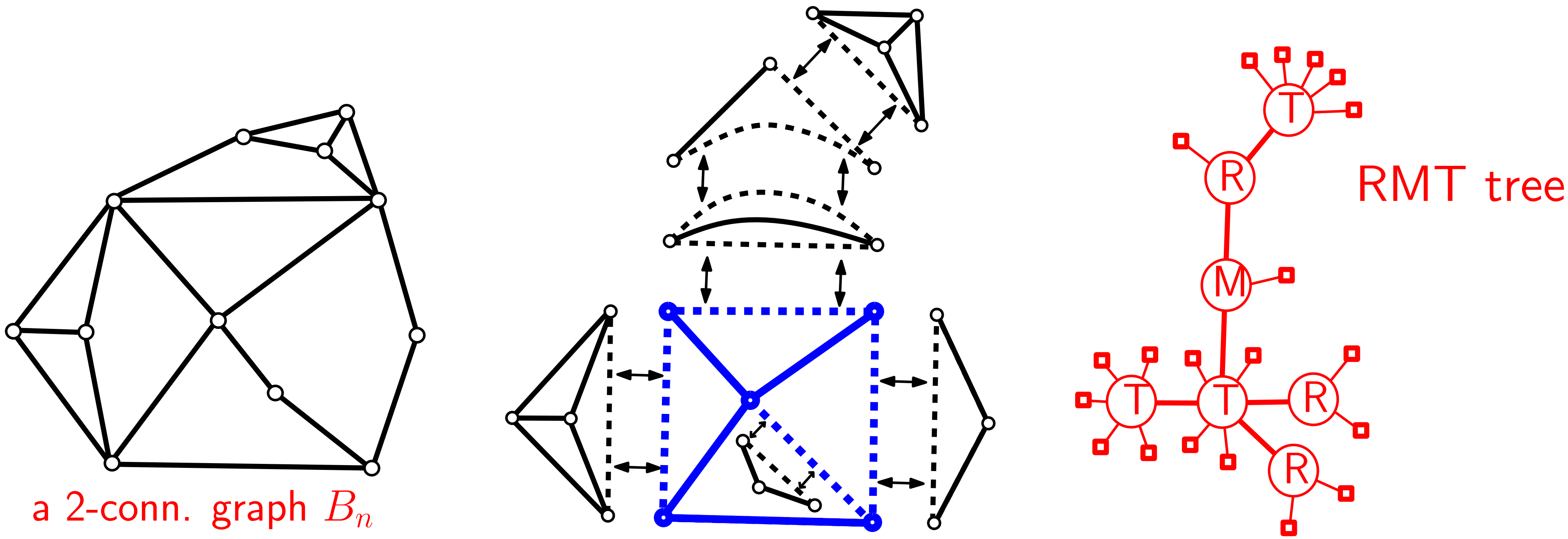


Same idea:

- there exists a T-component Y_n of linear size w.h.p.
- the diameter of the RMT-tree is $n^{o(1)}$ w.h.p.
- The extra-length due the edge substitution is also $n^{o(1)}$

Conclusion (I)

- **Thm** [C, Fusy, Giménez, Noy 2010+]
  Let G_n be the uniform random planar graph with n vertices.

  Then $\mathrm{Diam}(G_n) = n^{1/4+o(1)}$ w.h.p.

  More precisely $\mathbb{P}\Big(\mathrm{Diam}(G_n) \notin \big[n^{1/4-\epsilon}, n^{1/4+\epsilon}\big]\Big) = O(e^{-n^{\Theta(\epsilon)}})$.

- The proof relies both on exact generating functions and magical bijections: we couldn't do anything without this (or maybe something much weaker like $O(\sqrt{n})$?)
- The general picture is quite clear but the analysis is a bit tedious... (need to work with bivariate generating functions and prove estimates with enough uniformity)
- No way to obtain the convergence of $\frac{\mathrm{Diam}(G_n)}{n^{1/4}}$ - even for planar maps this is very difficult!
- Same result for the uniform random graph with n vertices and $\lfloor \mu n \rfloor$ edges for $1 < \mu < 3$.

Conclusion (II)

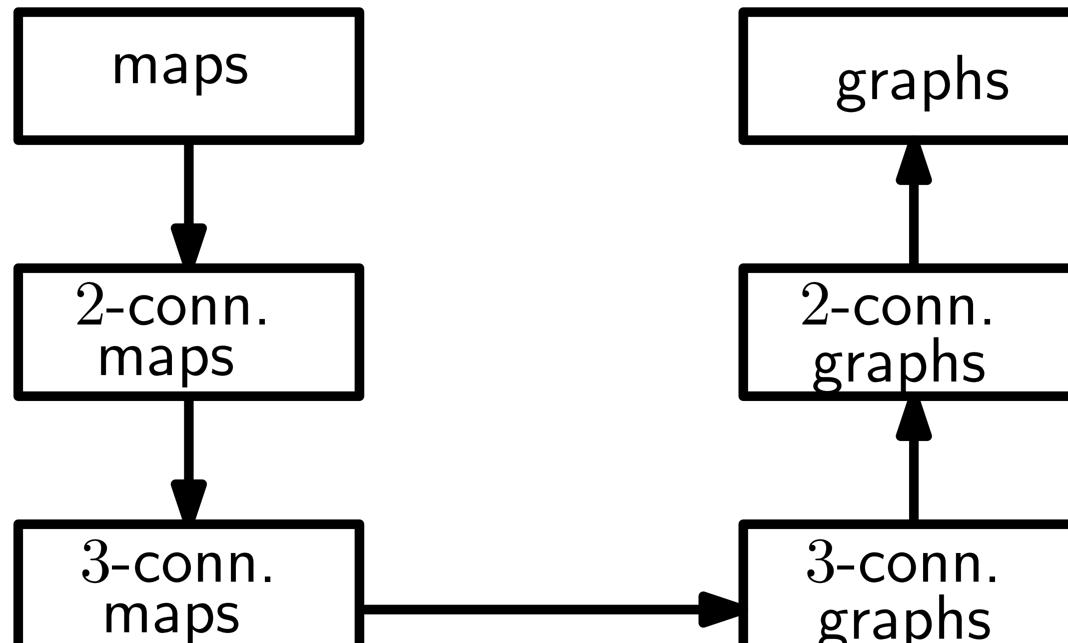
- We generalized the Giménez-Noy enumeration result to graphs embeddable on a surface of genus $g \geq 0$

Thm [C, Fusy, Giménez, Mohar, Noy 2011] [Bender-Gao 2011]

$$\#\{n\text{-vertex genus } g \text{ graphs}\} \sim c_g \cdot n! \cdot \gamma^n \cdot n^{\frac{5}{2}g-7/2} \qquad \gamma \approx 27.\ldots$$

Same kind of proof but Whitney's theorem (uniqueness of embedding) now requires that there is no short non-contractible cycle.
(but we could prove that)

The result on the diameter should be the same but this is not (and won't be) written.



The fact that non-contractible cycles are small imply the following:

Thm [C, Fusy, Giménez, Mohar, Noy 2011]

Fix $g \geq 1$. The random graph of genus g and size n has chromatic number in $\{4, 5\}$ and list chromatic number 5 w.h.p.

Thank you!