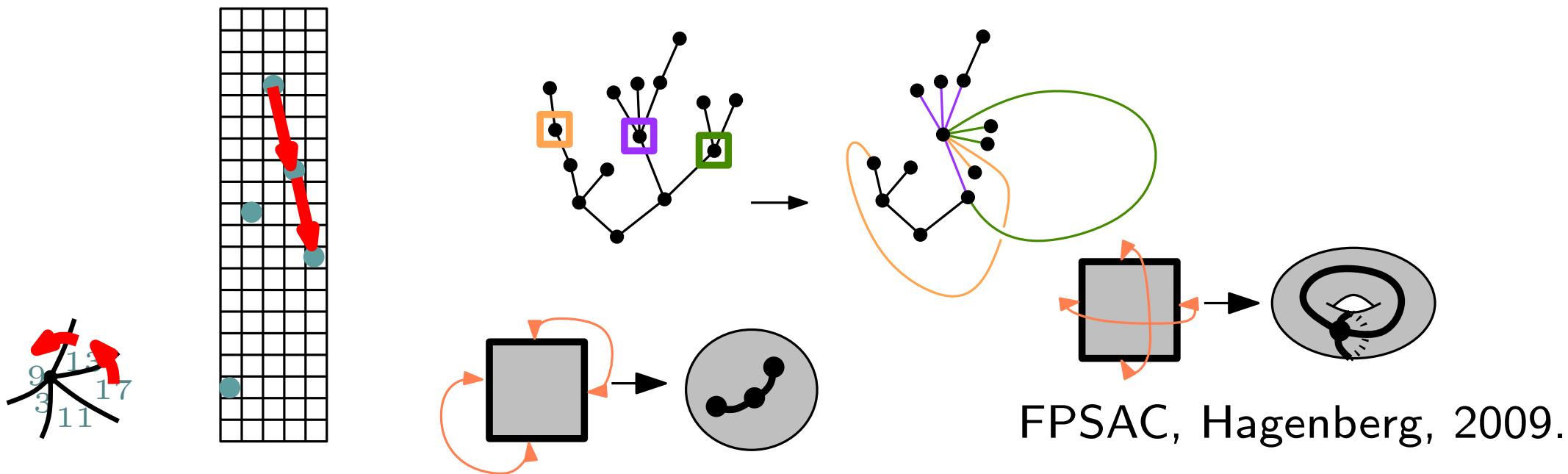


A new combinatorial identity for unicellular maps, via a direct bijective approach

Guillaume Chapuy,
École Polytechnique (France)

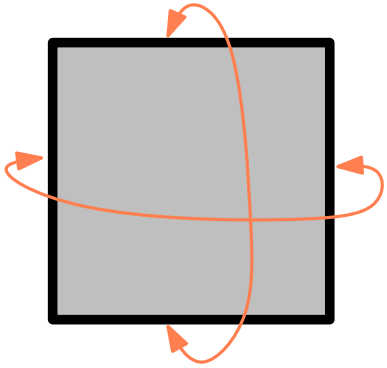


Unicellular maps as polygon gluings

We start with a $2n$ -gon, and we paste the edges pairwise in order to form an orientable surface.

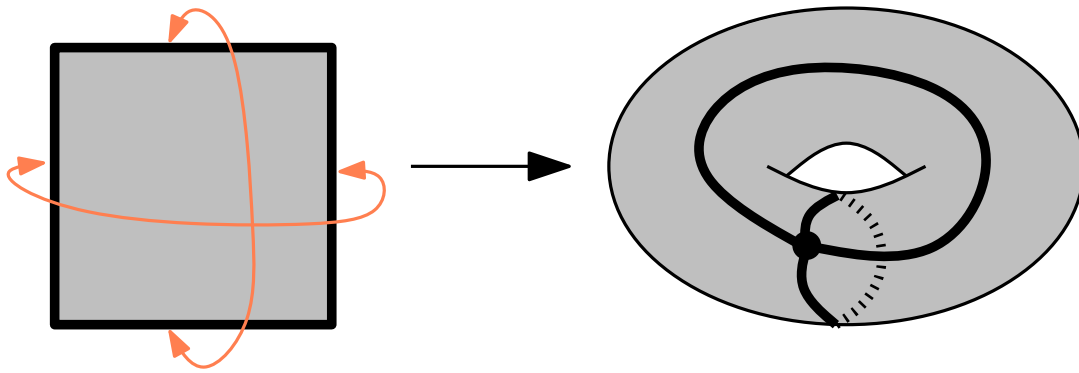
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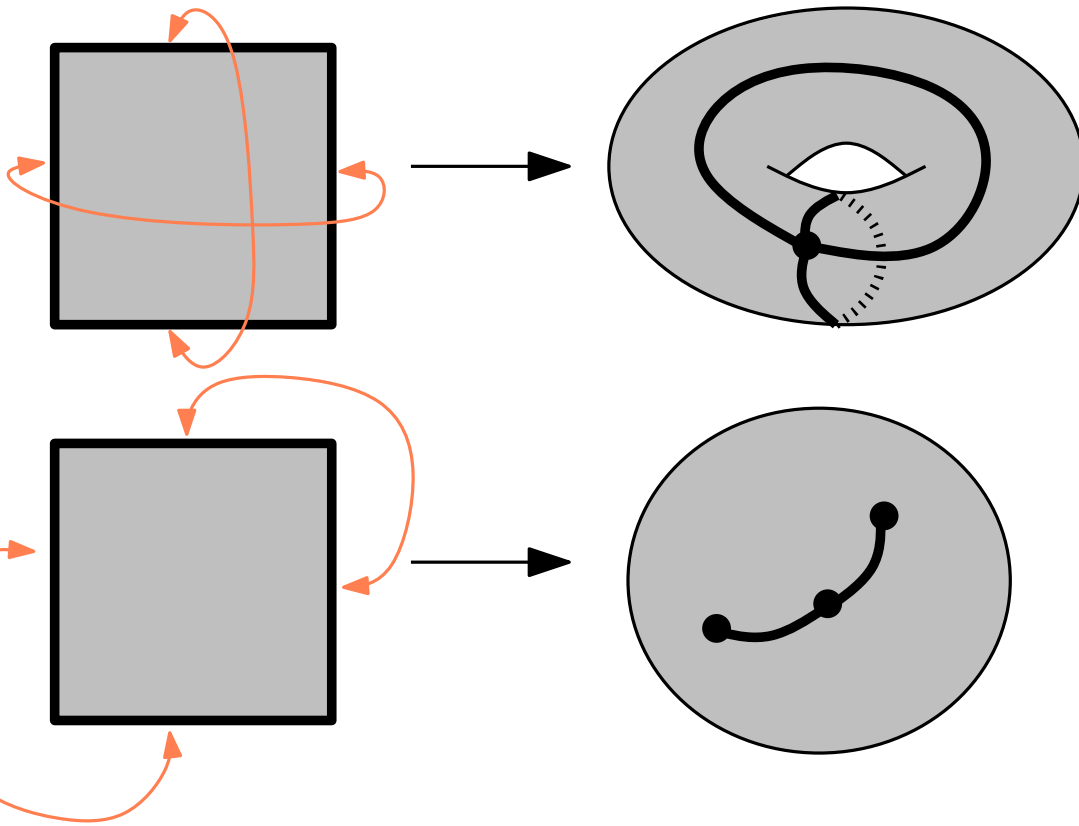
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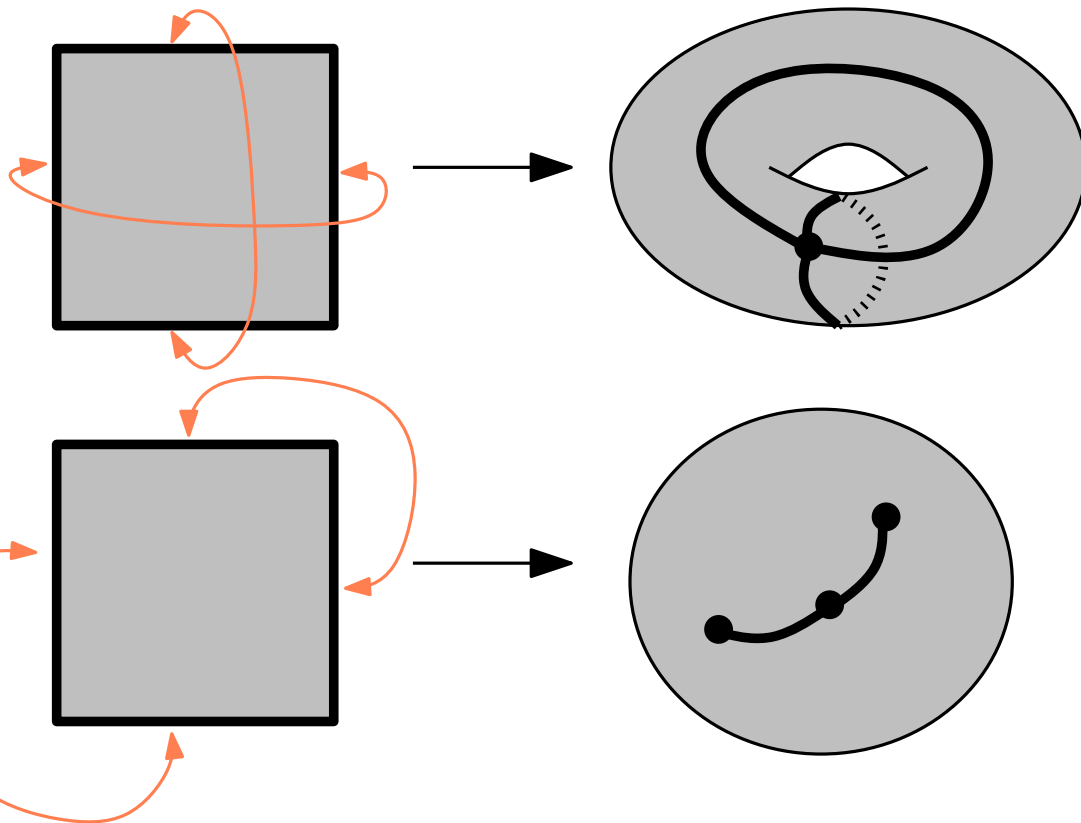
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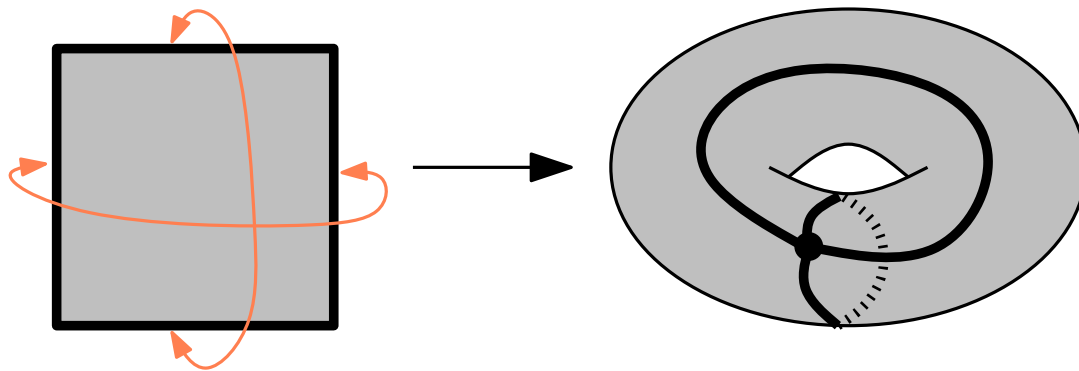
The image of the polygon forms the drawing of an n -edge graph on the surface.

Euler's formula relates the number of vertices to the genus of the surface :

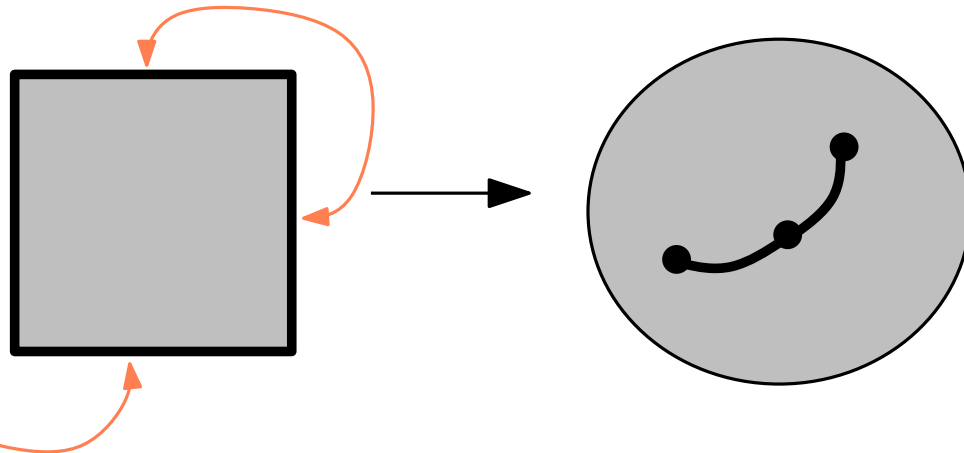
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1 vertex, genus 1



3 vertices, genus 0

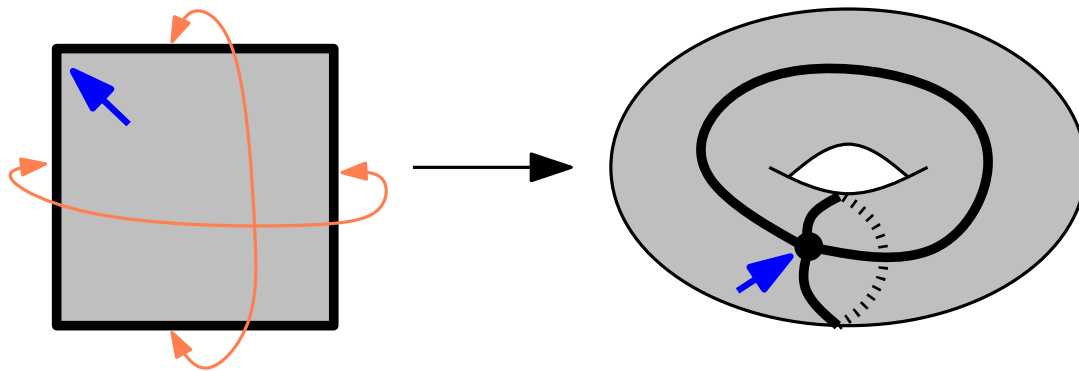
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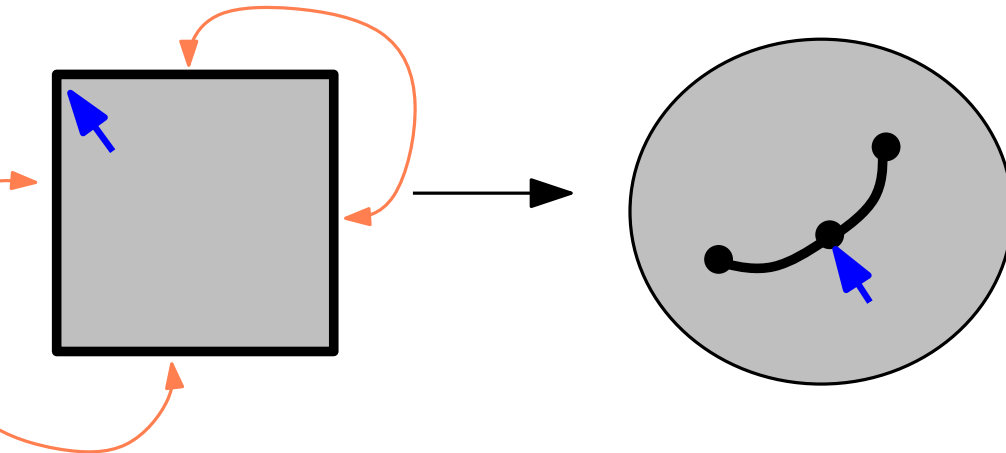
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Counting

The number of unicellular maps with n edges is equal to the number of distinct matchings of the edges : $\frac{(2n)!}{2^n n!}$.

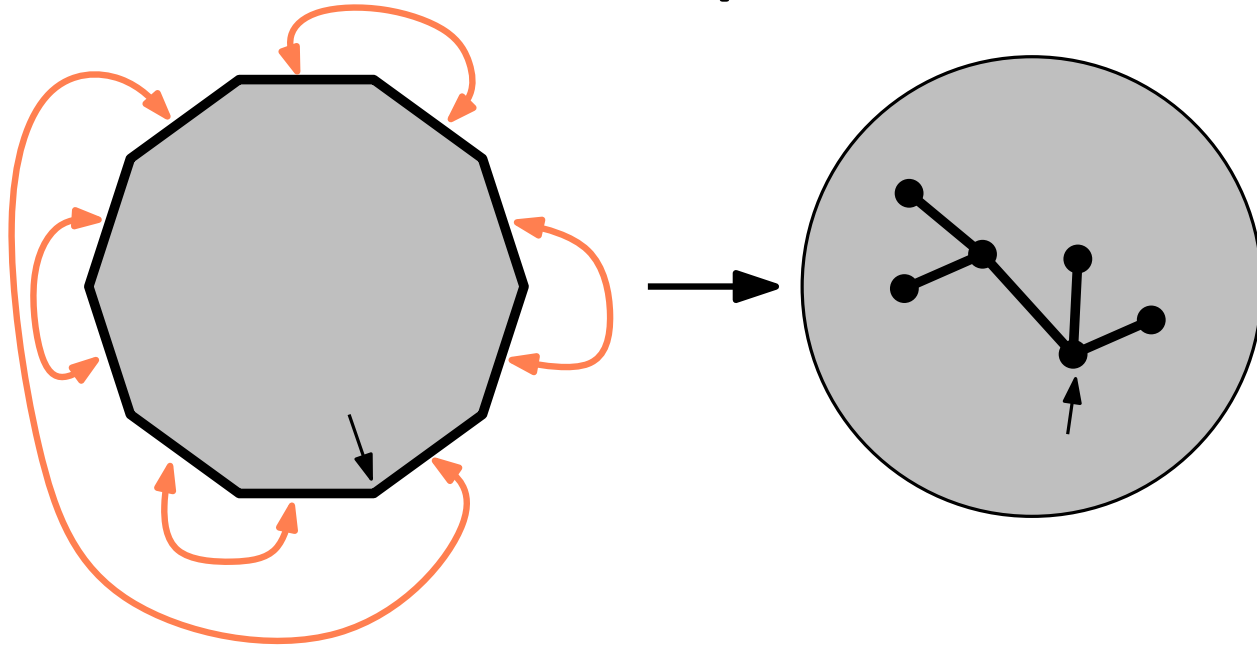
Aim: count unicellular maps of **fixed genus**.

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Aim: count unicellular maps of **fixed genus**.

For instance, in the planar case...



Unicellular maps are exactly **plane trees**.

Therefore the number of n -edge unicellular maps of genus 0 is :

$$\epsilon_0(n) = \text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$$

Higher genus ?

For each g the number of n -edge unicellular maps of genus g has the (beautiful) form :

$$\epsilon_g(n) = (\text{some polynomial}) \times \text{Cat}(n)$$

For instance :

$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12} \text{Cat}(n)$$

$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440} \text{Cat}(n)$$

References : [Lehman and Walsh 72](#) (formal power series), [Harer and Zagier 86](#) (matrix integrals).

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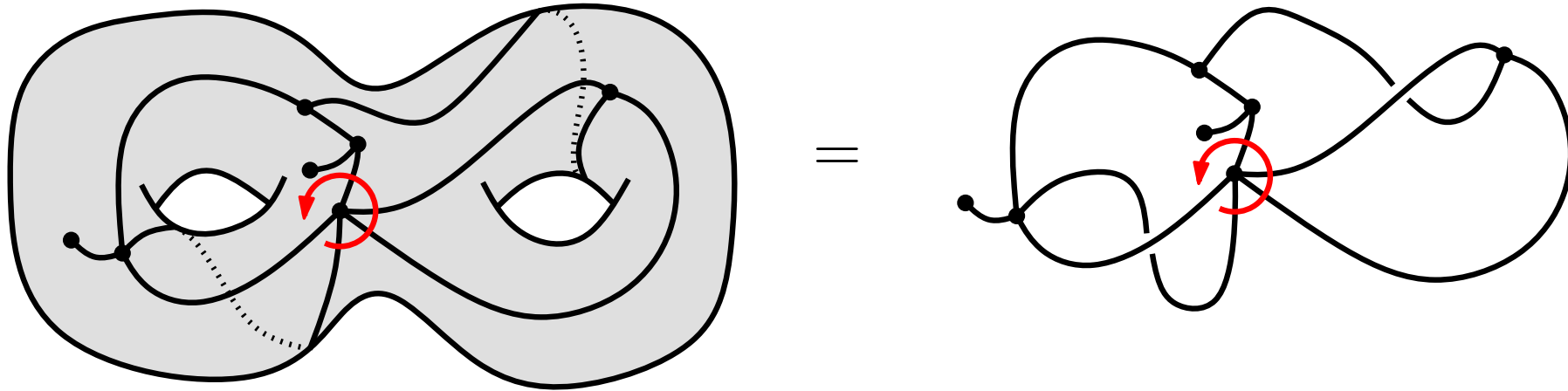
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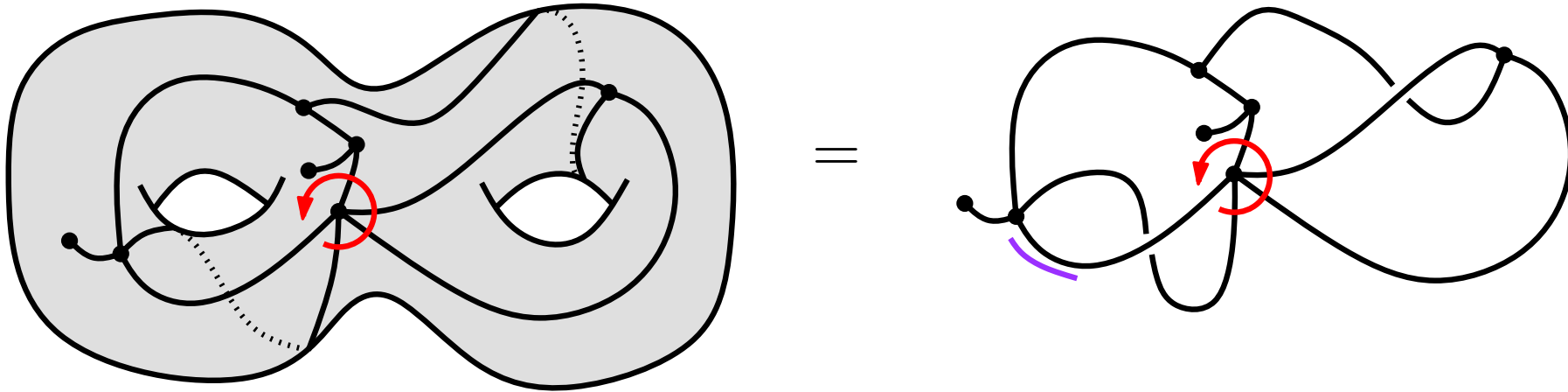
Note for experts: the Goulden-Nica bijection does not solve the same problem (it solves a "Poissonized" version of the problem).

Map = graph + rotation system



All the information is contained in the pair formed by the **graph** and the **cyclic ordering** of edges around each vertex.

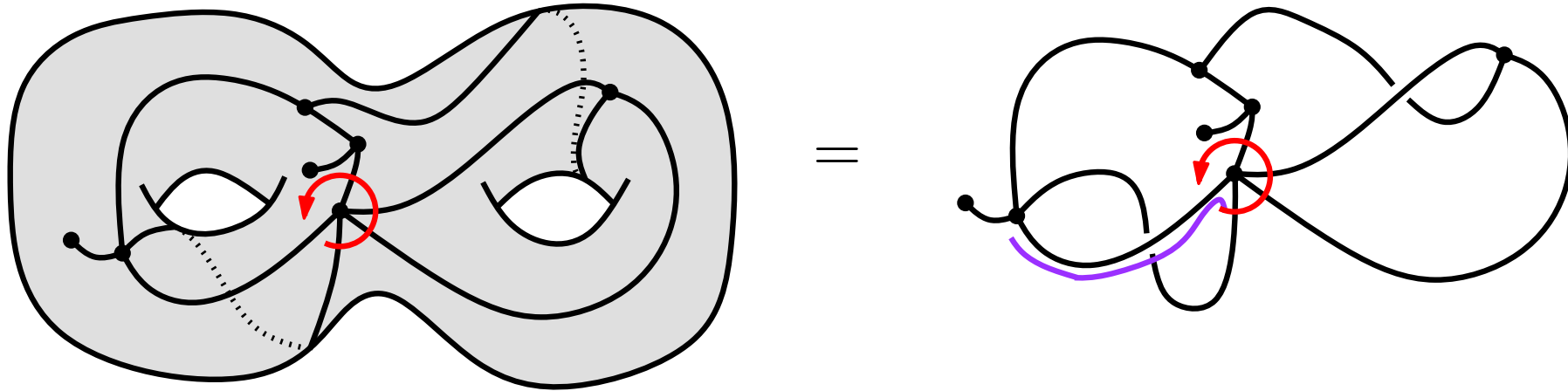
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 \Rightarrow the graph has **only one border**

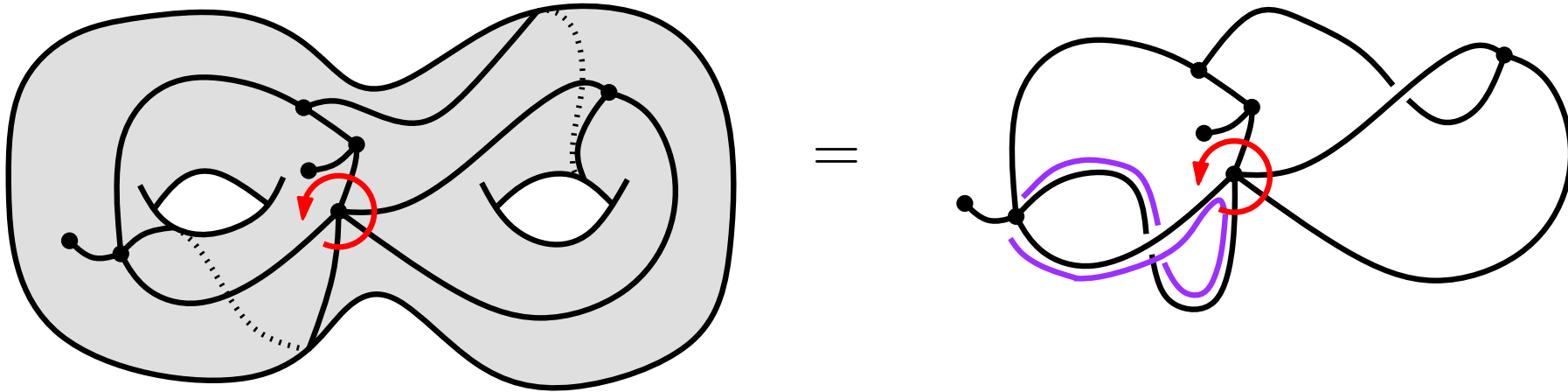
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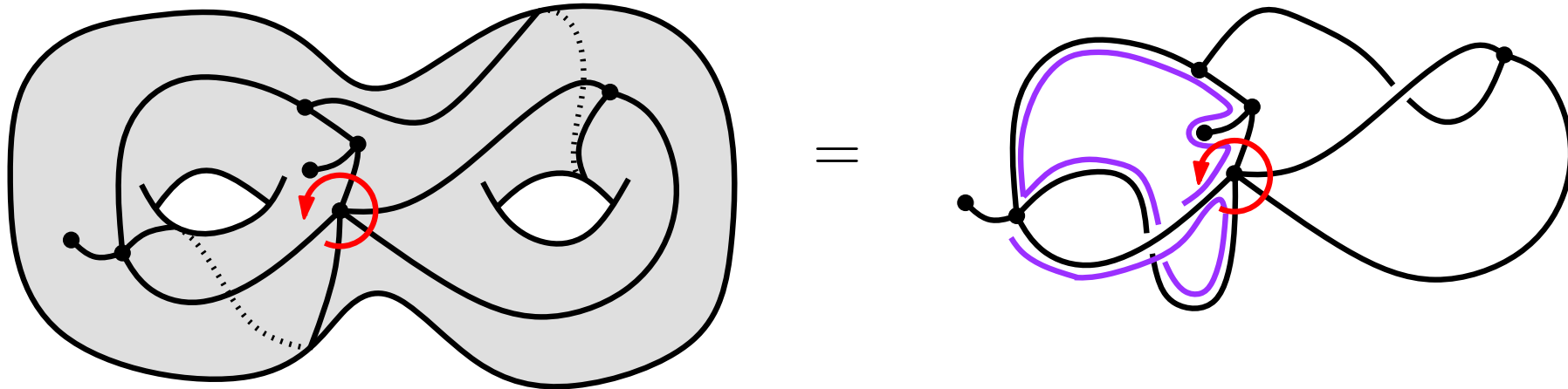
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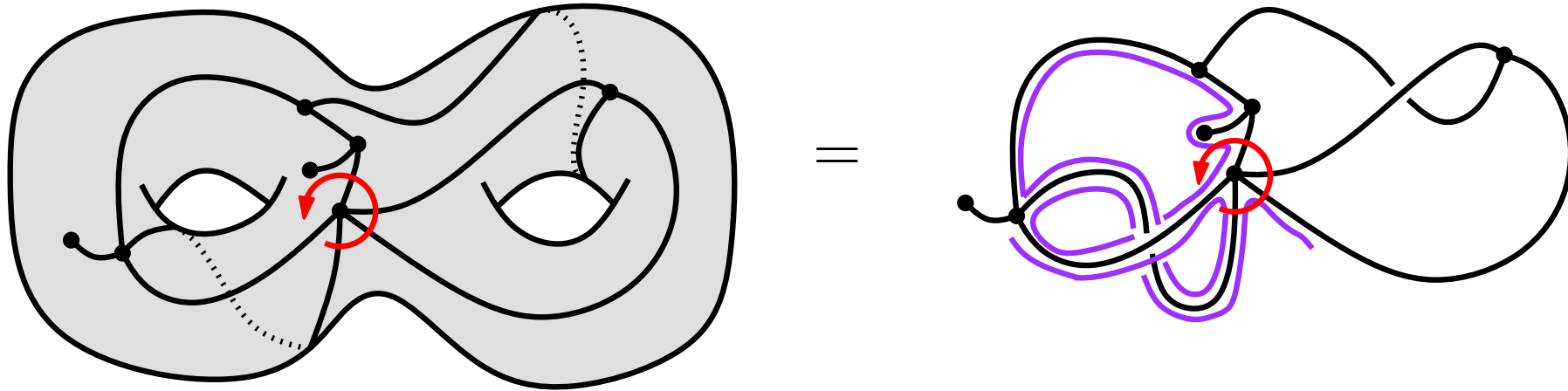
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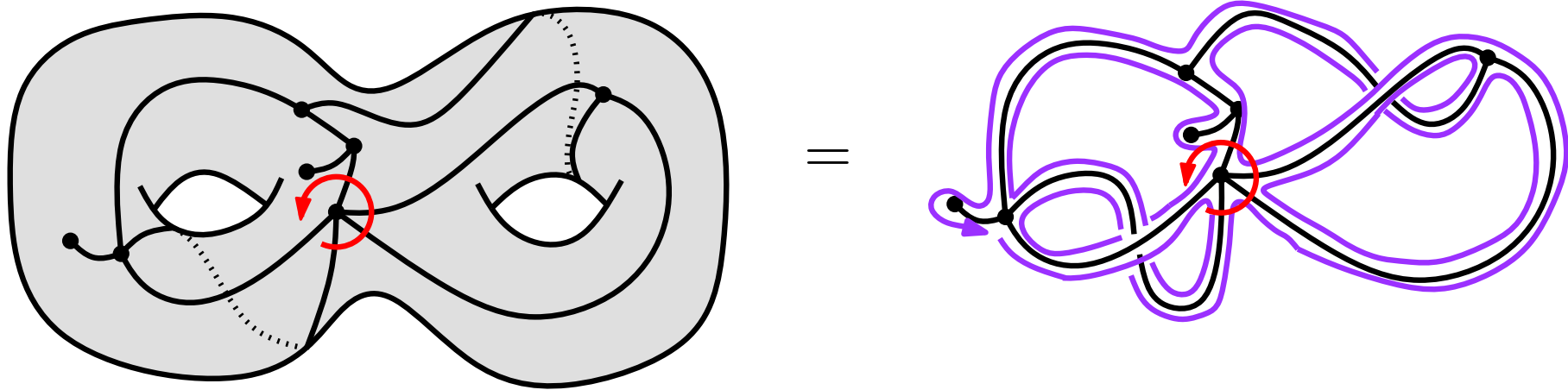
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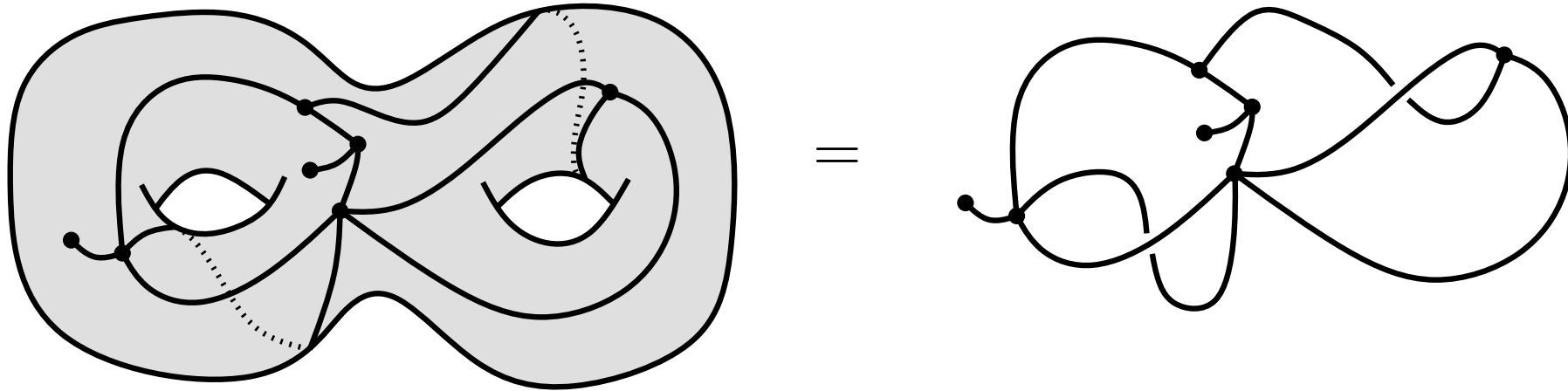
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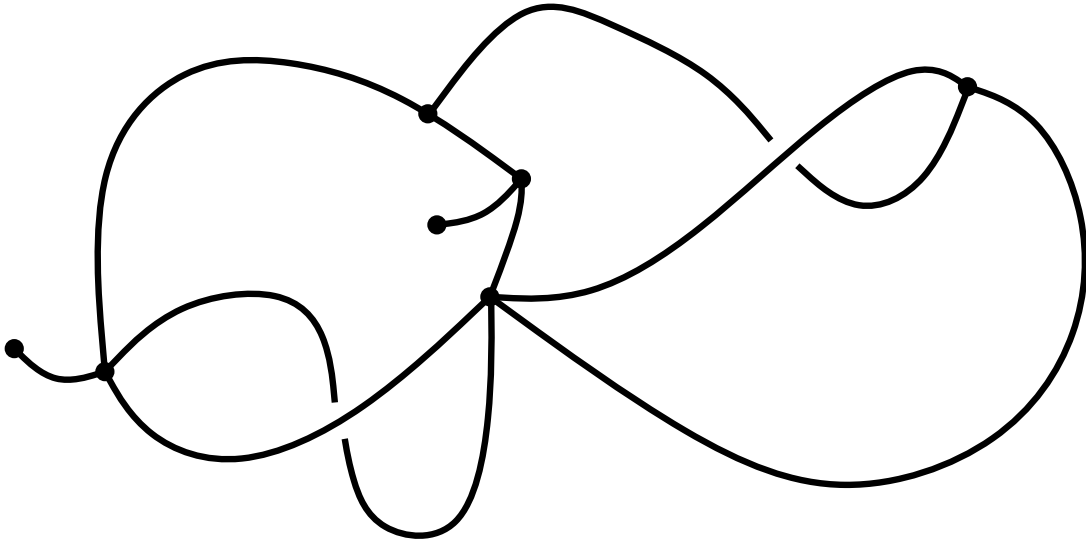
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To do: **cut** the **$2g$ independant cycles** of this graph in order to obtain a tree. Problem: **where** to cut ?

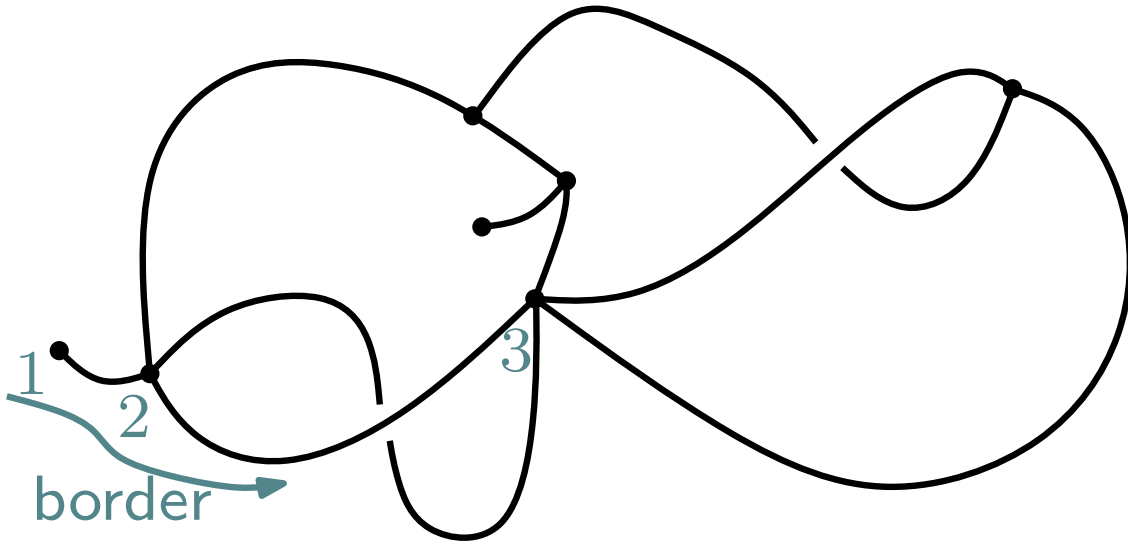
Numbering the corners.

We follow the border of the map starting from the root, and we **number the corners** from 1 to $2n$.



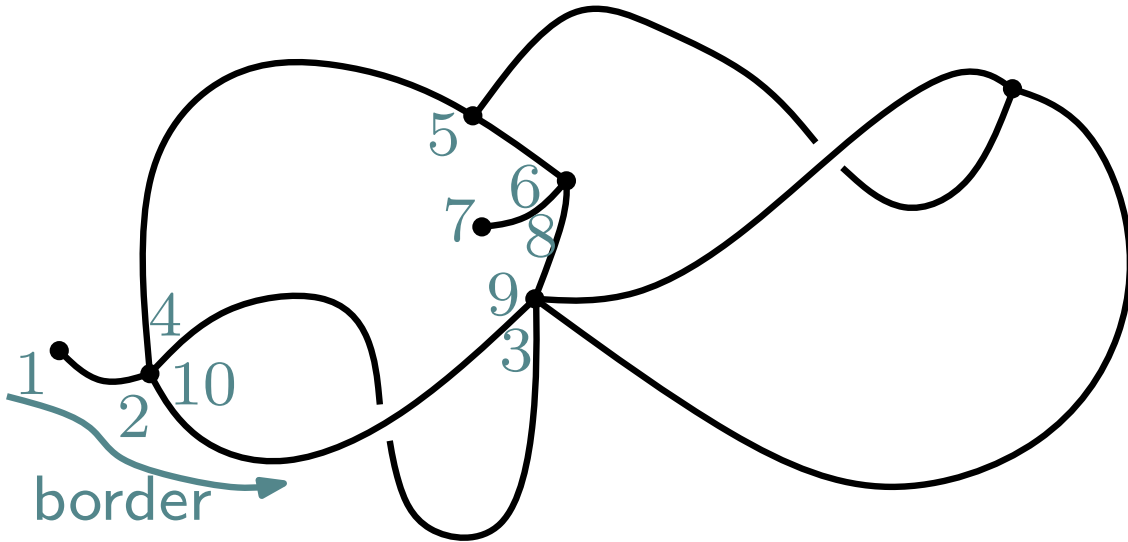
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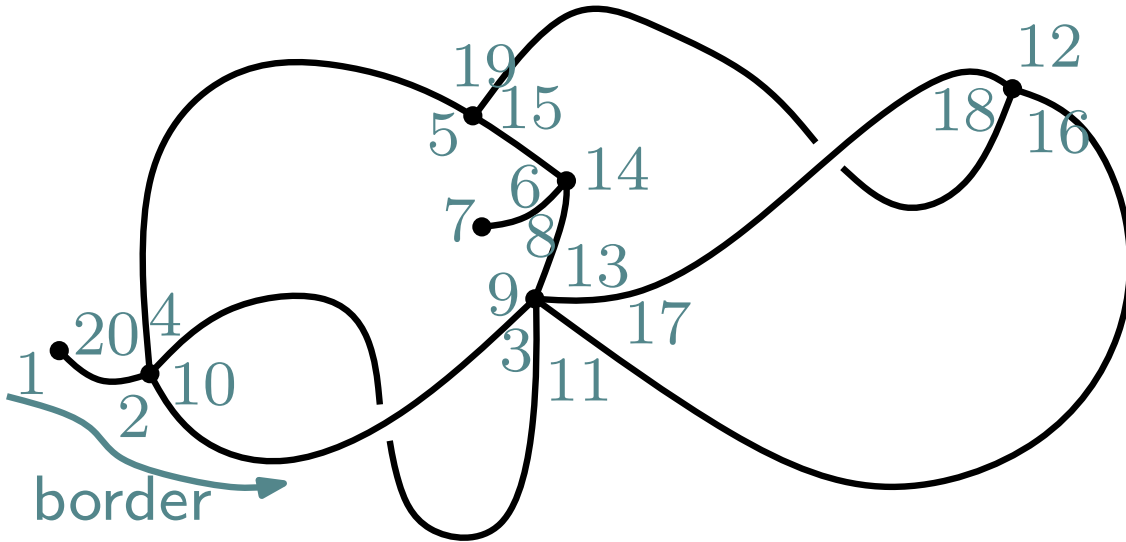
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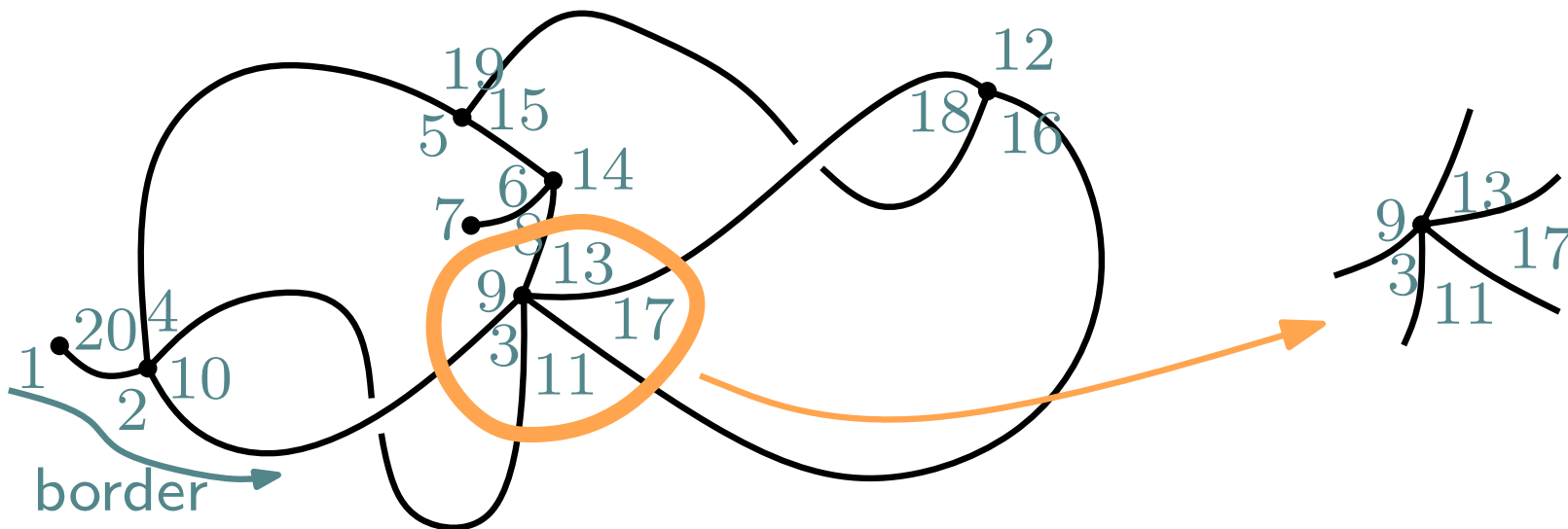
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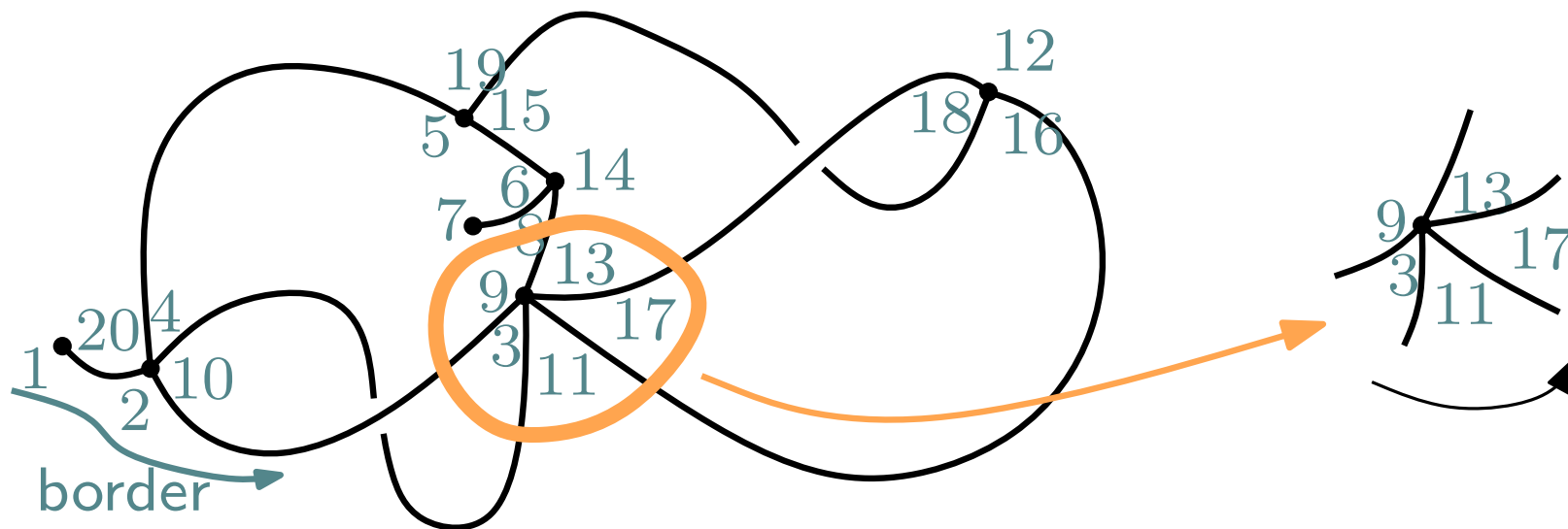
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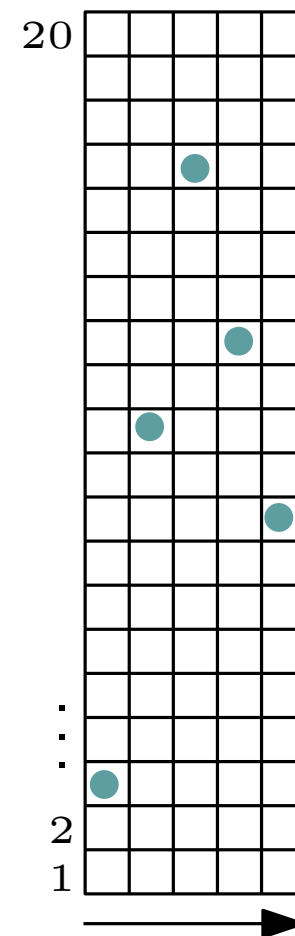
We compare the two natural orderings of corners **around one vertex**: this gives a diagram.

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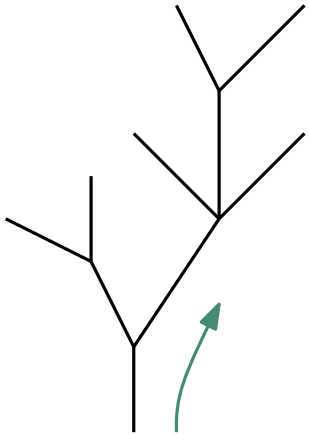


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Planar case

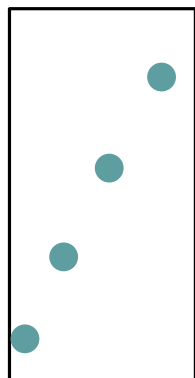
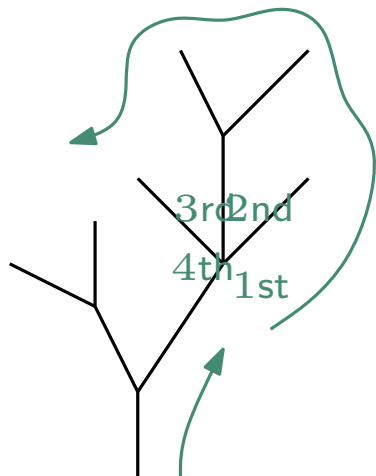
In the planar case, the border-numbering and the cyclic ordering **always coincide**:



Planar case

In the planar case, the border-numbering and the cyclic ordering **always coincide**:

For each vertex, the diagram is **increasing**:



The trisection lemma

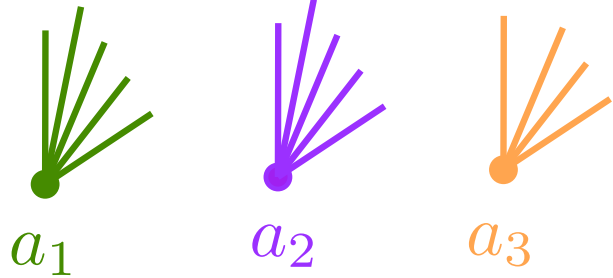
A unicellular map of genus g always has exactly $2g$ trisections.

Proof: simple counting argument.

→ It is an equivalent problem to count unicellular maps with a distinguished trisection.

How to build a trisection : first method.

- Start with a map of genus $(g - 1)$ with three marked vertices.
- Let $a_1 < a_2 < a_3$ be the labels of their **minimal corners**.
- **Glue** these three corners together as follows :



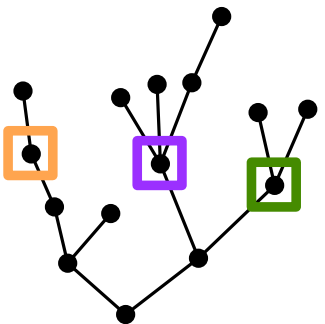
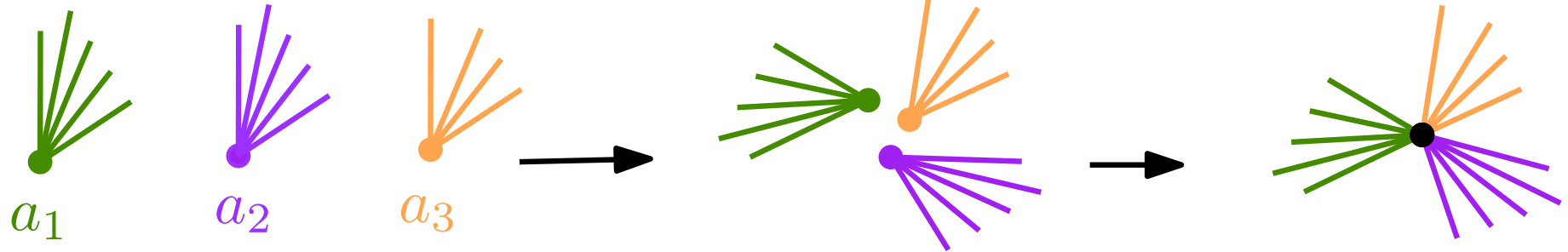
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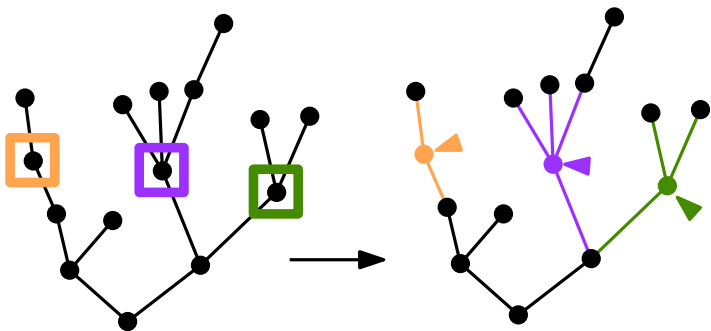
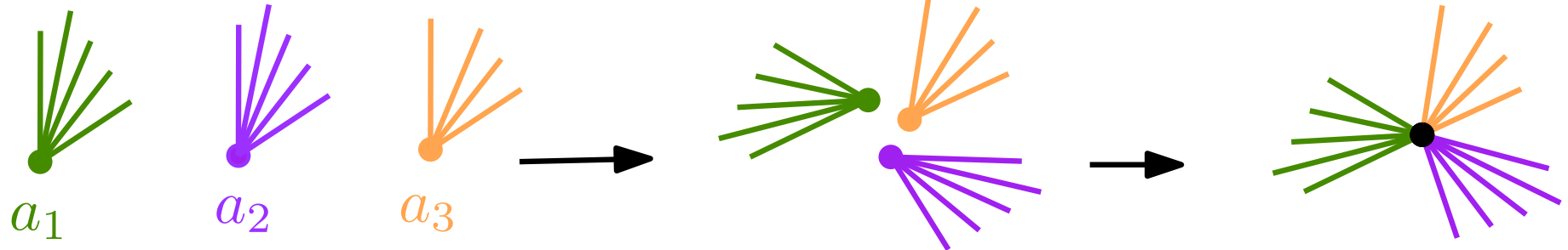
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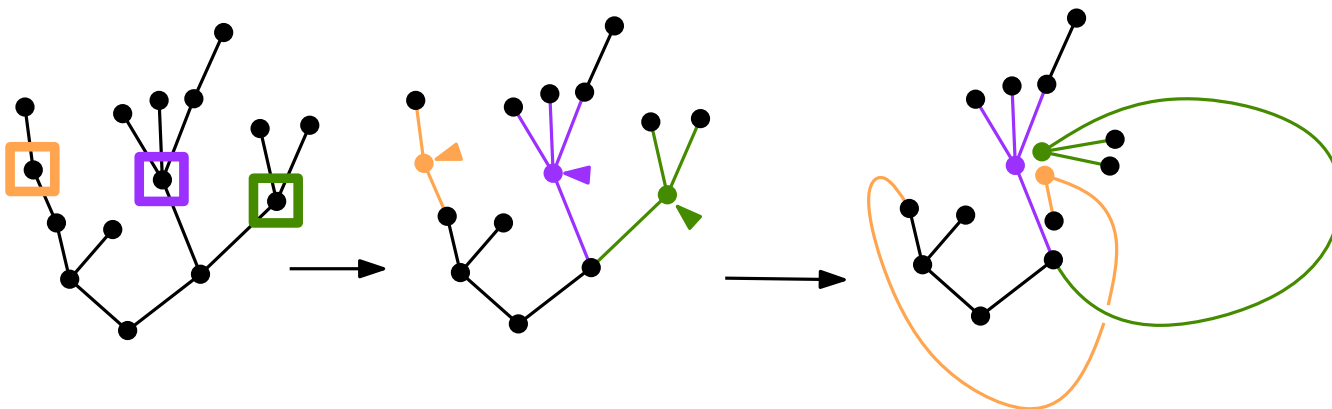
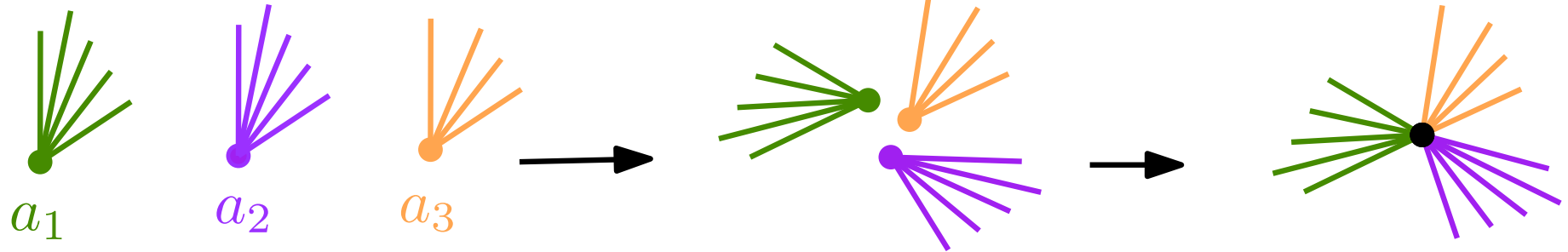
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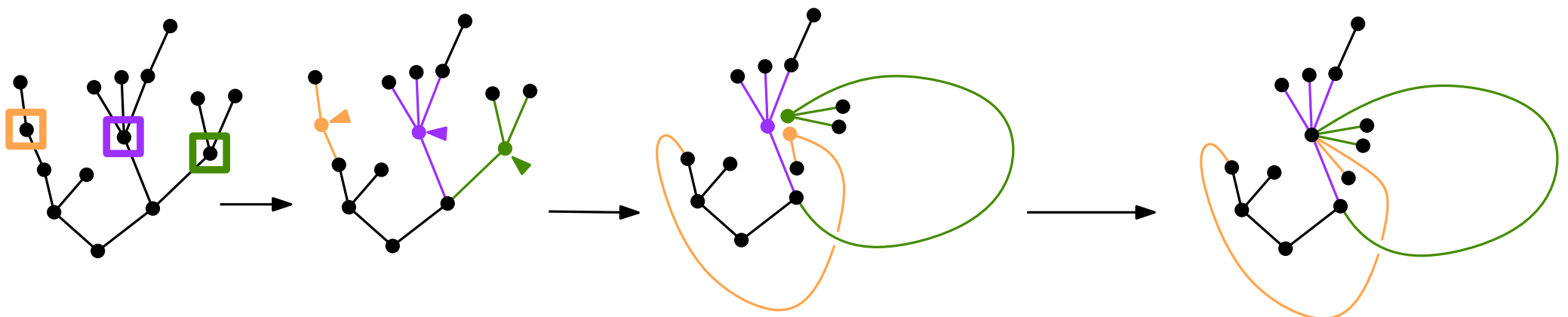
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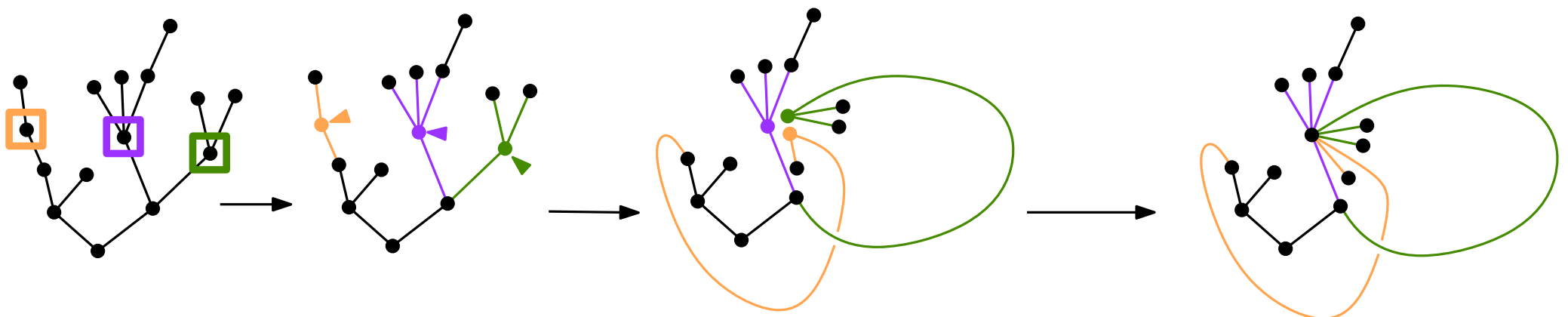
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- The resulting map has **only one border** :

$$1 \rightarrow 2 \rightarrow \dots \rightarrow a_1 \rightarrow \dots \rightarrow a_2 \rightarrow \dots \rightarrow a_3 \rightarrow \dots \rightarrow 2n$$

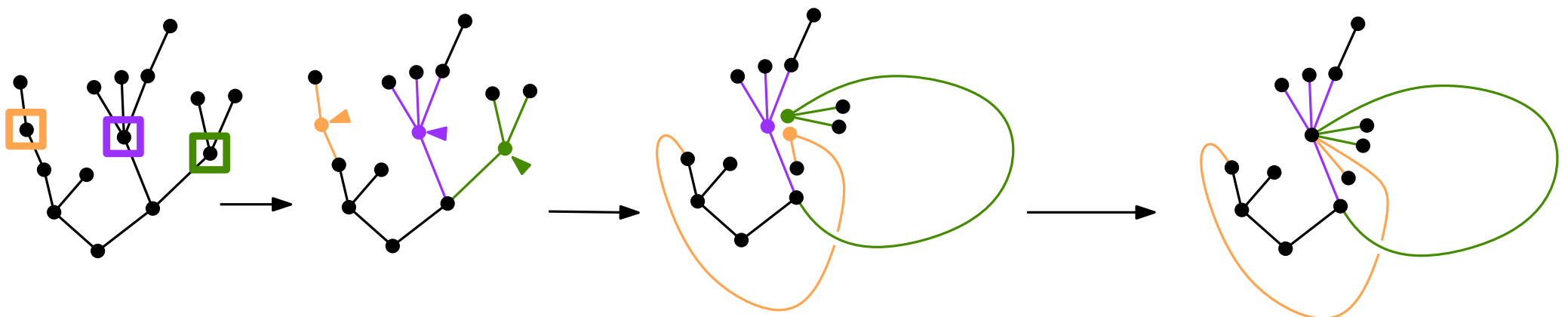
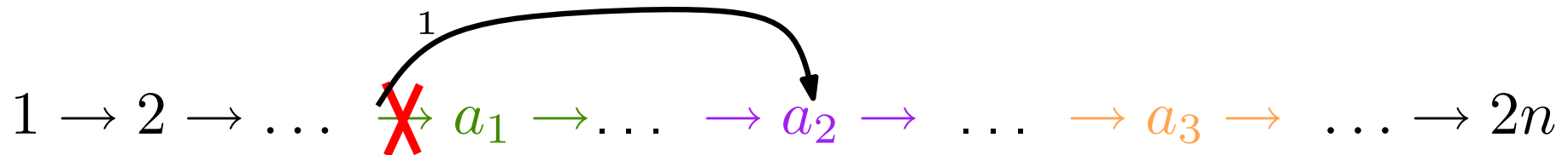


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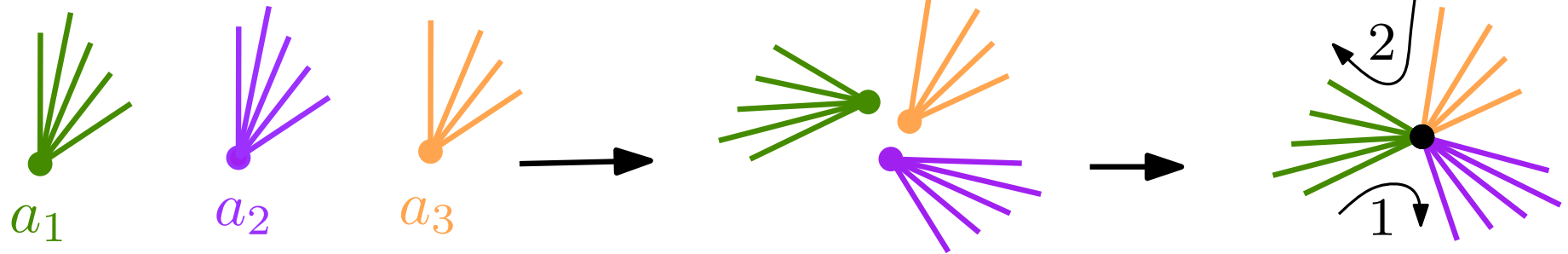


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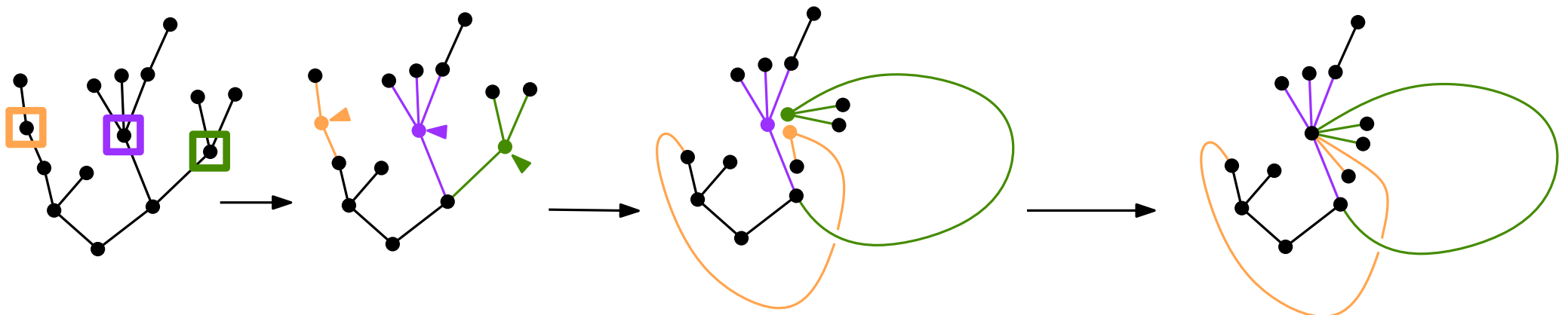
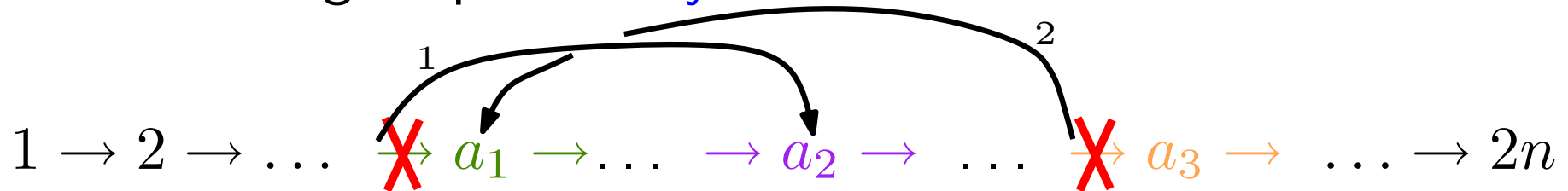


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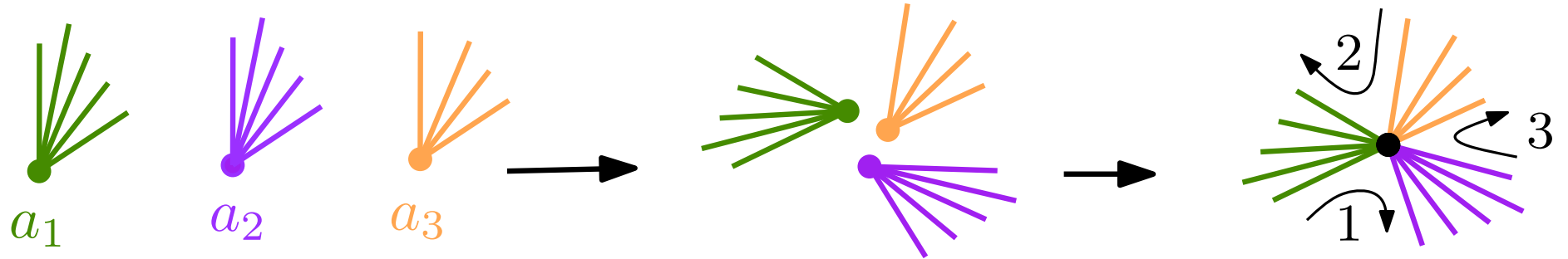


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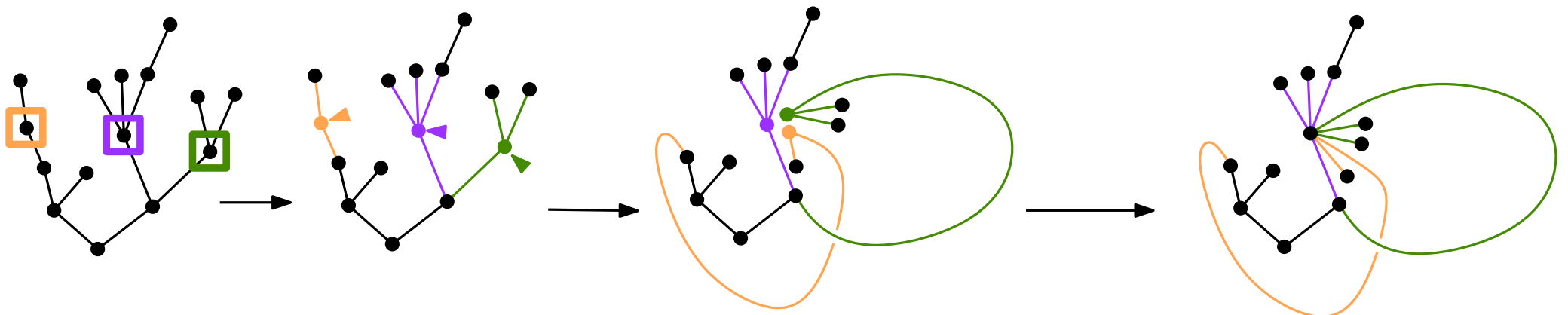
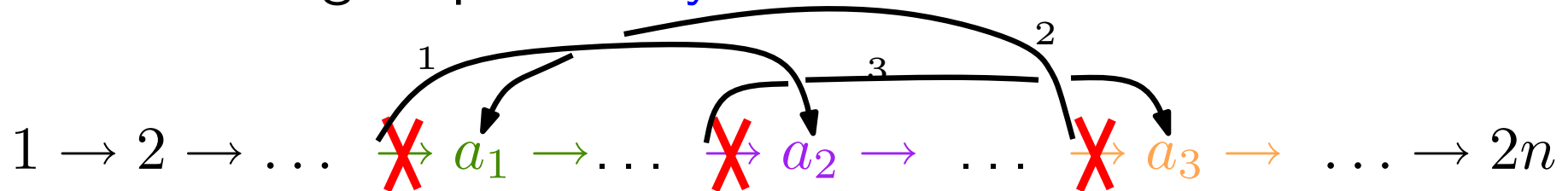


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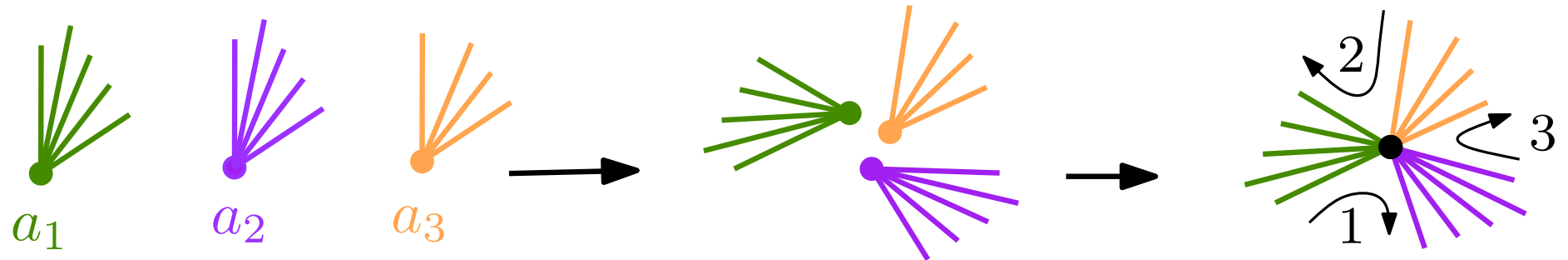


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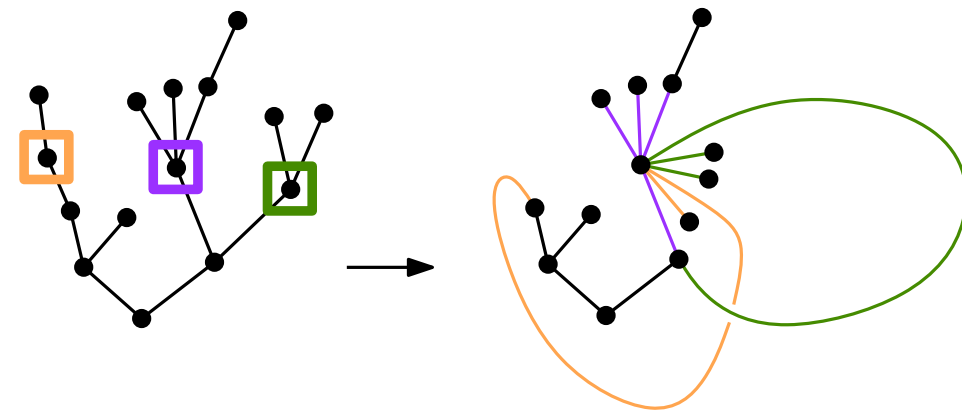
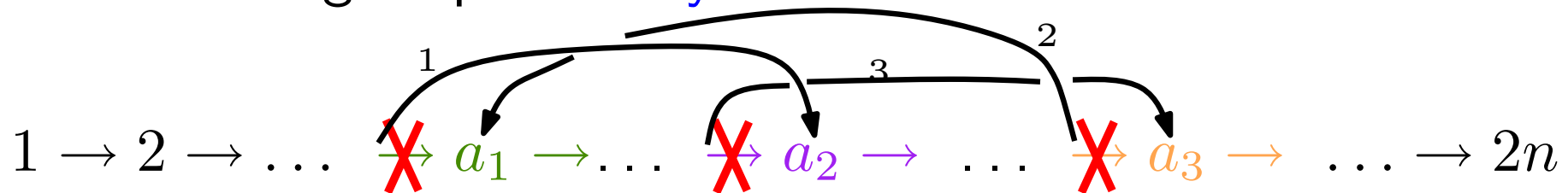


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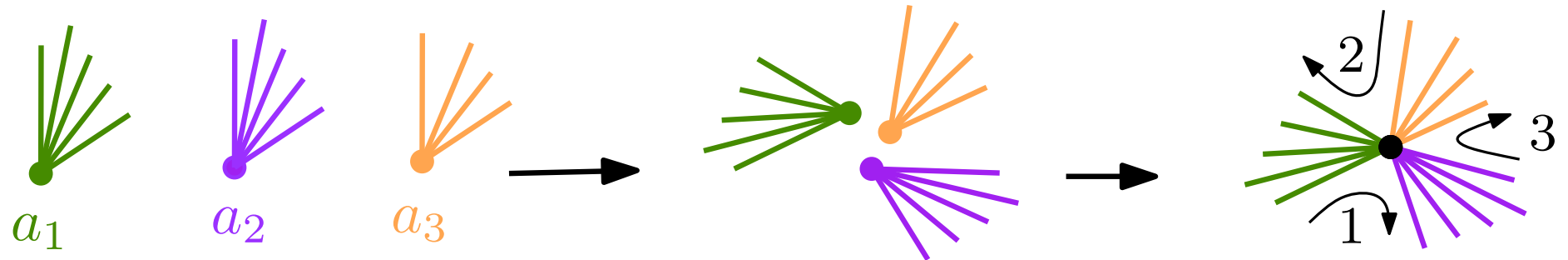


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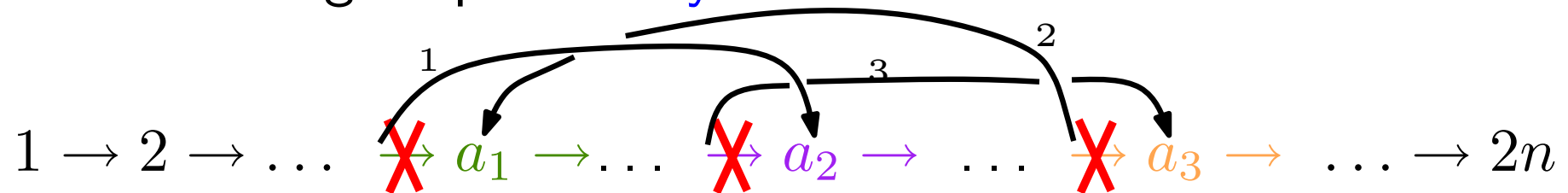


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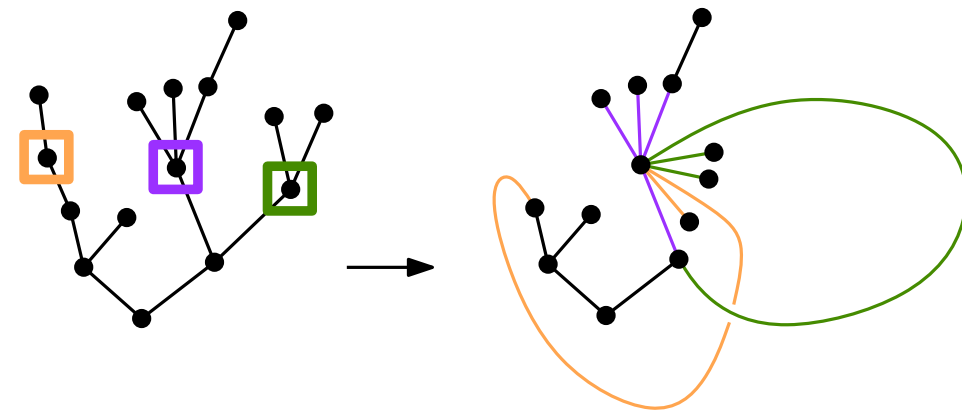
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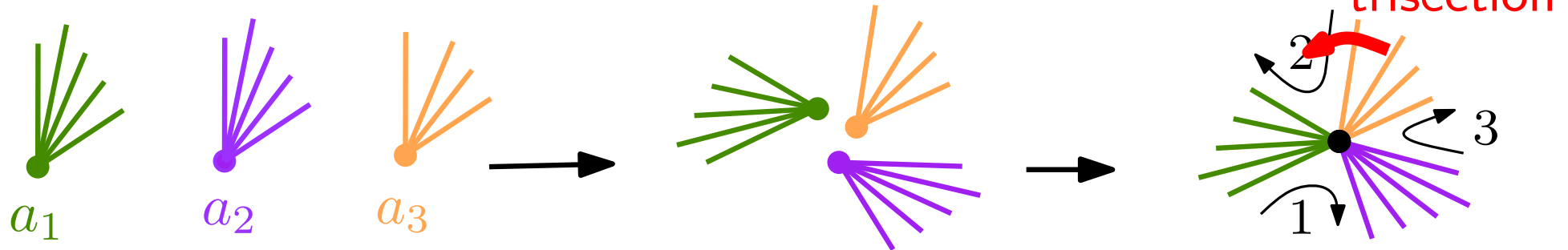


- By Euler's formula, it has **genus g** .

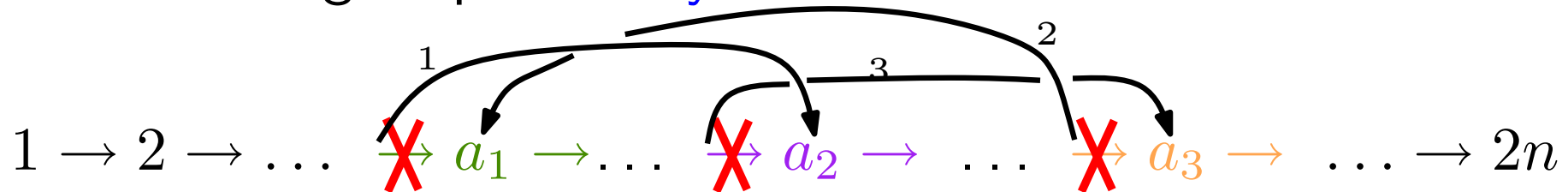


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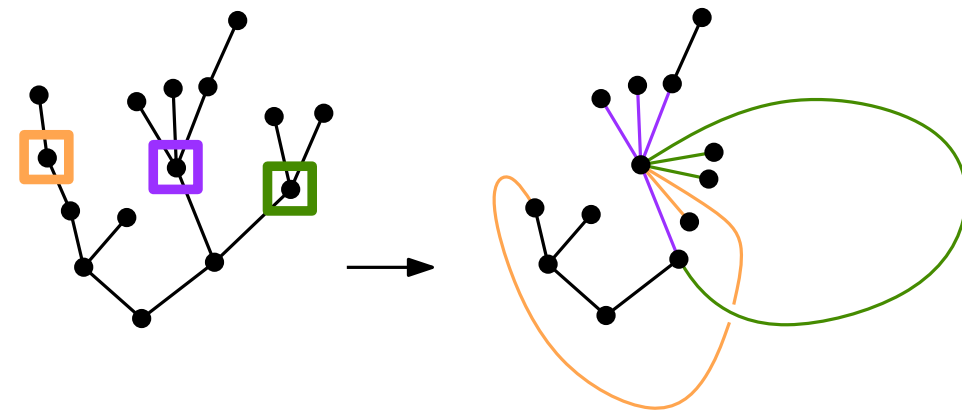


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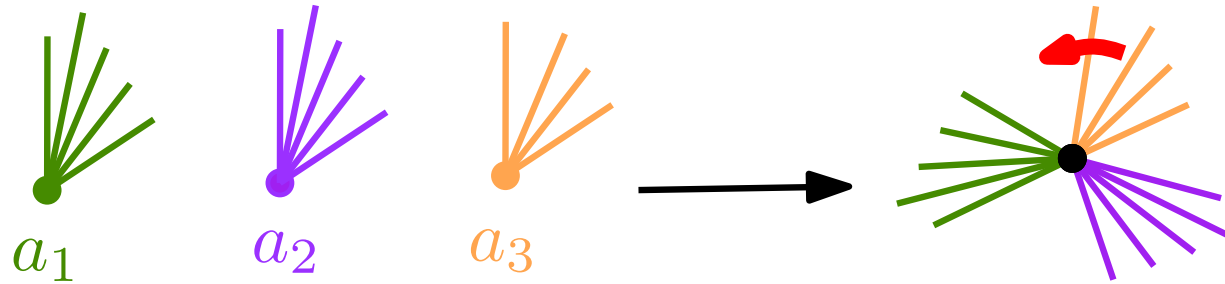


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- Moreover we have built a **trisection**.



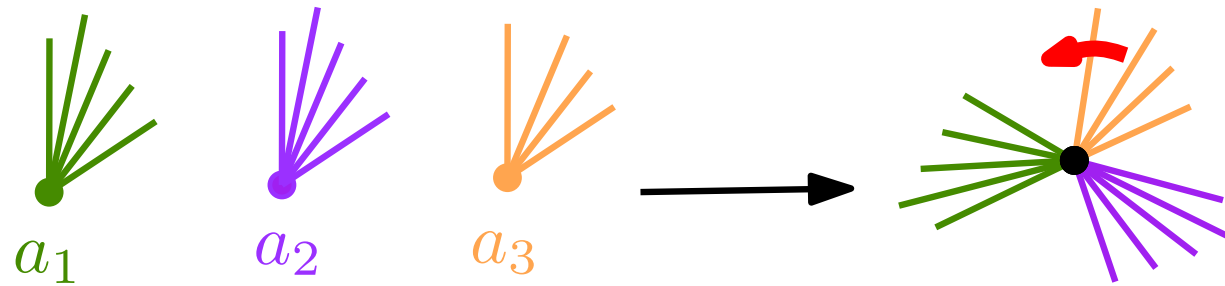
Therefore we have a mapping :



genus $g - 1$, three
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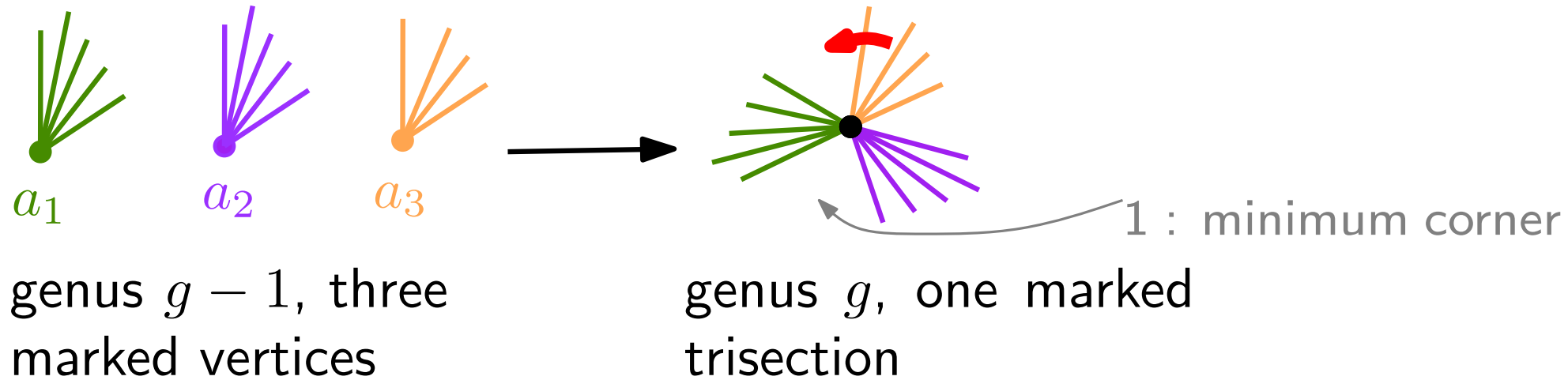


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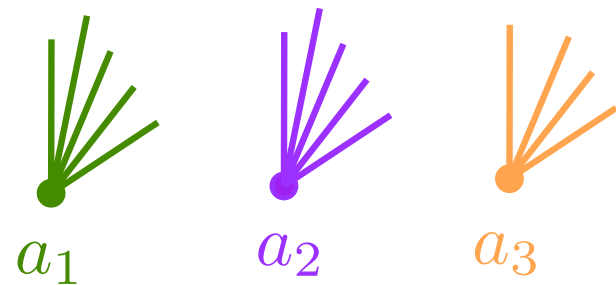
The mapping is **injective** because we can retrieve the three corners, and **cut** the vertex back.

Therefore we have a mapping :

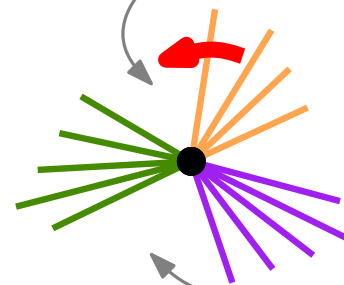


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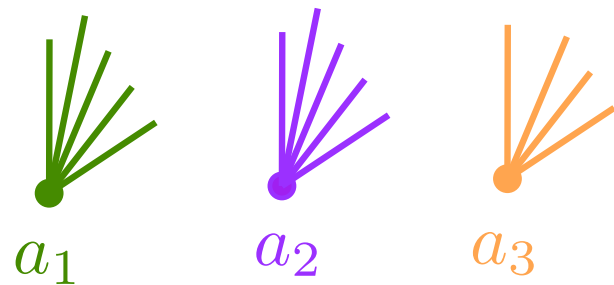
genus $g - 1$, three
marked vertices



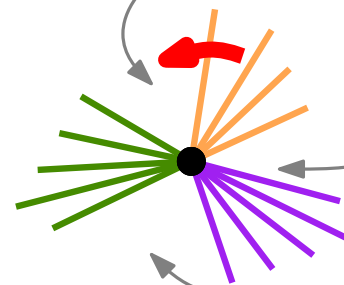
genus g , one marked
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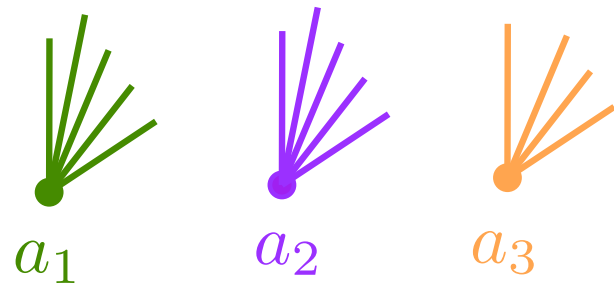
2: corner following the marked trisection

3: smallest corner between 2 and 1 which is greater than 2

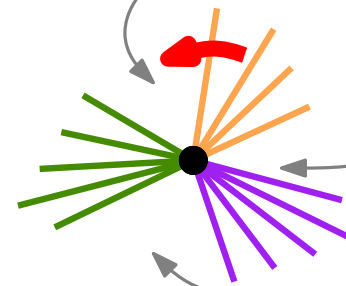
1 : minimum corner

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genus $g - 1$, three marked vertices



genus g , one marked trisection

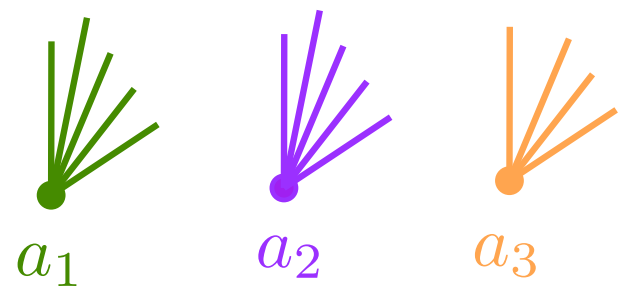
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Hence :

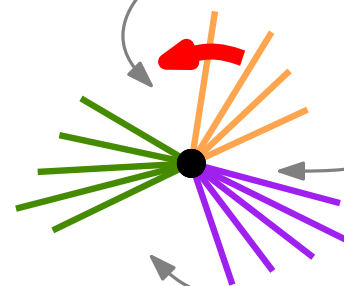
$$2g \cdot \epsilon_g(n) = \binom{n + 3 - 2g}{3} \epsilon_{g-1}(n) + \dots$$

↑
↑
 genus g genus $g - 1$
 marked trisection 3 marked vertices

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genus $g - 1$, three marked vertices



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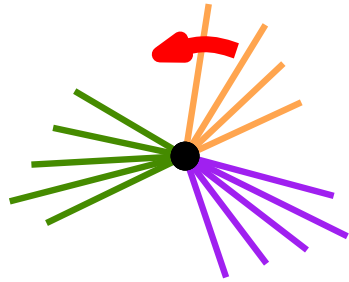
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genus g
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↑
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3 marked vertices

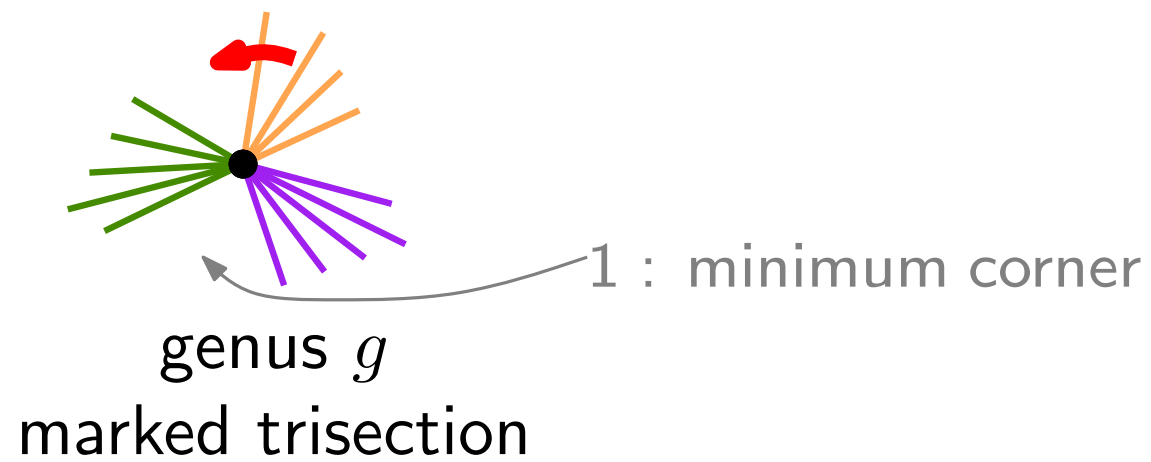
...
?

Let's try the reverse mapping...

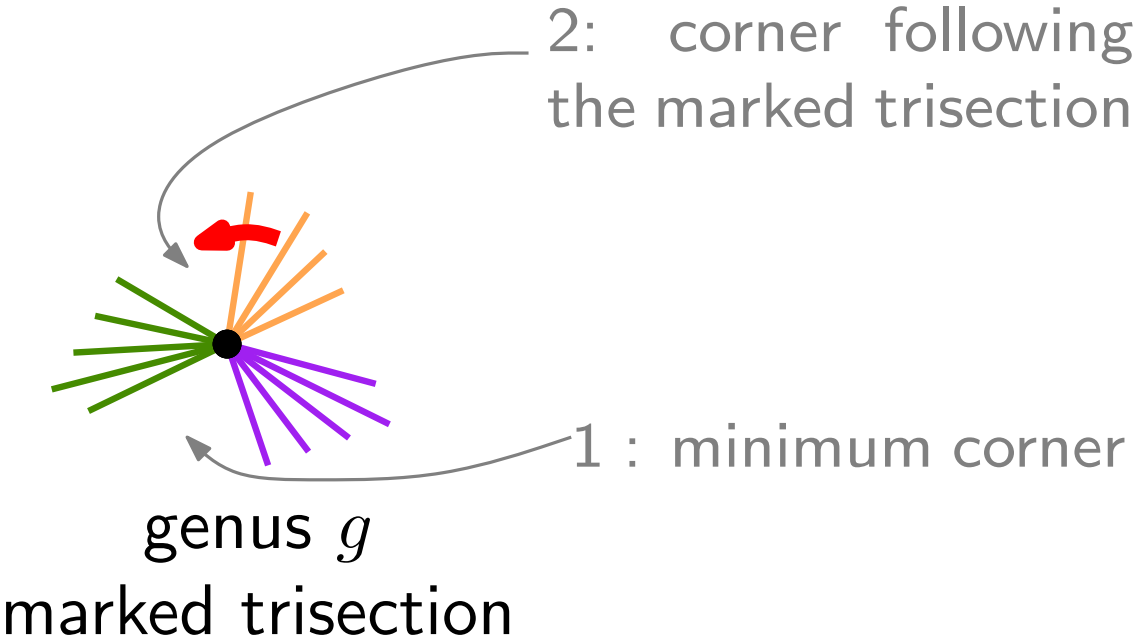


genus g
marked trisection

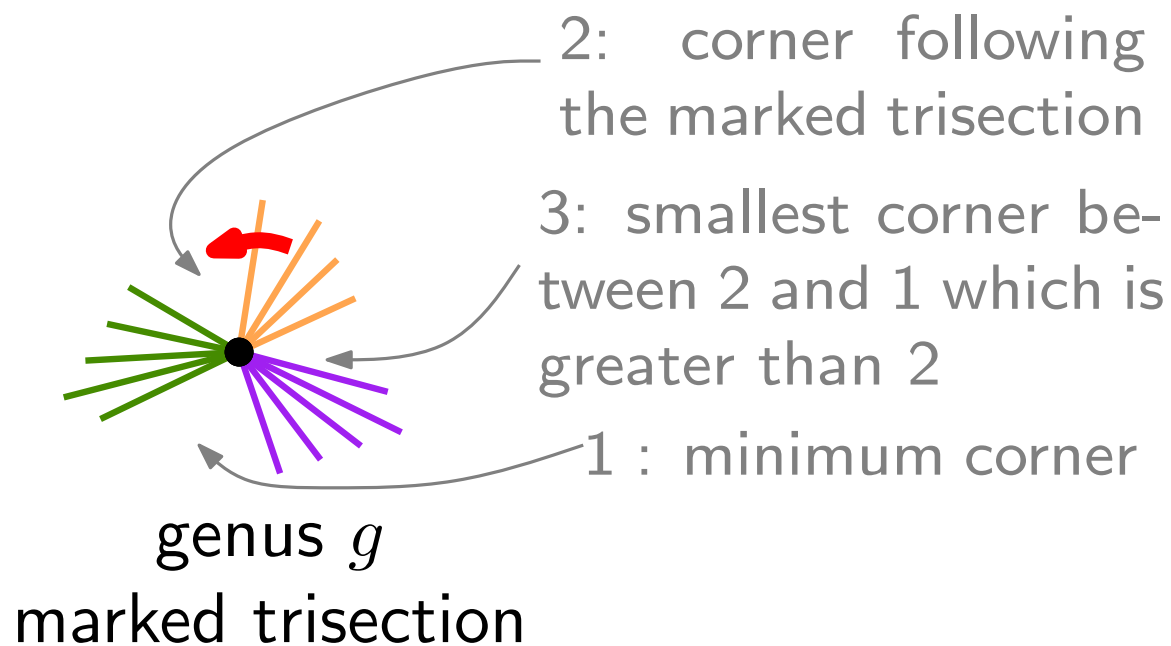
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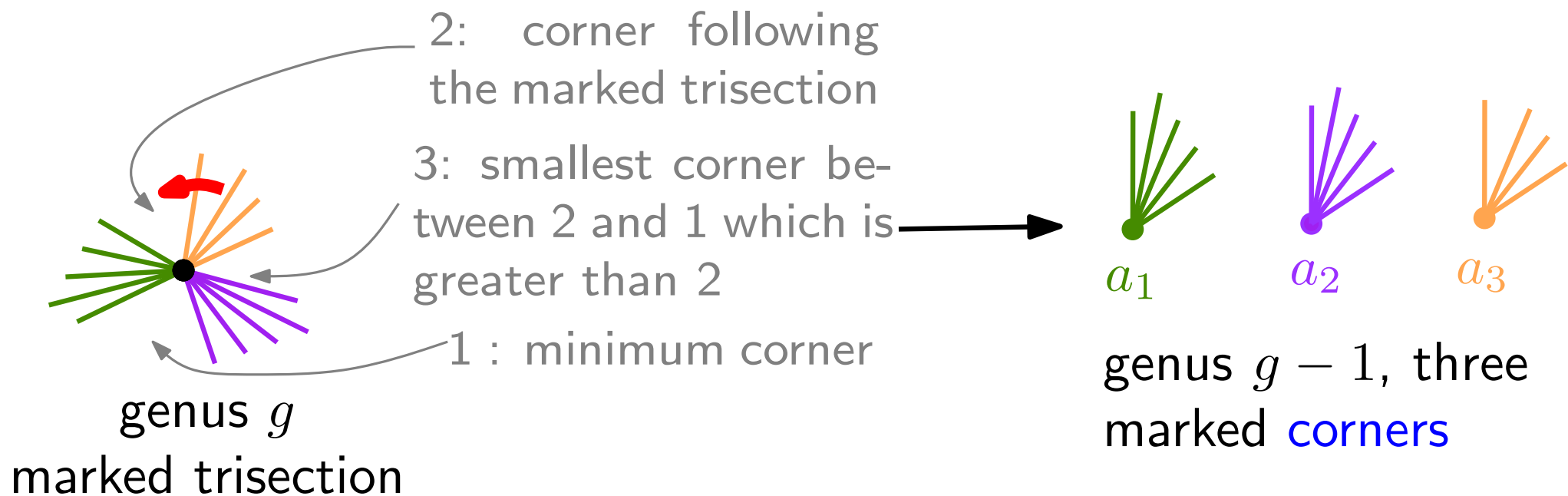
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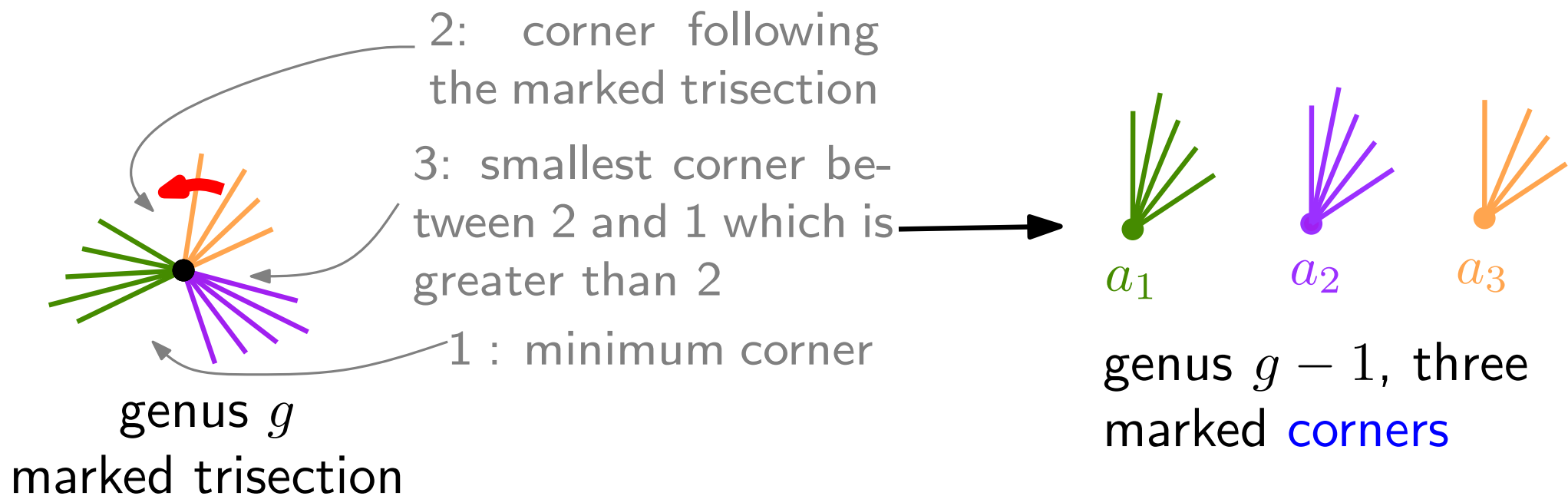
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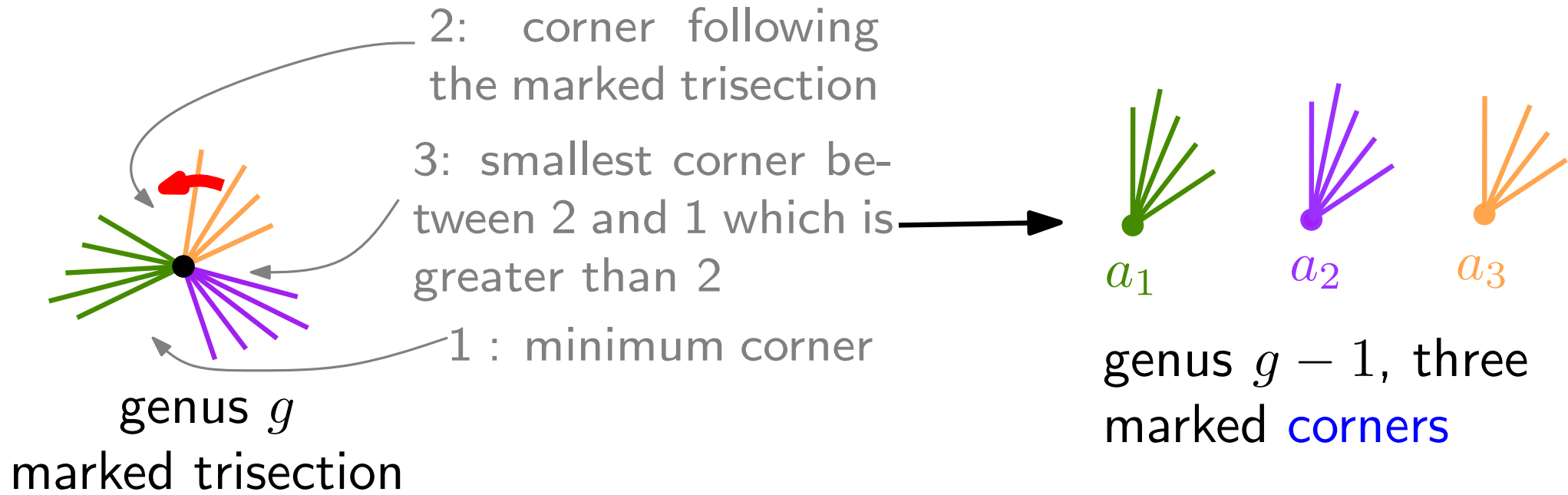


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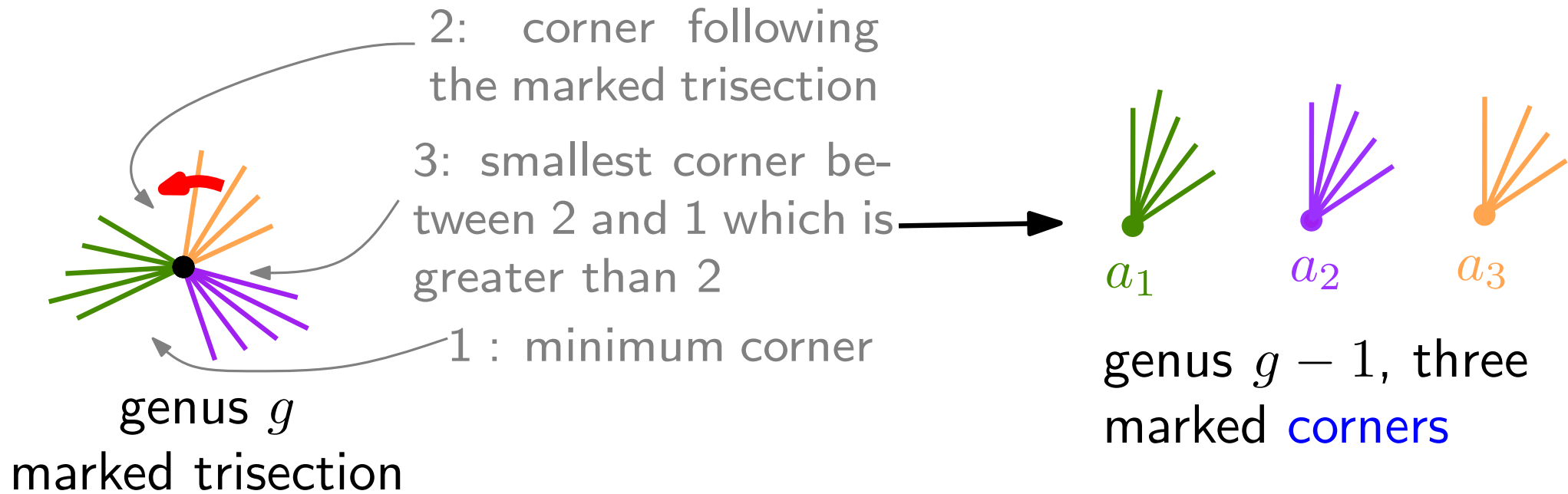
- We still have $a_1 < a_2 < a_3$ in the map of genus $(g - 1)$.

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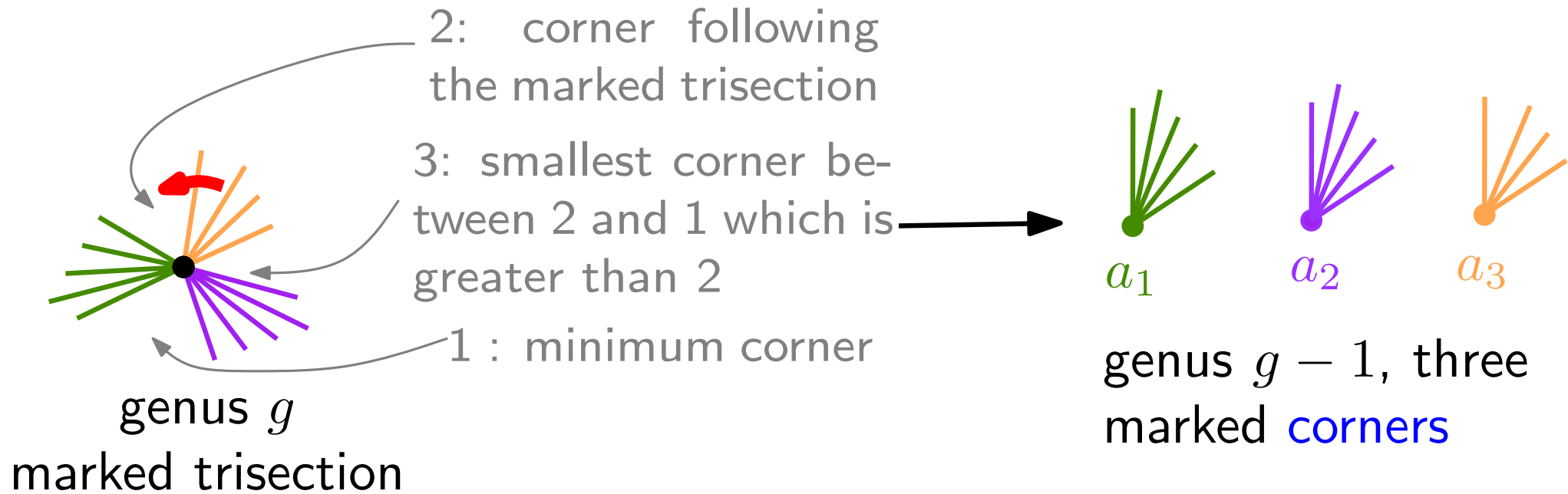
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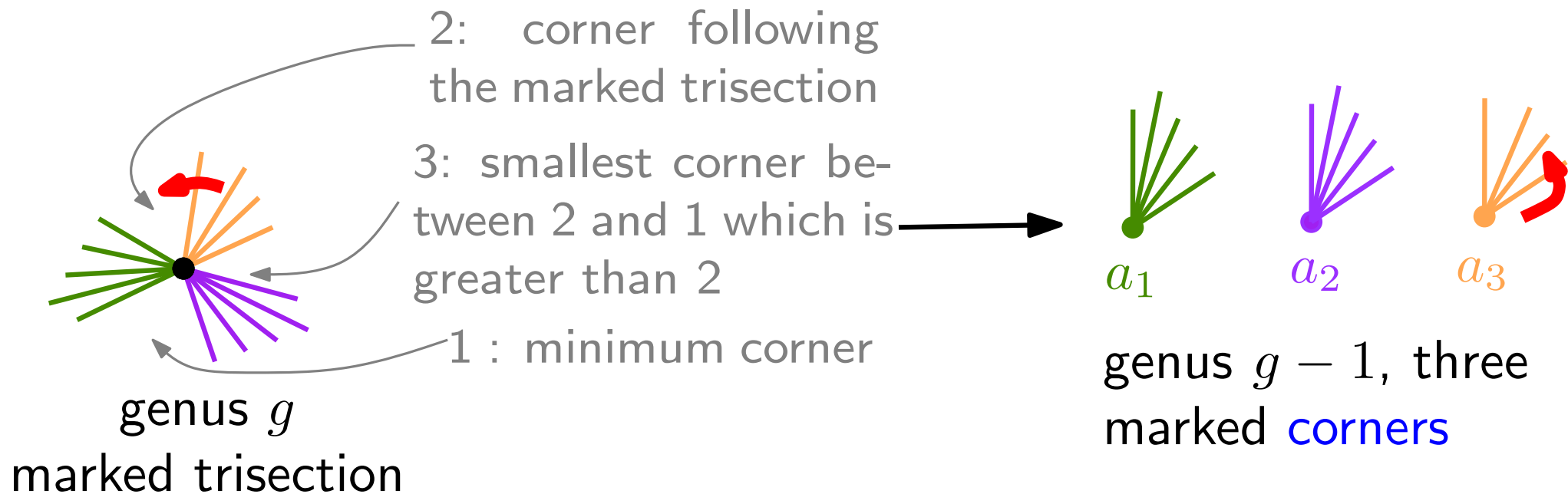
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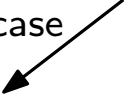
Let's try the reverse mapping...



- We still have $a_1 < a_2 < a_3$ in the map of genus $(g - 1)$.
- a_1 and a_2 are both the **minimum corner** in their vertex.
- This is **not always** the case for a_3 :
 - If a_3 is the **minimum** of its vertex : we are in the image of the previous construction.
 - Else a_3 is incident to a **trisection** of the map of genus $(g - 1)$.

Therefore :

genus g , one marked
trisection

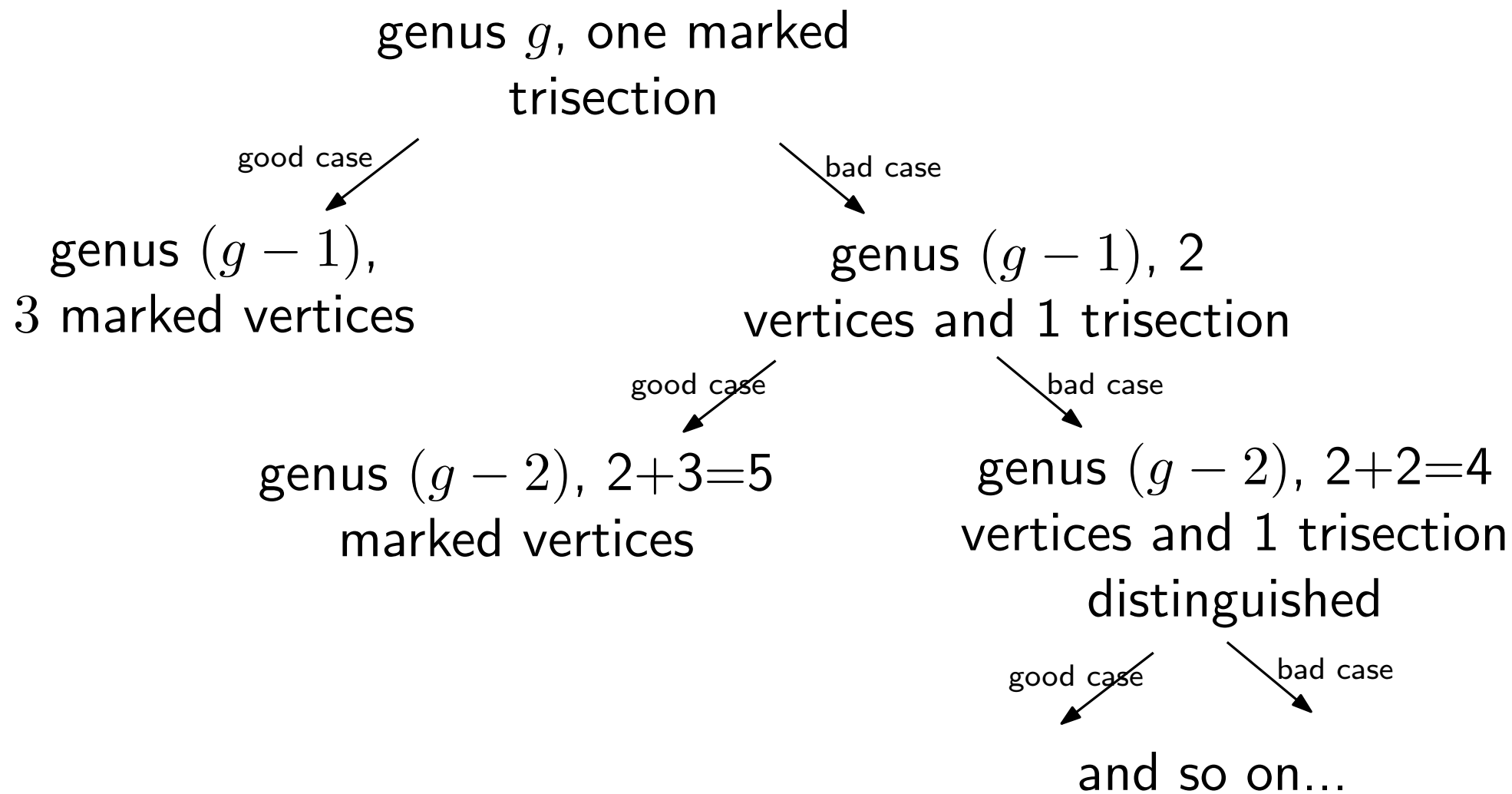
good case


genus $(g - 1)$,
3 marked vertices

bad case


genus $(g - 1)$, 2
vertices and 1 trisection

Therefore :



Our main result:

genus g ,
one marked trisection

$$\stackrel{\text{bij.}}{=} \bigcup_{i > 0} \left(\begin{array}{l} \text{genus } g-i \text{ and } 2i+1 \\ \text{marked vertices.} \end{array} \right)$$

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Everything boils down to plane trees:

$$\epsilon_g(n) = \underbrace{(\text{some polynomial})}_{\text{number}} \times \text{Cat}(n)$$

= "number" of possibilities for the successive choices of vertices.

$$= \sum_{0=g_0 < g_1 < \dots < g_r = g} \prod_{i=1}^r \frac{1}{2g_i} \binom{n+1-2g_{i-1}}{2(g_i - g_{i-1}) + 1}$$

For instance :

$$2 \cdot \epsilon_1(n) = \frac{(n+1)n(n-1)}{6} \text{Cat}(n)$$

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$$\begin{aligned} 4 \cdot \epsilon_2(n) &= \frac{(n-1)(n-2)(n-3)}{6} \epsilon_1(n) + \frac{(n+1)n(n-1)(n-2)(n-3)}{5!} \text{Cat}(n) \\ &= \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440} \text{Cat}(n) \end{aligned}$$

Extensions

- The formula leads to a differential equation which enables to recover the known **closed formulas for the generating functions** (Harer-Zagier, Itzykson-Zuber).
- Works the same for **bipartite** unicellular maps.
- The Marcus-Schaeffer bijection relates **general maps** on surfaces to **labelled unicellular maps**. The composition of the two bijections leads to a description of general maps of given genus in terms of **labelled trees with distinguished vertices**. This gives information about the **continuum limit of maps on surfaces** (Brownian map of genus g).

Thank you!

Formules d'Harer et Zagier :

Récurrance :

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2)$$

Version sommatoire :

$$\sum_{g \geq 0} \epsilon_g(n) y^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$