

A simple model of trees for unicellular maps

Guillaume Chapuy (LIAFA, Paris-VII)

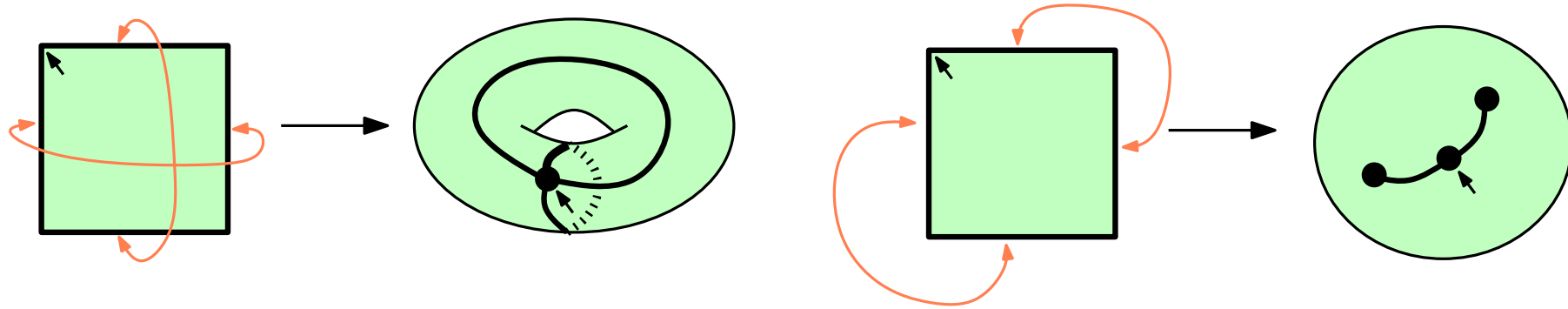
joint work with

Valentin Féray (LaBRI, Bordeaux-I)

Éric Fusy (LIX, Polytechnique)

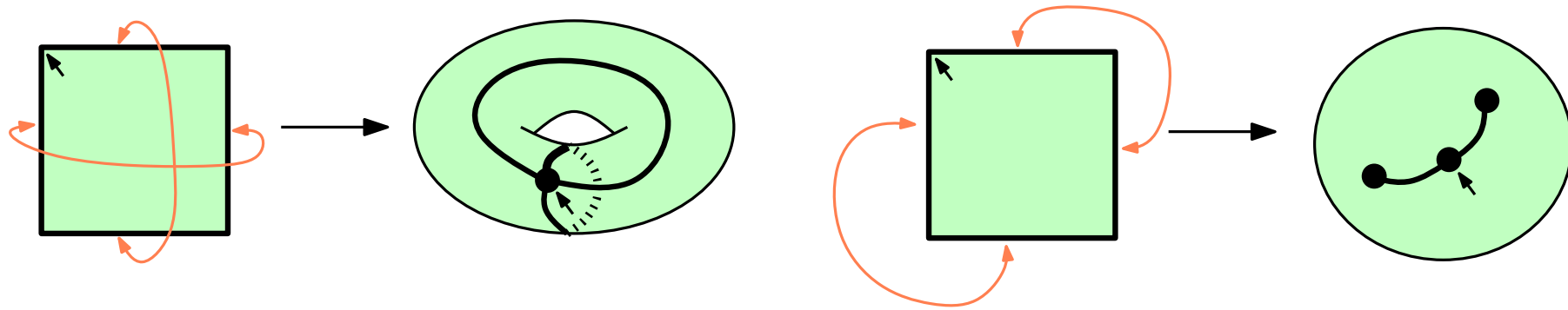
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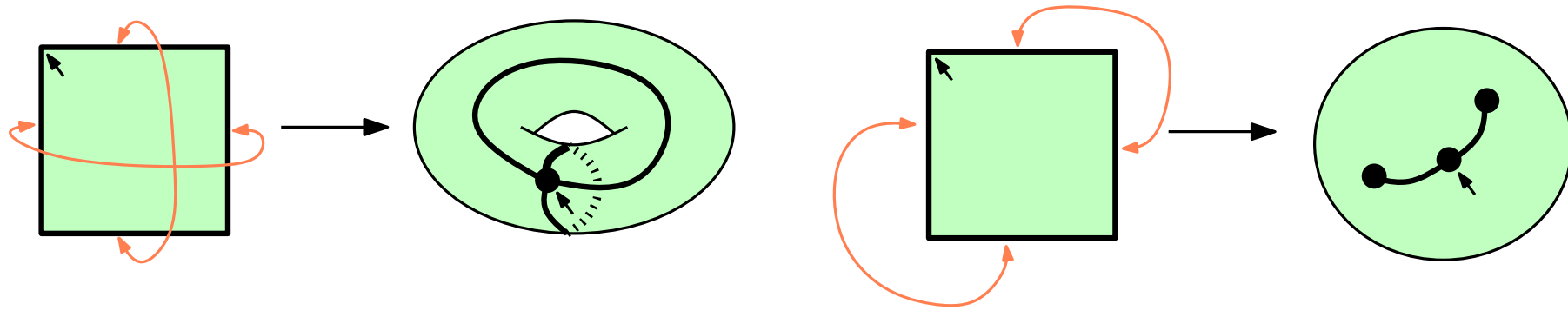


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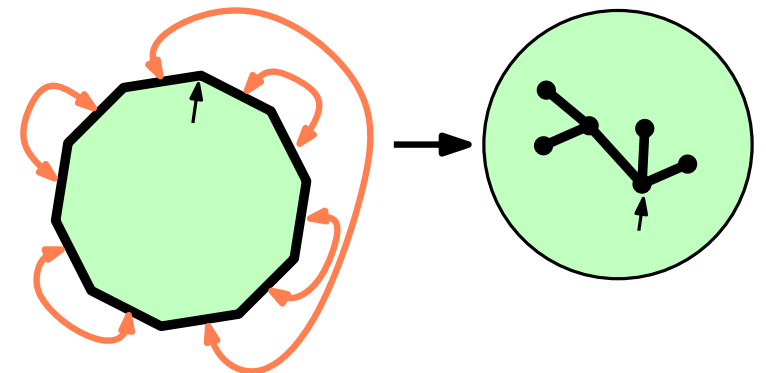


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- The number of unicellular maps of size n is $(2n - 1)!!$

- What if we fix the genus? For example, on the sphere (genus 0), unicellular maps = plane trees... so there are $\text{Cat}(n)$ of them.



Unicellular maps: counting!

- Let $\epsilon_g(n)$ be the number of unicellular maps with n edges and genus g .

- Are these numbers interesting ? Yes!

$$\epsilon_0(n) = \text{Cat}(n)$$

$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12} \text{Cat}(n)$$

$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440} \text{Cat}(n)$$

...

- These numbers are **connection coefficients** in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ (all map numbers are - but this is not really the point of this talk).

Unicellular maps: some chosen formulas

[Lehman-Walsh 72]

$$\epsilon_g(n) = \left(\sum_{\gamma \vdash g} \frac{(n+1-2g)2^{\ell(\gamma)+1}}{2^{2g} \prod_i m_i! (2i+1)^{m_i}} \right) \text{Cat}(n)$$

no bijective proof!

[Harer-Zagier 86]
(summation form)

$$\sum_{g \geq 0} \epsilon_g(n) y^{n+1-2g} = (2n-1)!! \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$

nice bijective proof [Bernardi10] building on [Lass 01, Goulden Nica 05]

[Harer-Zagier 86] (recurrence form)

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[Goupil-Schaeffer 98] for $\lambda \vdash 2n$, $\lambda = 1^{m_1} 2^{m_2} \dots$:

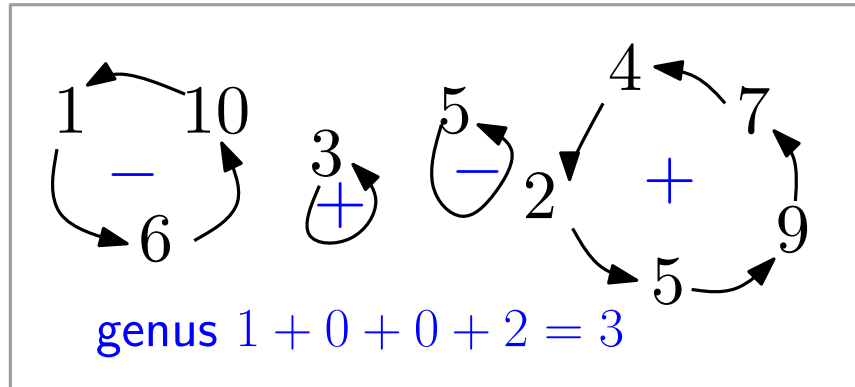
$$\epsilon_g(n; \lambda) = \frac{(l+2g-1)!}{2^{2g-1} \prod_i m_i!} \sum_{\gamma_1 + \gamma_2 + \dots + \gamma_l = g} \prod_i \frac{1}{2^{\gamma_i} + 1} \binom{\lambda_i - 1}{2\gamma_i}$$

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vertex degrees ↗

Our result, in short

- A C-permutation of a set S :

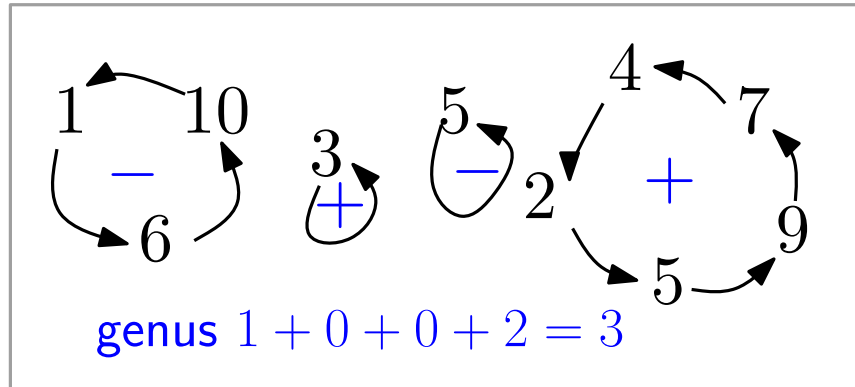


- all cycles have odd length
- each cycle carries a sign in $\{+, -\}$
- its genus is $g := \sum_i k_i$ where $(2k_i + 1)$ are the cycle-lengths

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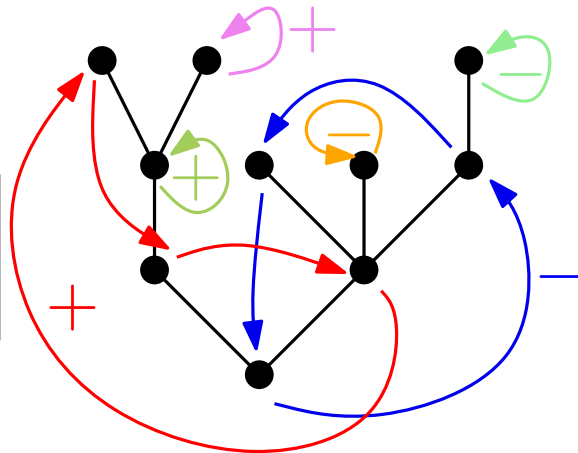


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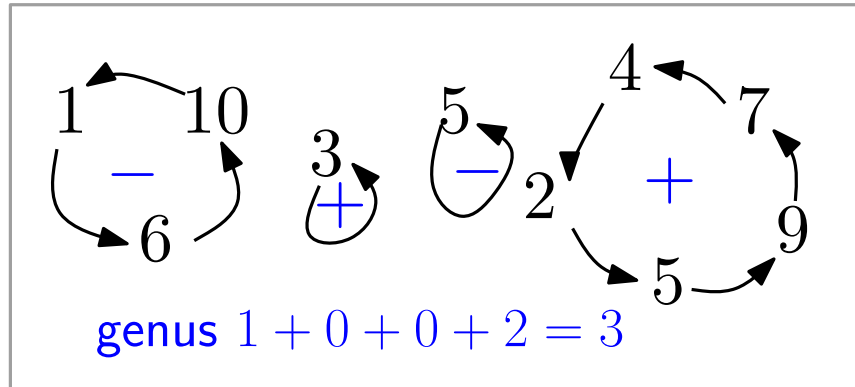
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a **C-decorated tree** of genus 2



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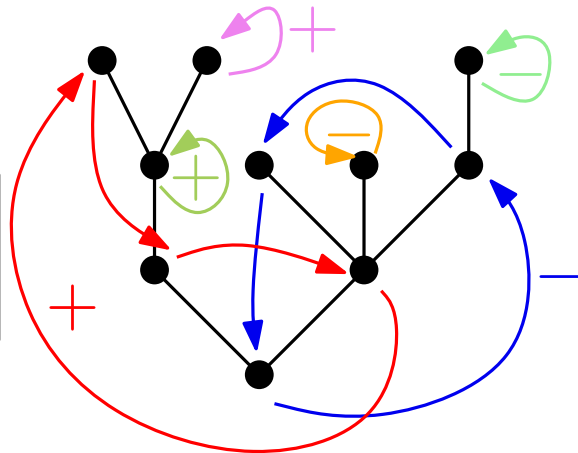


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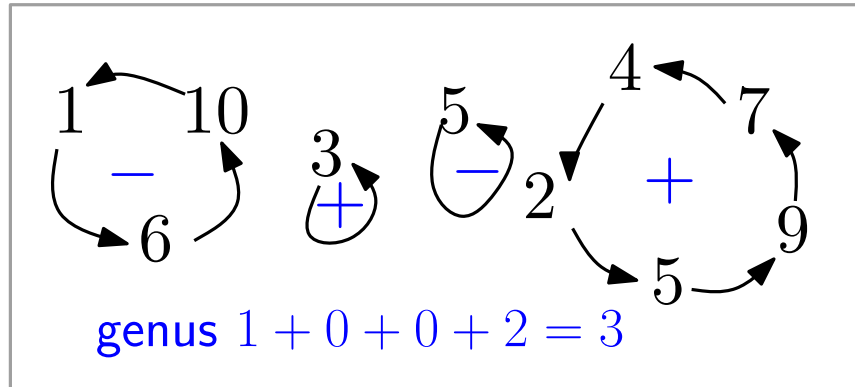


- **Theorem** [C., Féray, Fusy] (our main result!)

There is a 2^{n+1} -to-1-jection between **unicellular maps** of size n and **C-decorated trees** with n edges. It preserves both the genus and the underlying graph.

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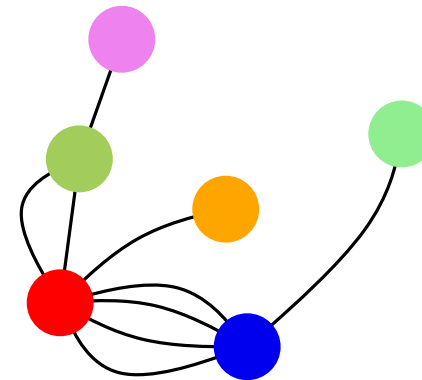
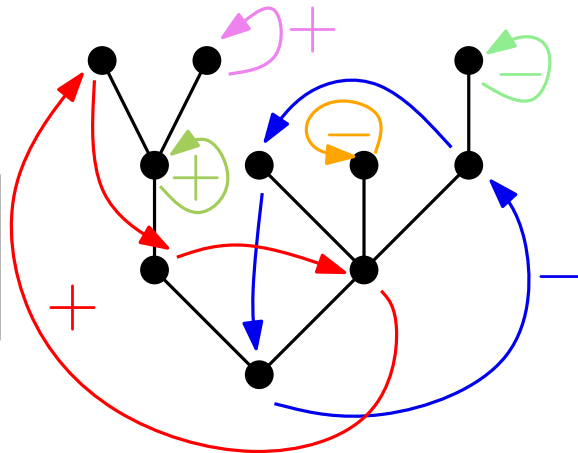


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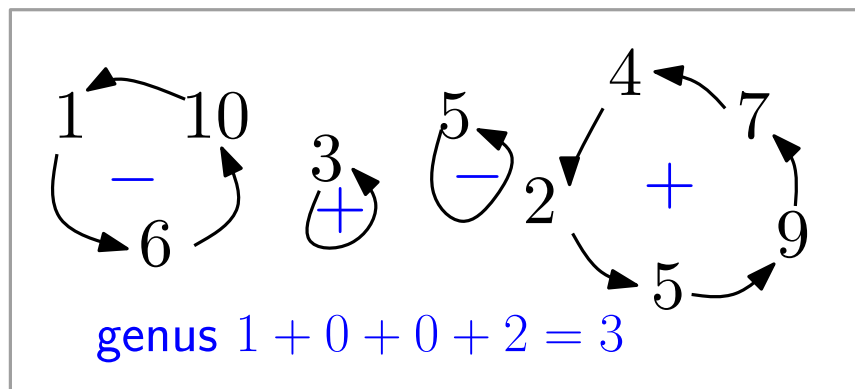
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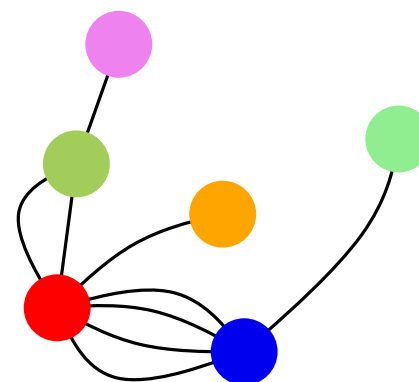
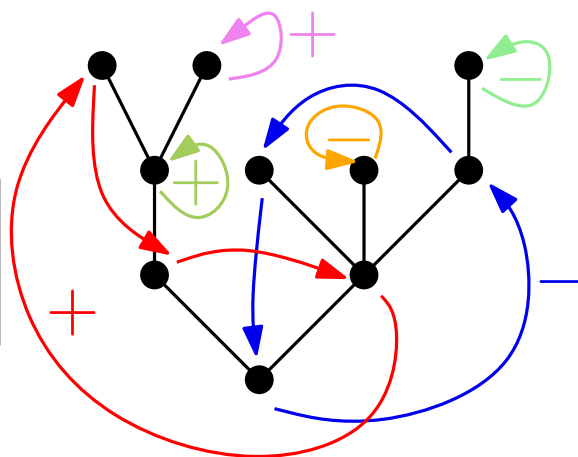


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FROM THERE ALL KNOWN FORMULAS FOLLOW ON, BIJECTIVELY!

A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g , n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$ -to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

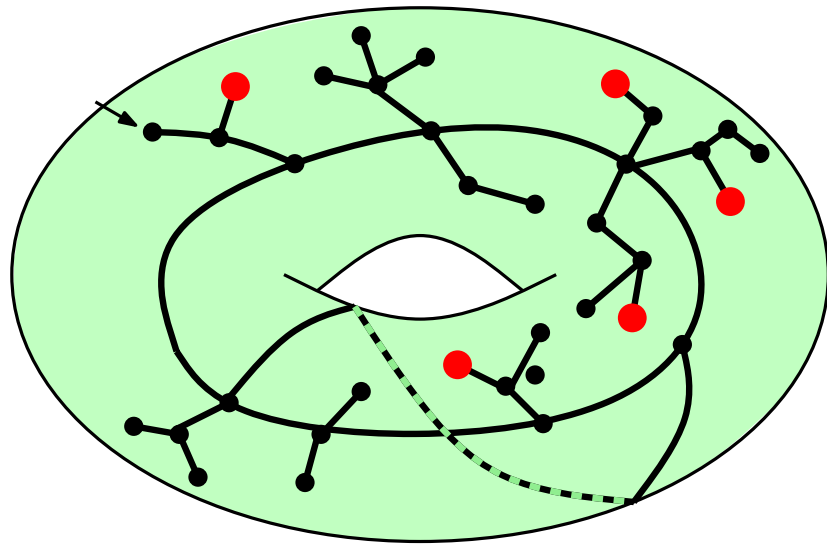
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- **Sketch:** Take $(2k + 1)$ vertices in a map of genus $g - k$ and glue them together preserving the “one-face” condition:



genus 1, 5 marked vertices

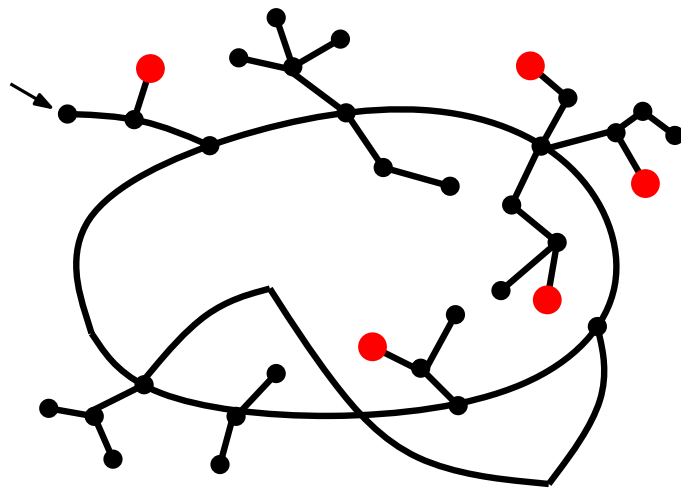
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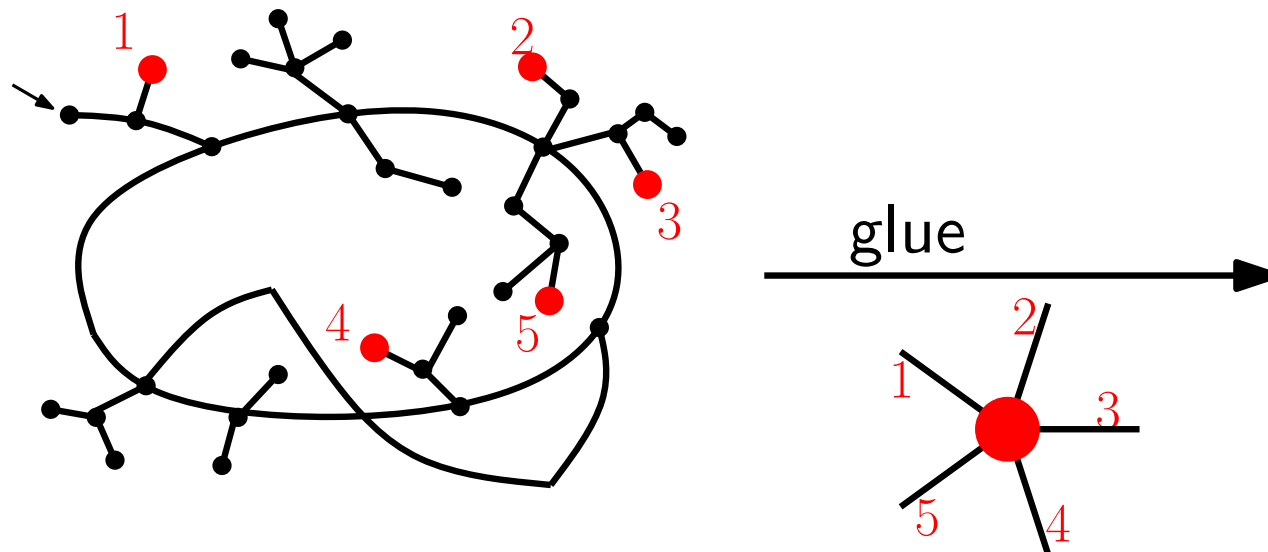
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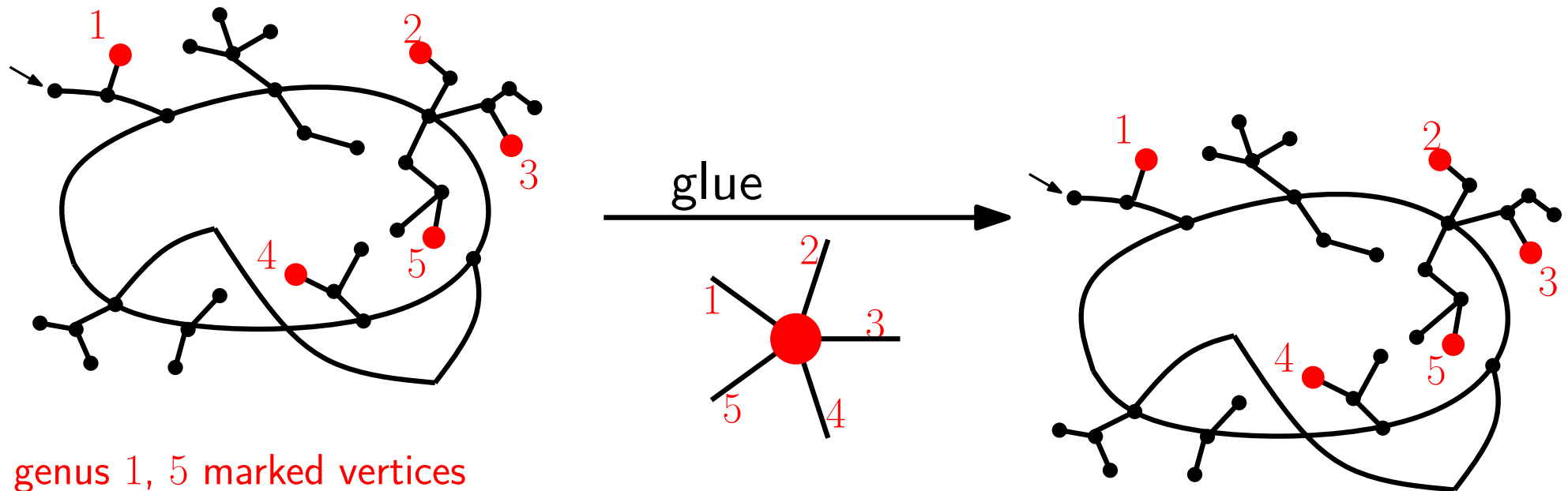
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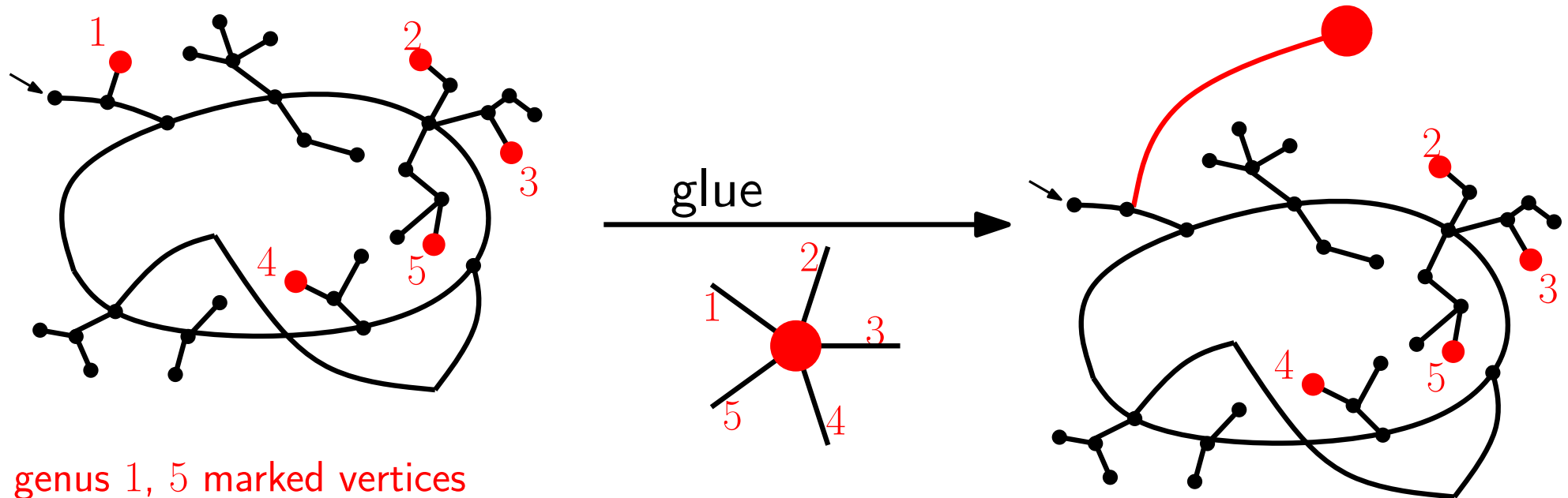
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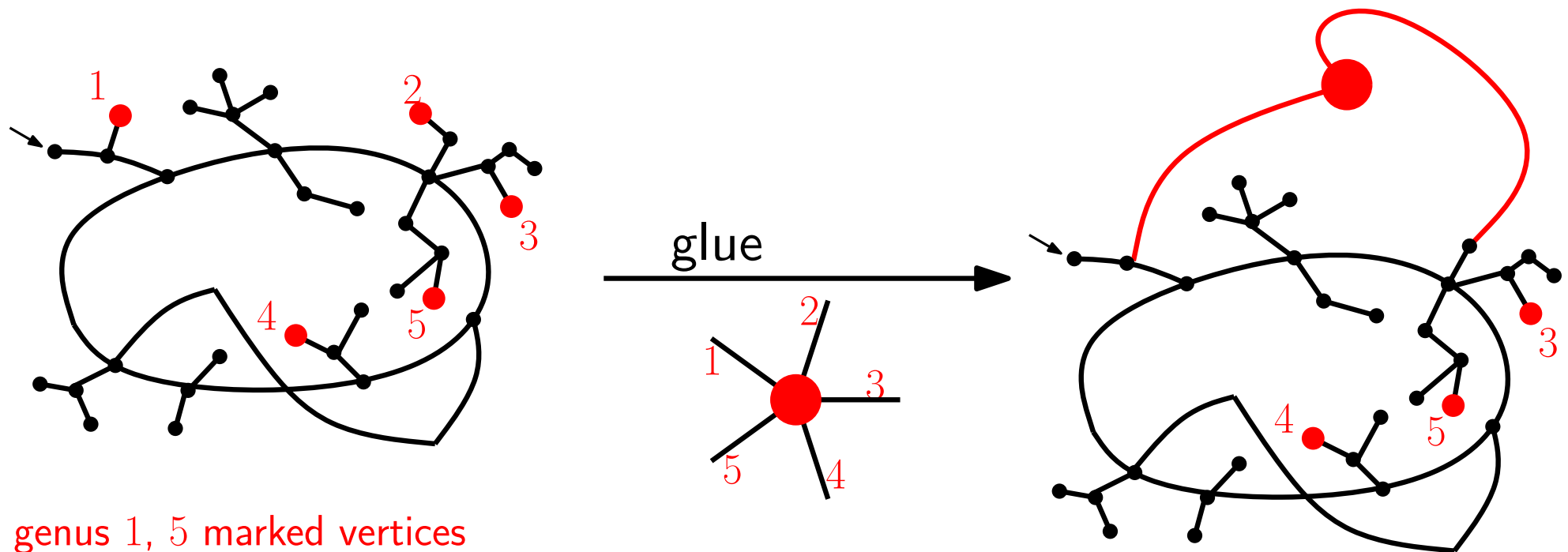
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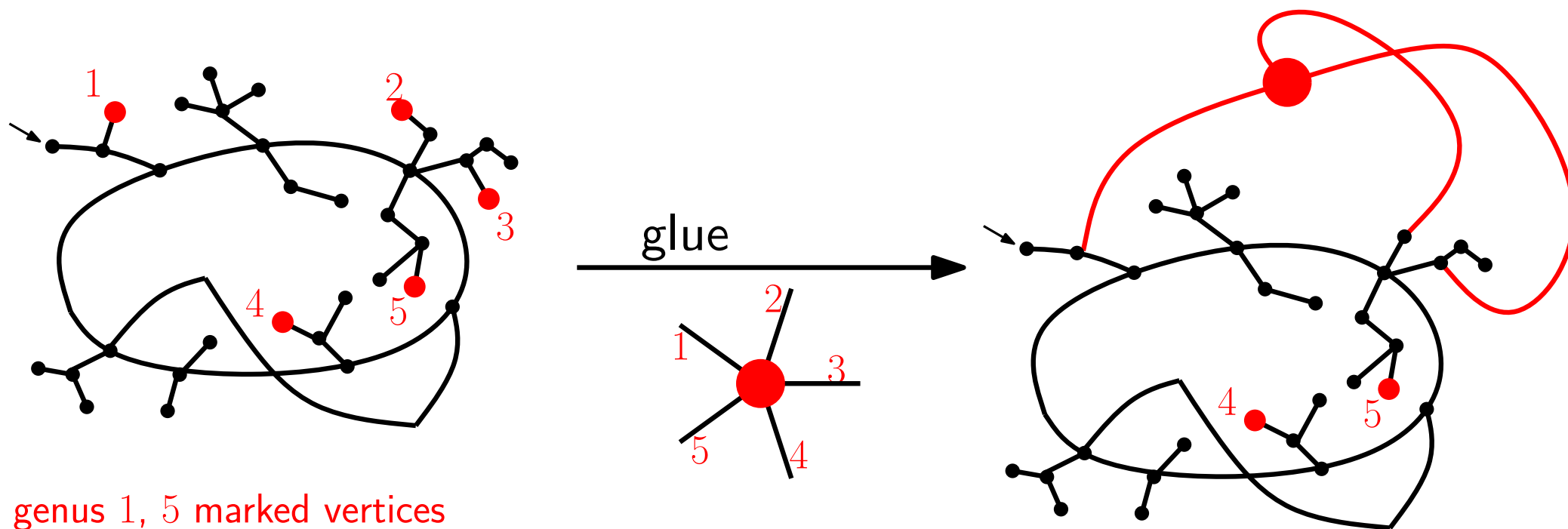
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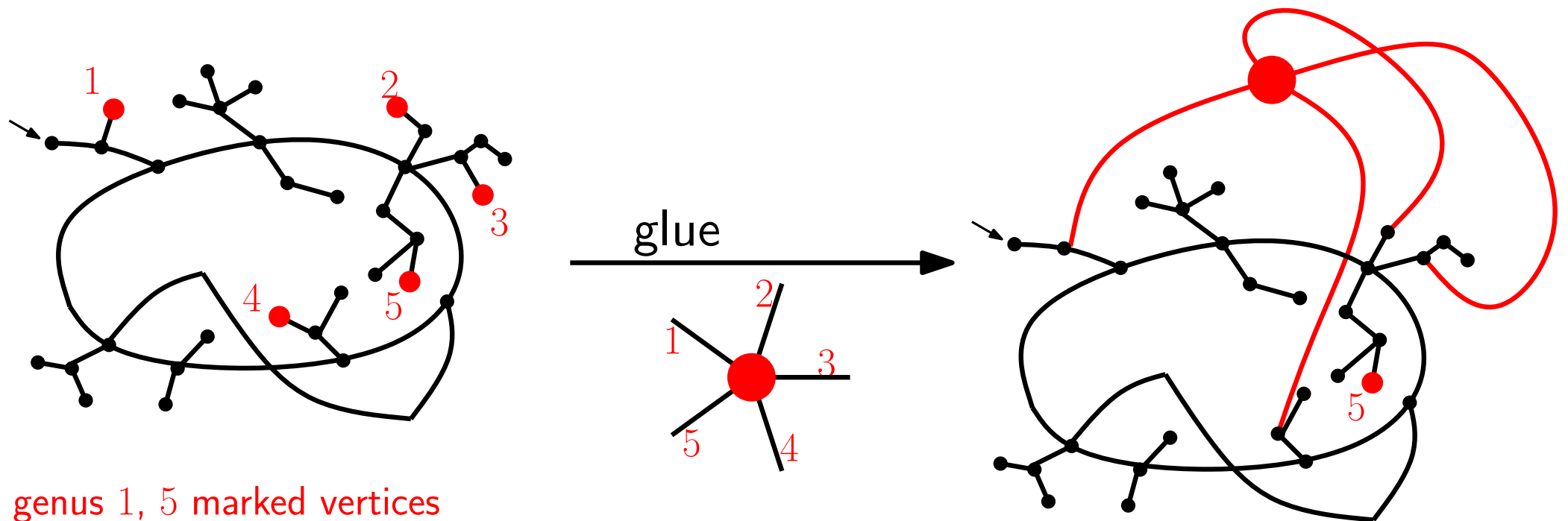
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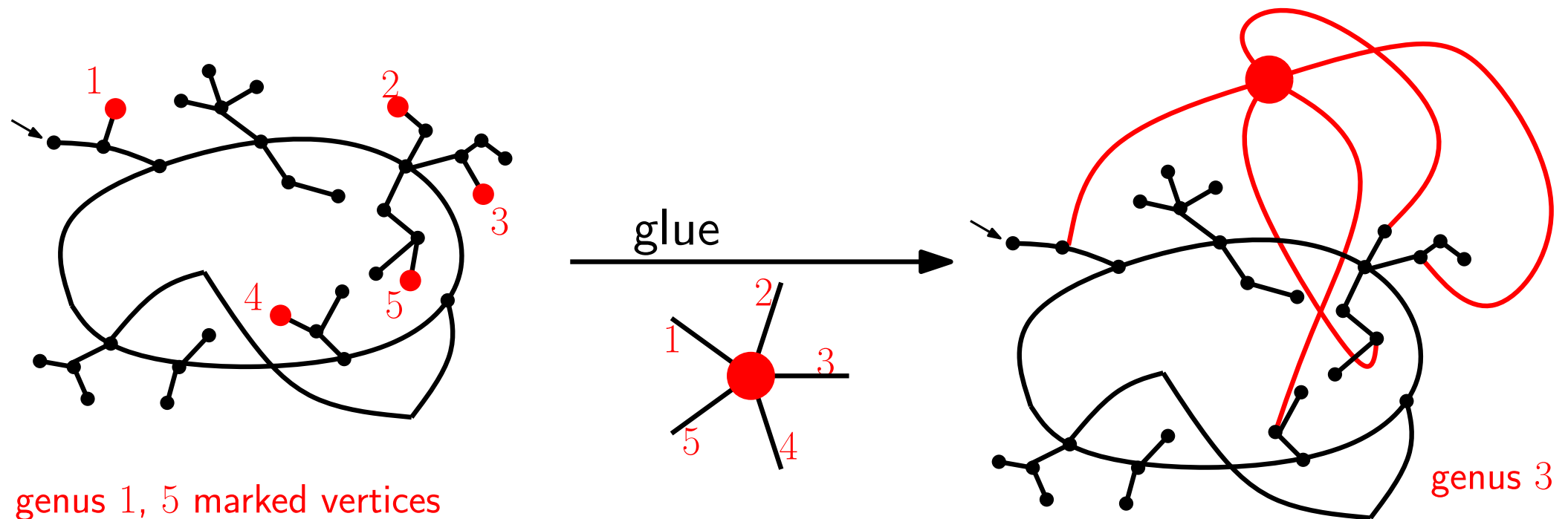
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- the genus increases by k (Euler's formula)

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- **Corollary:** $\epsilon_g(n) = P_g(n) \times \text{Cat}(n)$ where the polynomial P_g is defined recursively:

$$2g \cdot P_g(n) = \binom{n+3-2g}{3} P_{g-1}(n) + \binom{n+5-2g}{5} P_{g-2}(n) + \cdots + \binom{n+1}{2g+1} P_0(n)$$

...but now we can say more !

C -permutations solve the recurrence combinatorially!

- **Fact:** C -permutations satisfy the same recurrence as unicellular maps:

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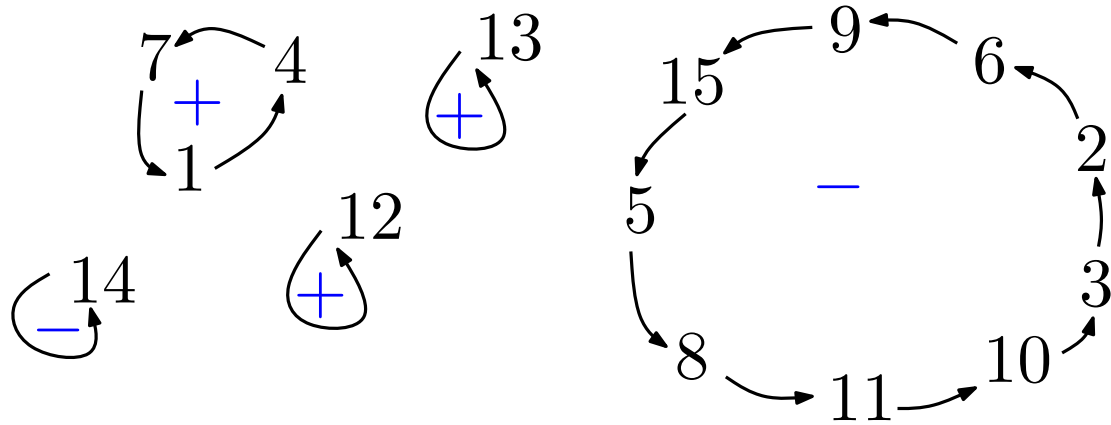
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 $2g = (\# \text{elements}) - \#(\text{cycles})$ is the number of **non-minimal elements**



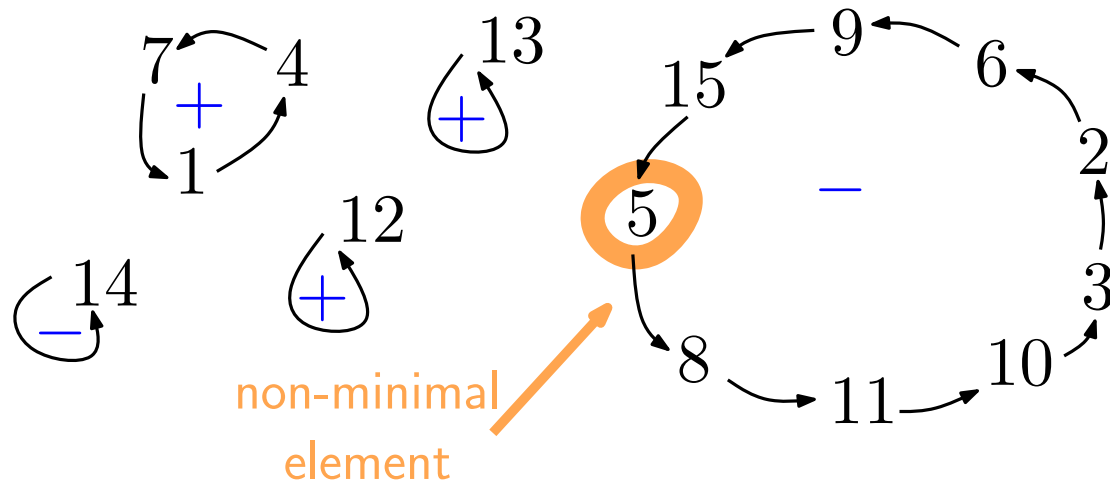
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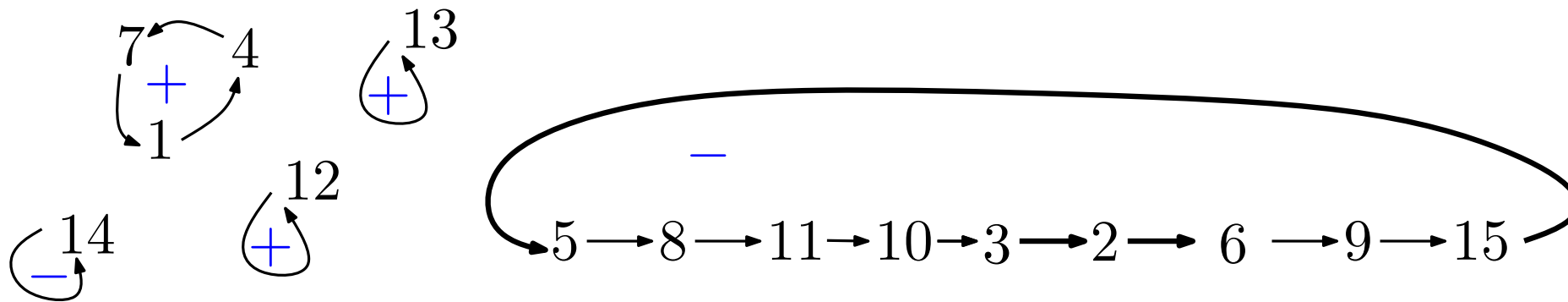
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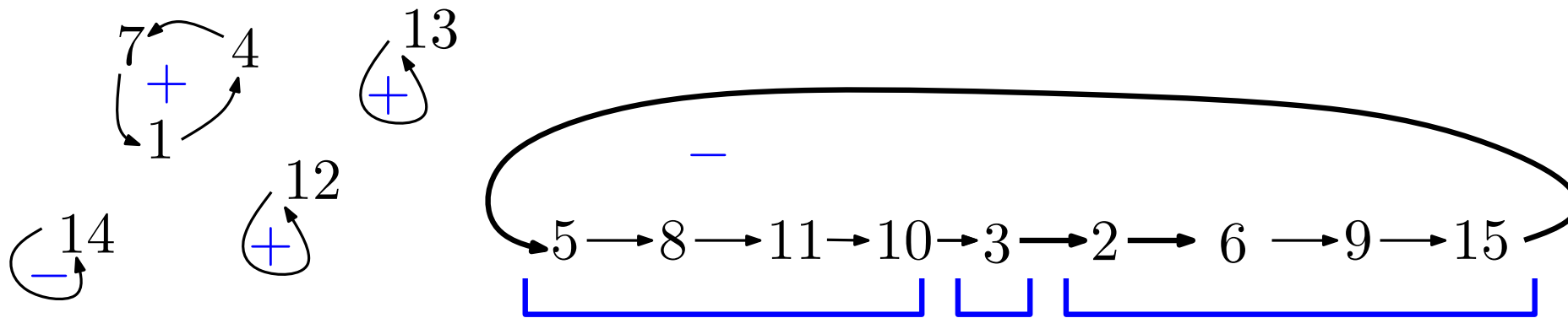
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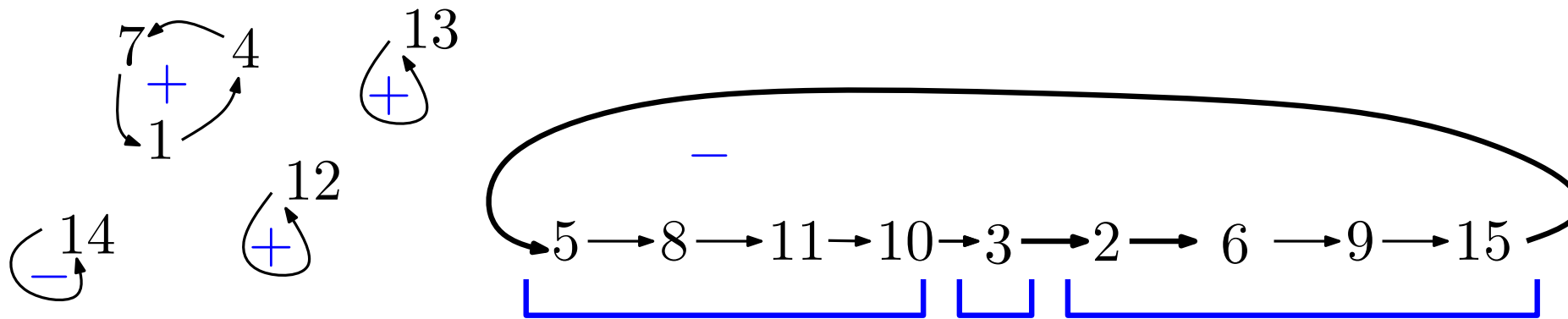
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 correct this by applying a small transposition when necessary
 and write a "+" to remember that you have done it

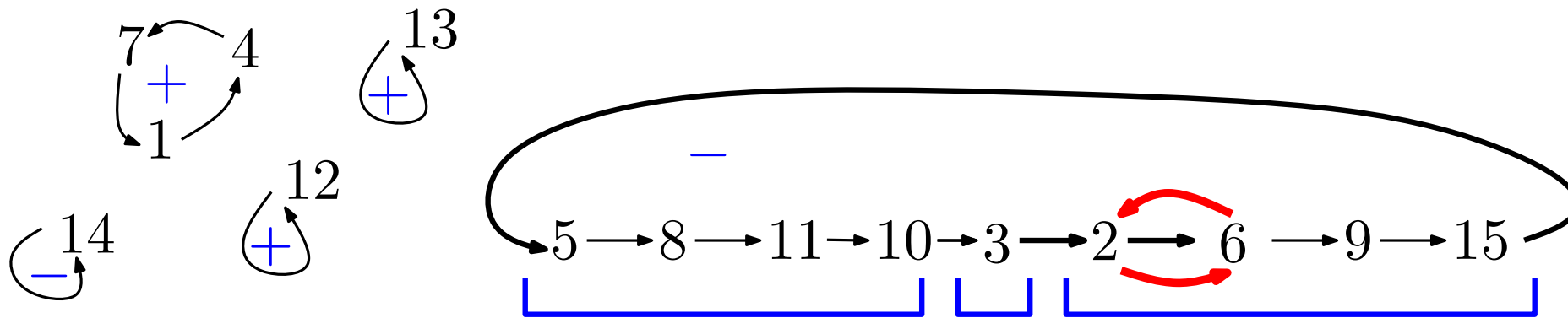
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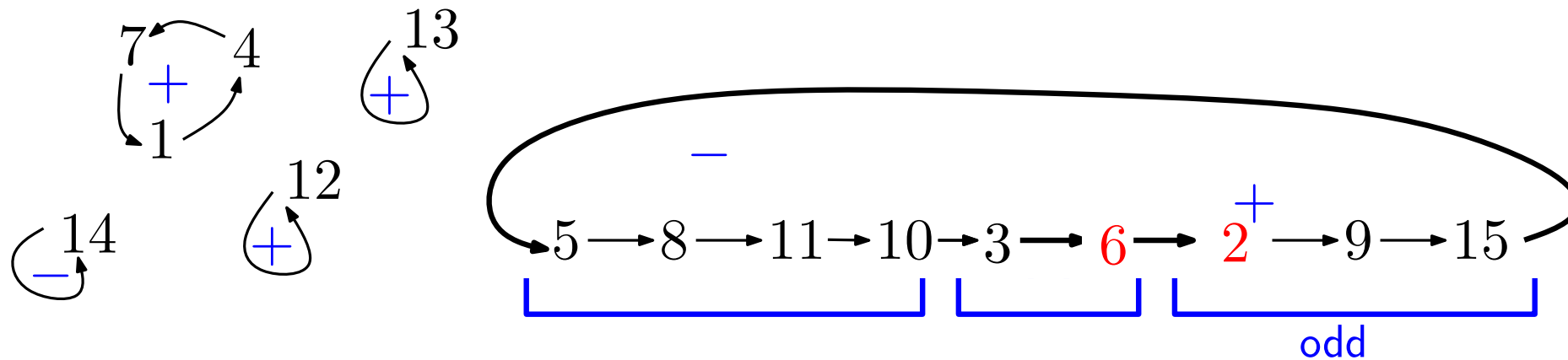
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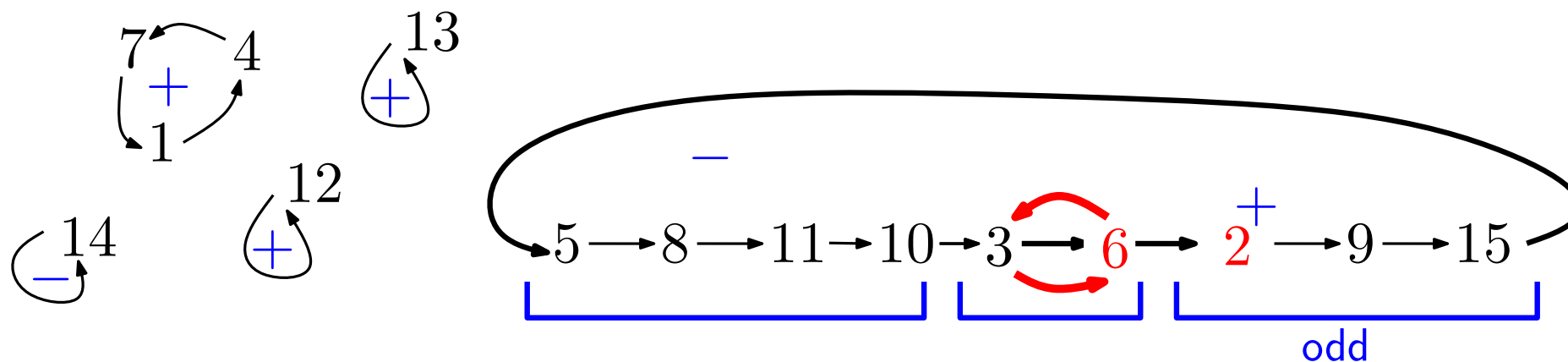
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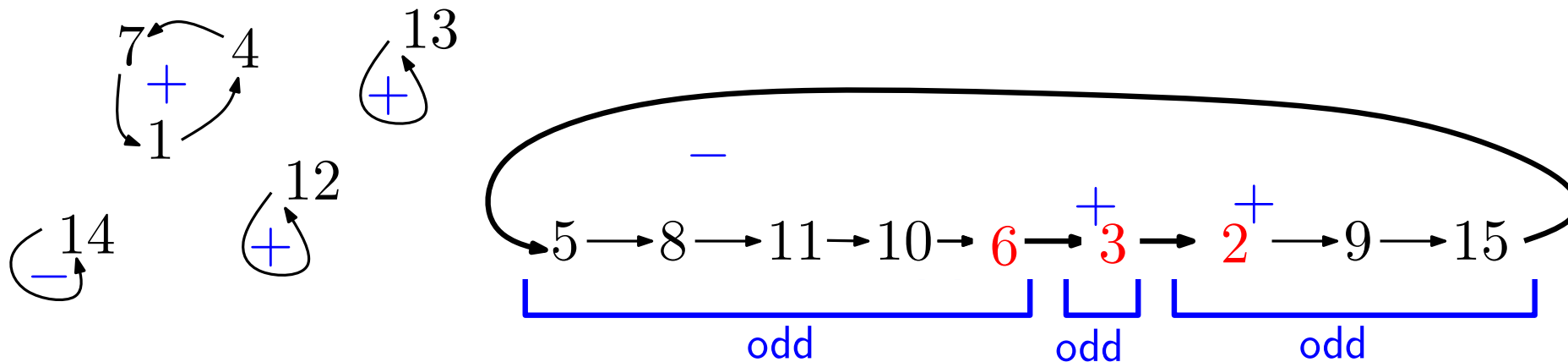
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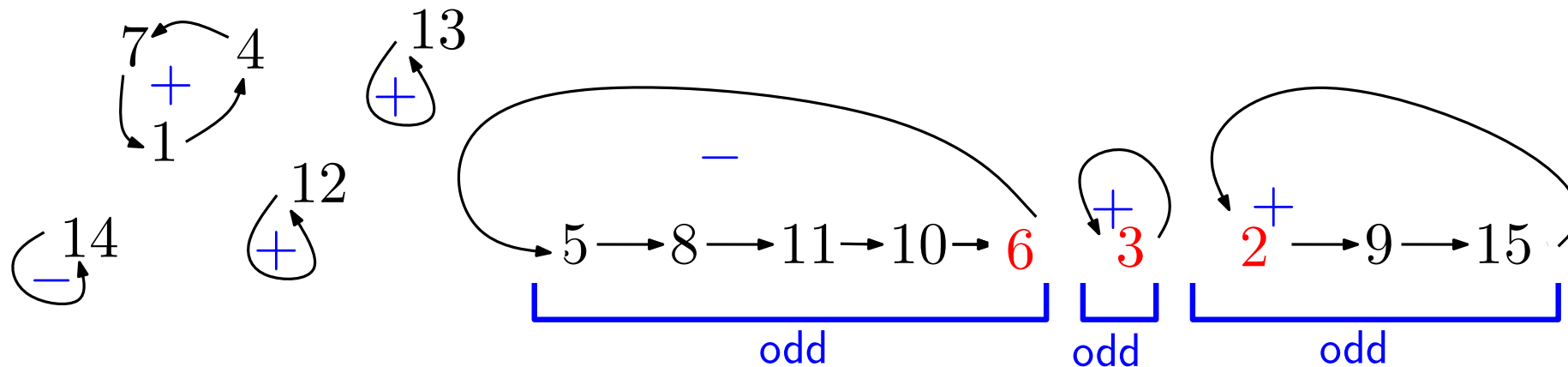
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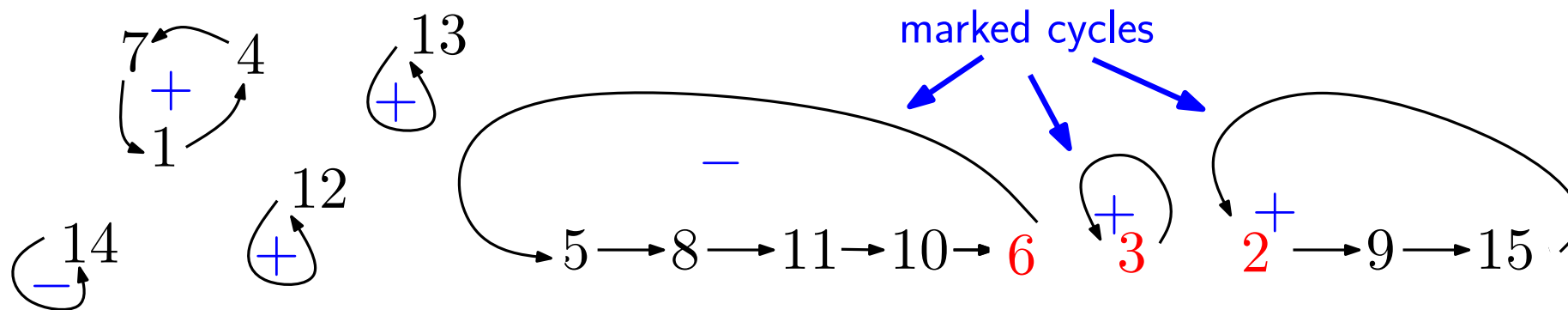
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Counting C -decorated trees is straightforward

- **Theorem** [C., Féray, Fusy]

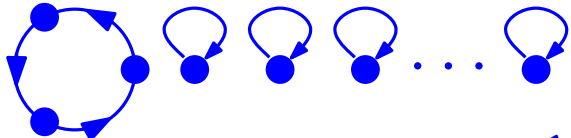
The number of unicellular maps of genus g with n edges satisfies:

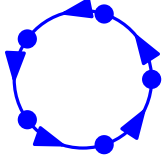
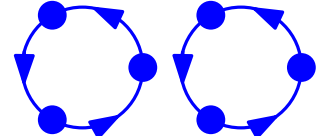
$$2^{n+1} \epsilon_g(n) = C_g(n+1) \text{Cat}(n)$$

where $C_g(n+1)$ is the number of C -perm. of genus g on $n+1$ elements.

- but $C_g(n+1) =$ easy numbers!

- $C_0(n+1) = 2^{n+1}$  ($n+1$ cycles)

- $C_1(n+1) = \frac{(n+1)n(n-1)}{3} 2^{n-1}$  ($n-1$ cycles)

- $C_2(n+1) = \left(4! \binom{n+1}{5} + 40 \binom{n+1}{6} \right) 2^{n-3}$ (either  or )

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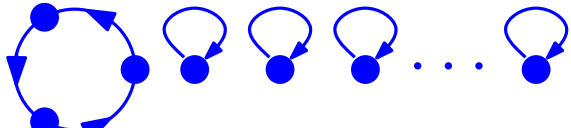
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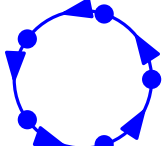
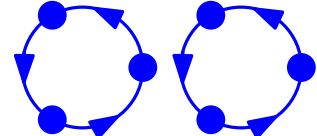
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- In general: $C_g(n+1) = \left(\sum_{\gamma \vdash g} \frac{(n+1-2g)^{2\ell(\gamma)+1}}{\prod_i m_i! (2i+1)^{m_i}} \right) 2^{n+1-2g}$

sum is over the cycle type
of the C -permutation:
($2\gamma_i + 1$) = cycle lengths.

✓ Lehman-Walsh formula !

Conclusion

- It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!
- For example the beautiful Harer-Zagier recurrence formula has been waiting for a combinatorial interpretation since 1986...

Harer-Zagier recurrence formula (1986)

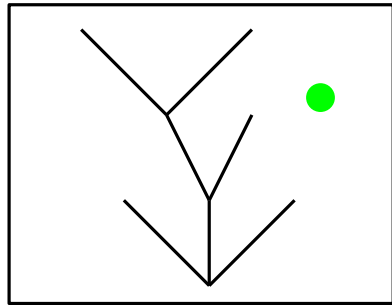
- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

$$(n + 1)\text{Cat}(n) = 2 \times (2n - 1)\text{Cat}(n - 1)$$

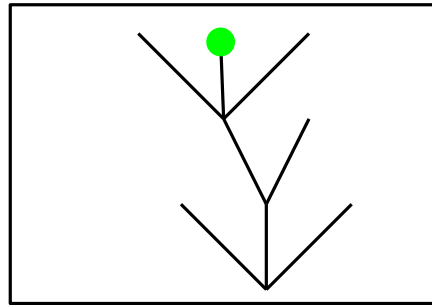
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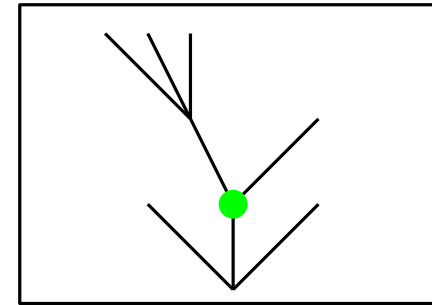
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rooted tree, n edges,
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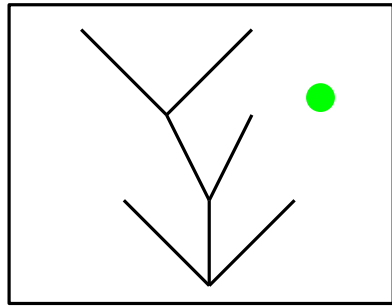


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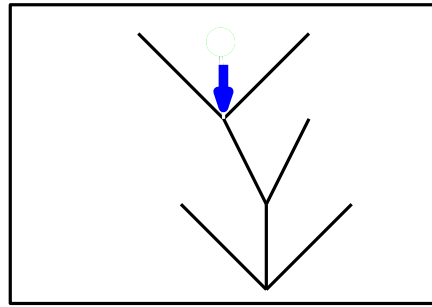
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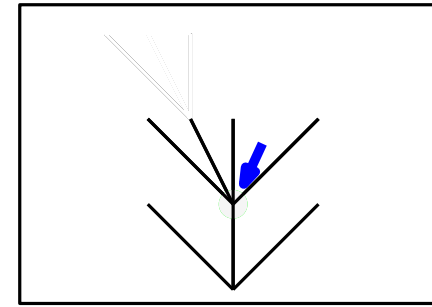
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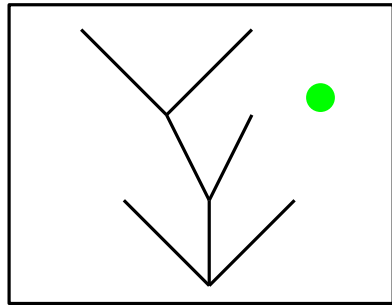


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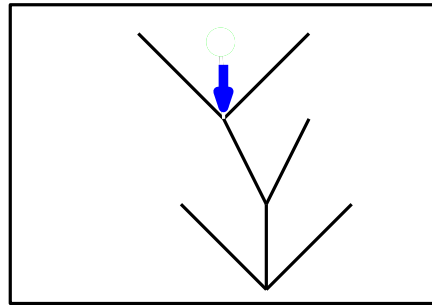
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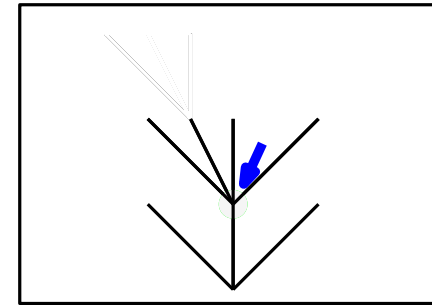
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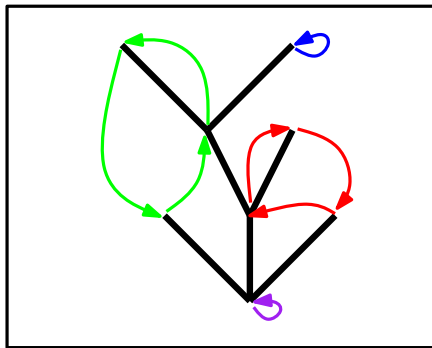
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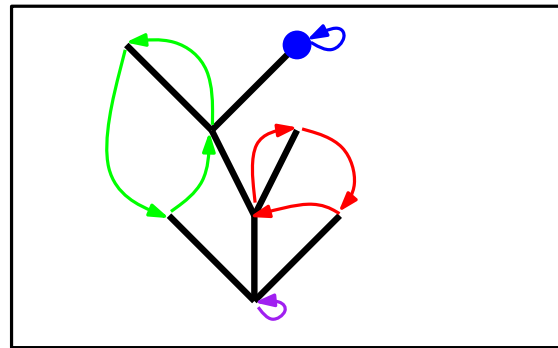
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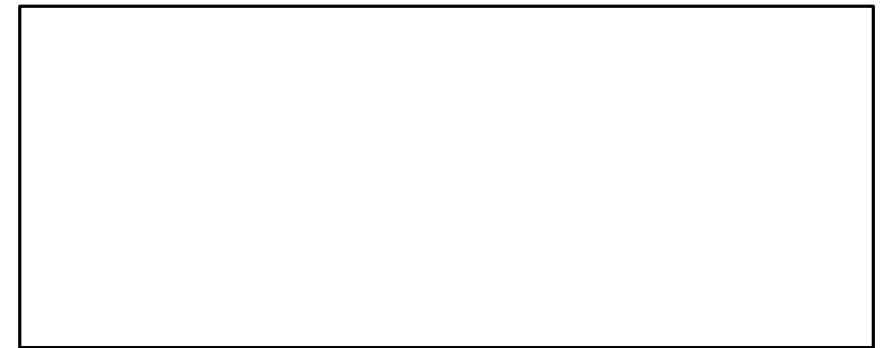
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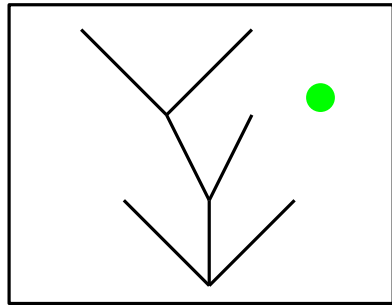


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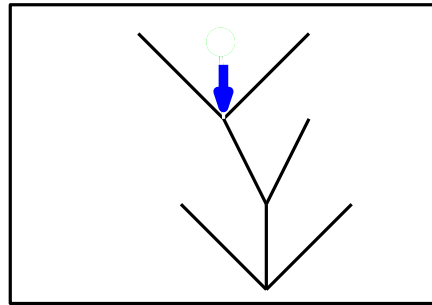
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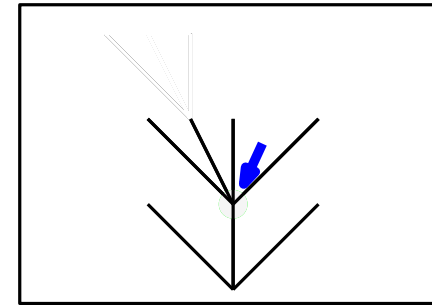
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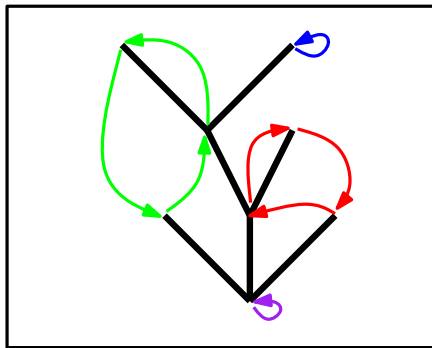
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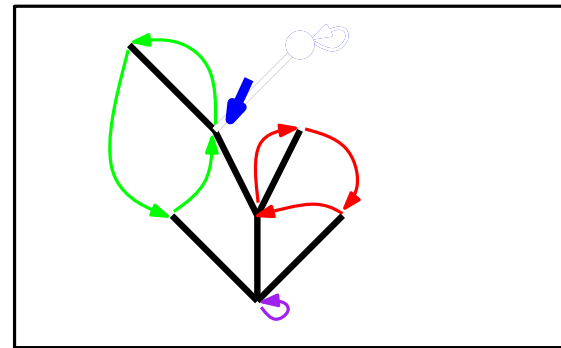
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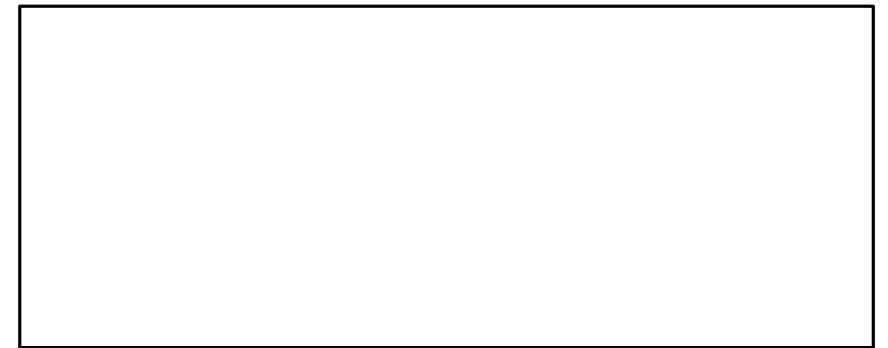
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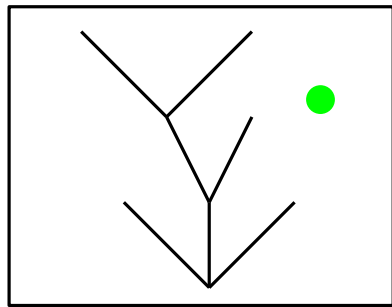


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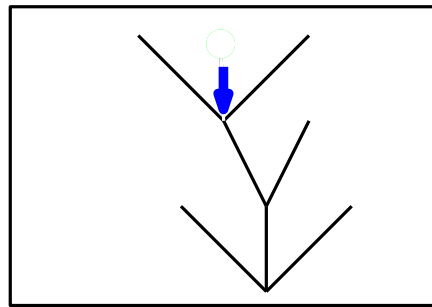
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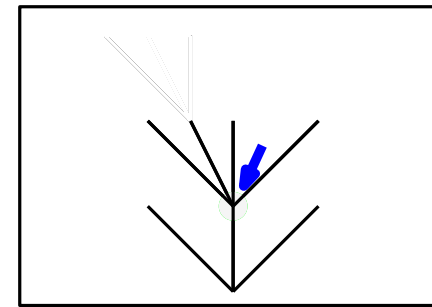
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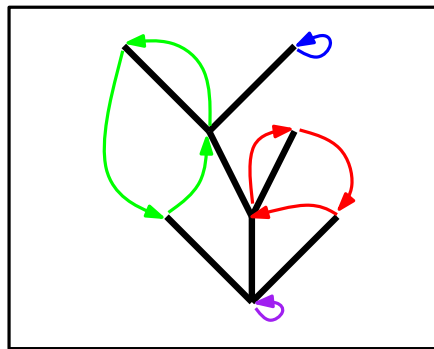
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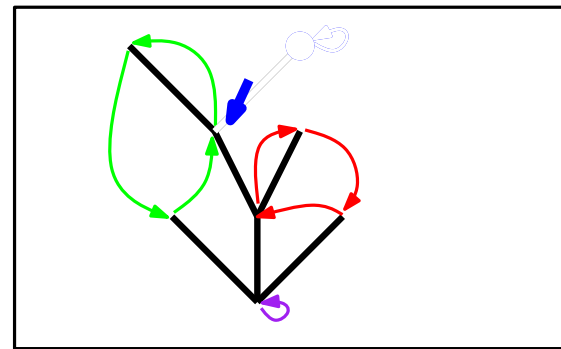
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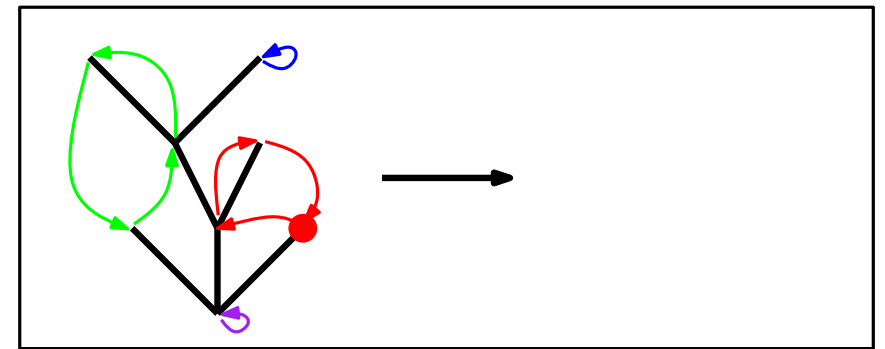
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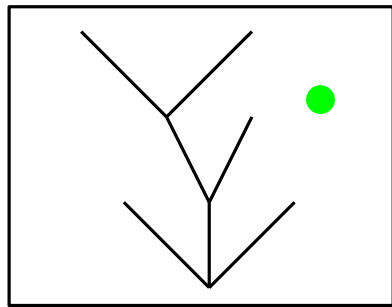


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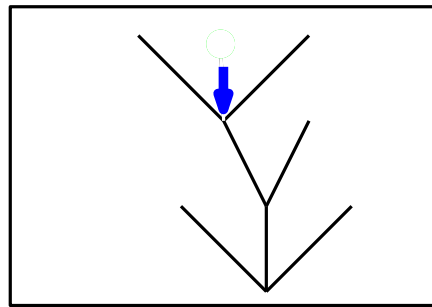
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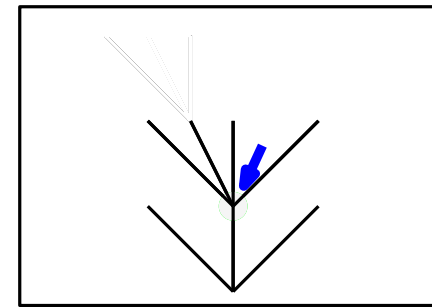
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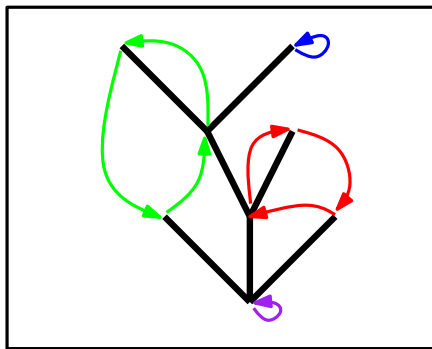
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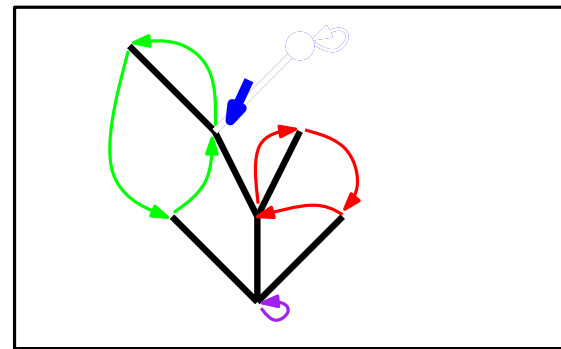
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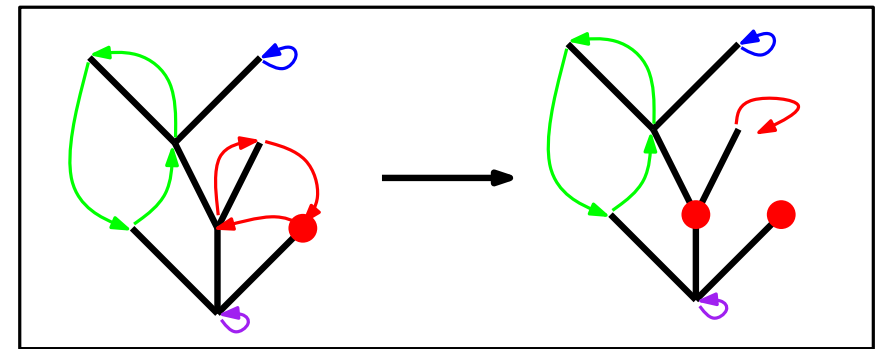
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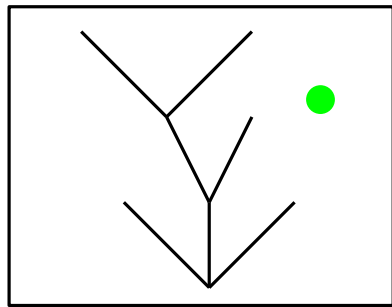


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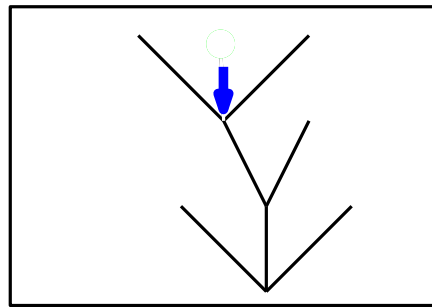
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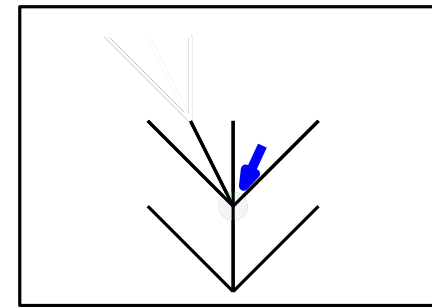
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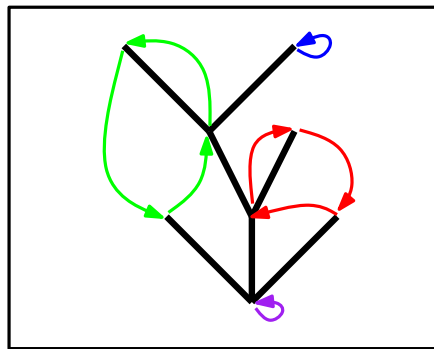
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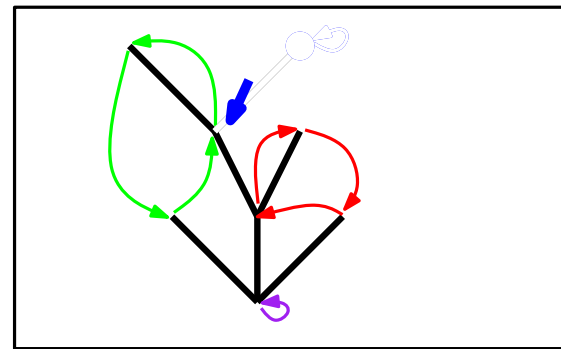
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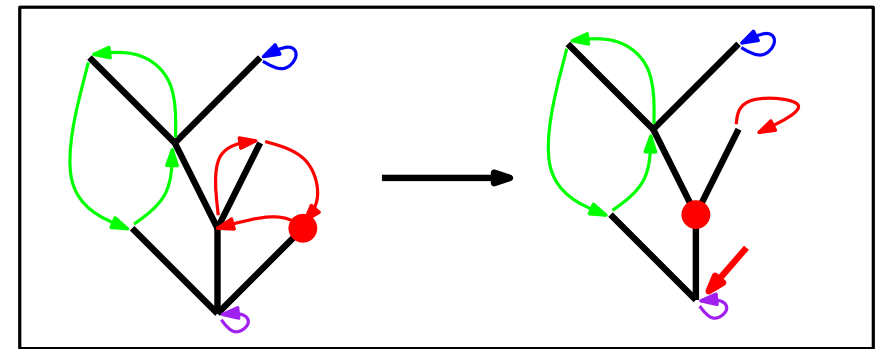
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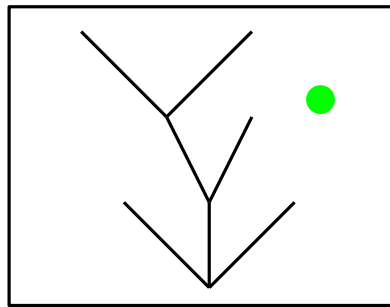


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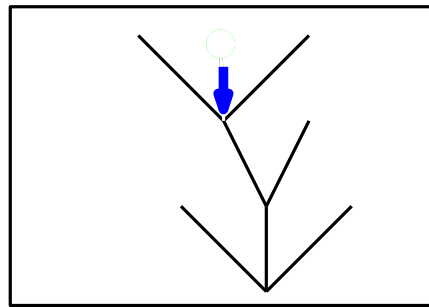
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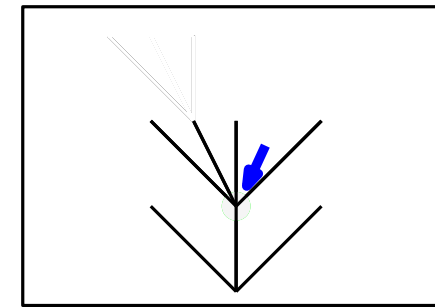
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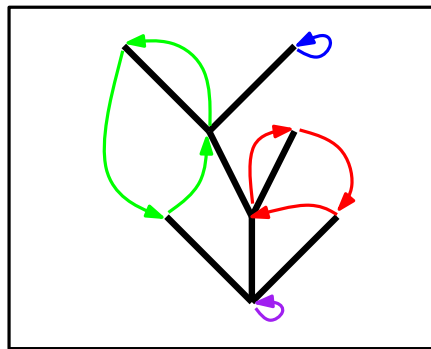
case a: vertex is
a leaf. Delete it.



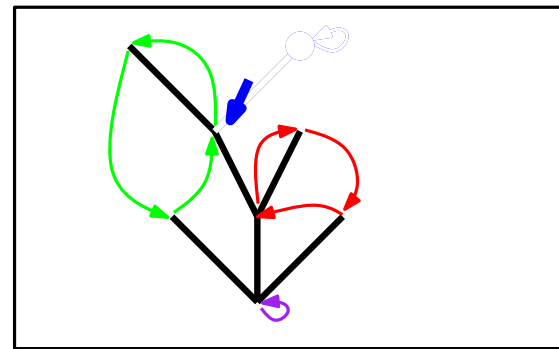
case b: vertex is not a leaf. Contract the
leftmost outgoing edge.

- Then for general g :

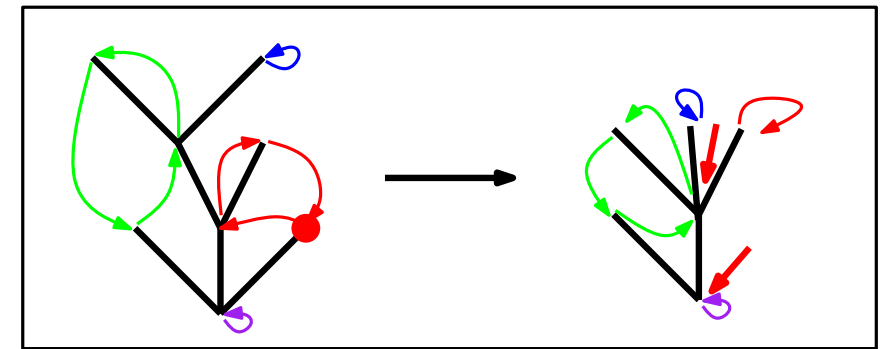
$$(n + 1)\epsilon_g(n) = 2(2n - 1)\epsilon_g(n - 1) + (n - 1)(2n - 1)(2n - 3)\epsilon_{g-1}(n - 2)$$



C -decorated tree, n edges,
genus g , one marked vertex



case 1: vertex is a **fixed point**:
apply **Rémy's bijection**
(one vertex disappears)



case 2: vertex is in a $(2k + 1)$ -**cycle**.
Apply **Rémy's bijection twice** (two vertices
disappear, cycle length decreases by 2)

Conclusion

- It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!
- The bijection also applies to Féray's expression of characters in terms of unicellular maps (we obtain a new expression - is it useful?)
- Next ?
 - unicellular constellations? ([Poulalhon-Schaeffer 02, Bernardi-Morales 11])
(problem: FPSAC'09 bijection does not work well)
(very partial results in the full version - take a look!)
 - many-face maps? (KP hierarchy?)
(problem: seems much harder!)
 - non-orientable surfaces?
(problem: FPSAC'09 bijection only exists in asymptotic version - [Bernardi-Ch., FPSAC'10])