A (very) simple formula to count maps on orientable surfaces

Guillaume Chapuy, CNRS & Université Paris Diderot

joint work with

Sean R. Carrell, University of Waterloo

FPSAC 2014, Chicago.

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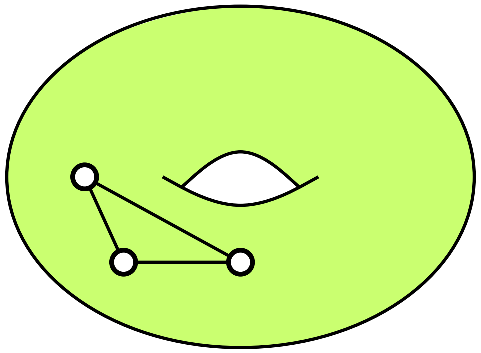
I. Maps

A map of genus g

= graph drawn (without edge-crossings) on the orientable surface of genus g, such that each face is homeomorphic to a polygon.



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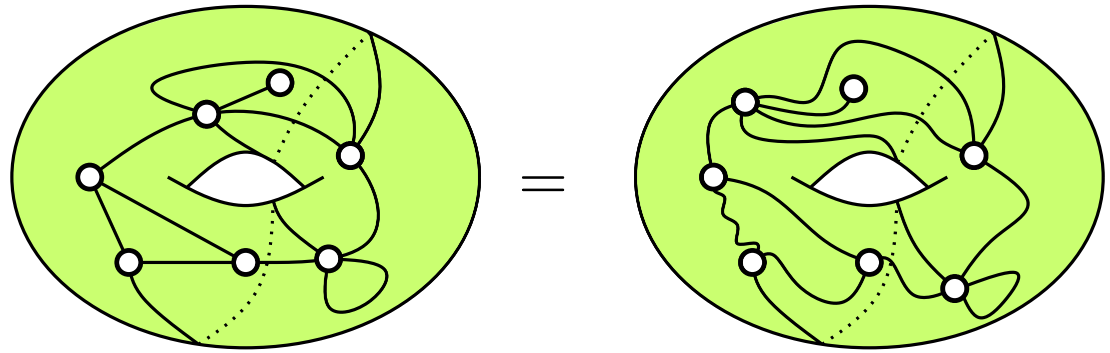

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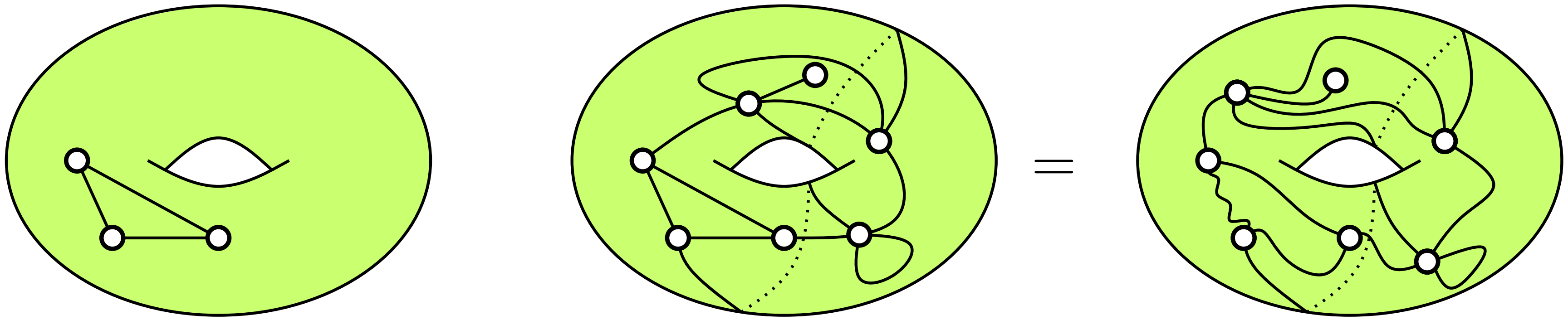

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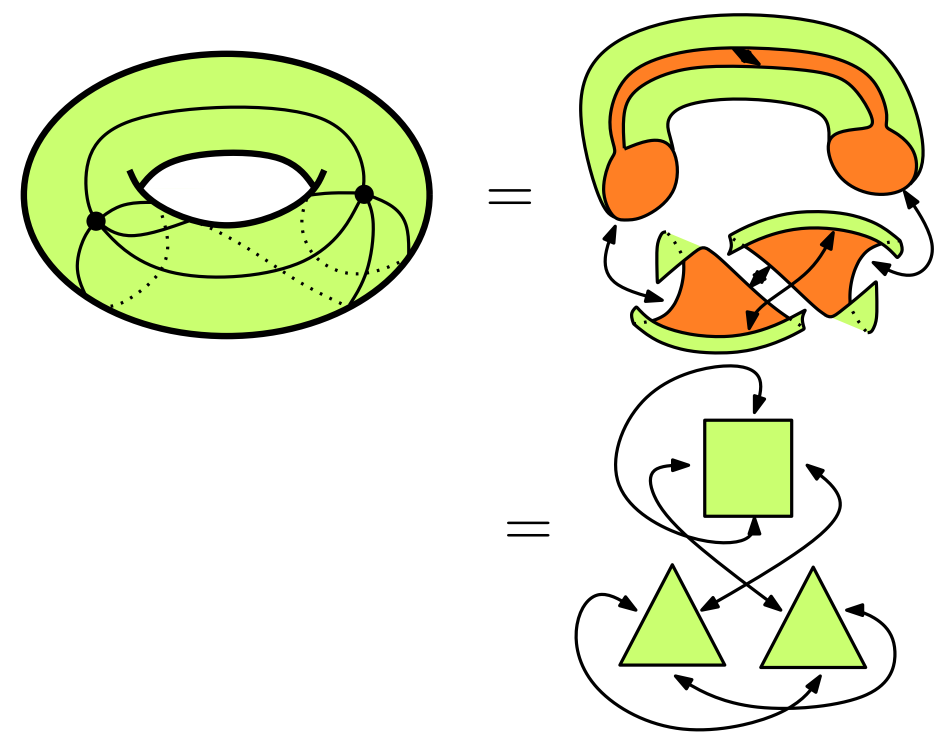
One can view a map as a surface made by the edge-by-edge gluing of a finite number of polygons.
(in this talk all the gluings are "orientable")



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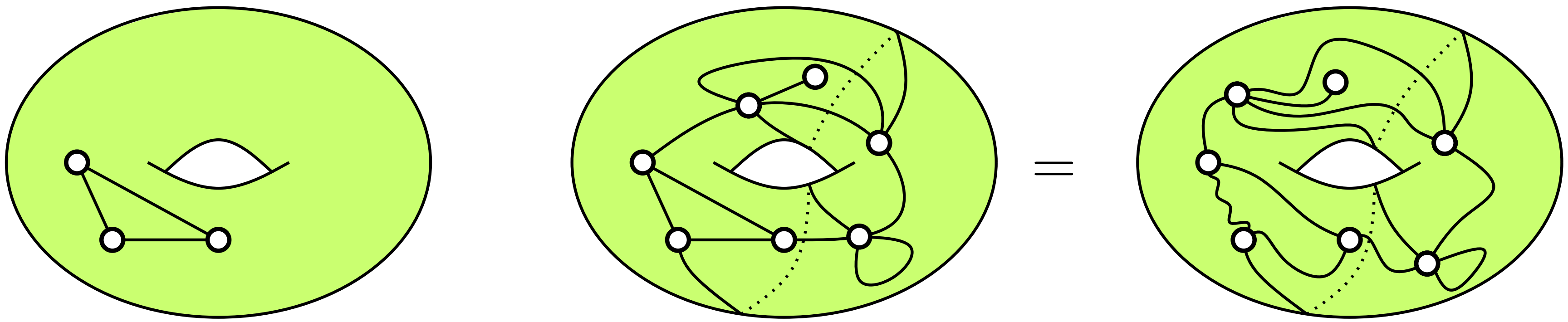

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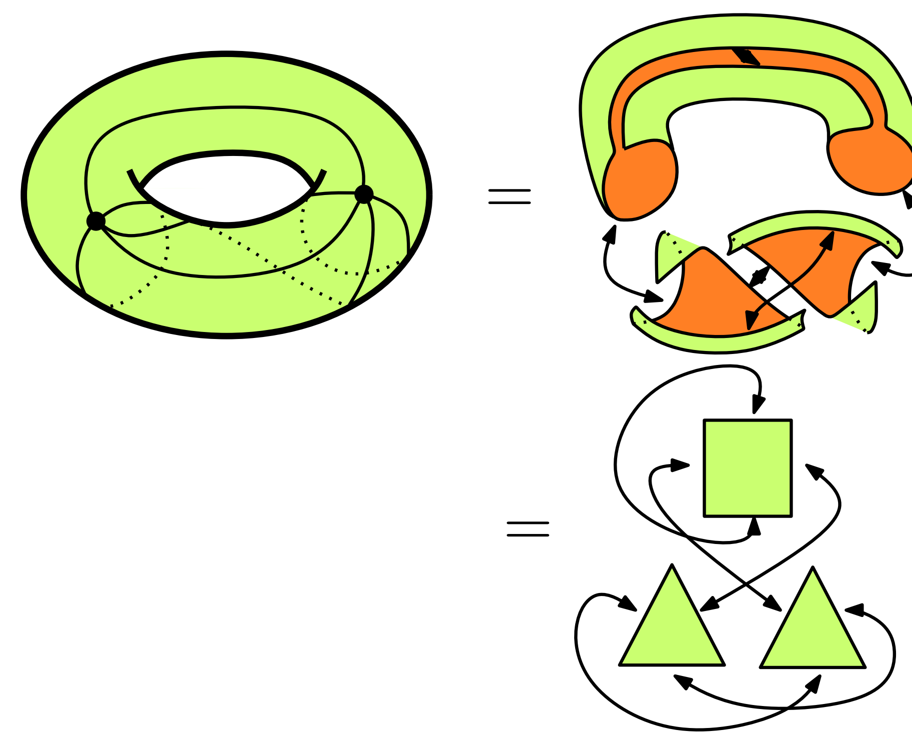

Euler's formula links the main parameters together:

$$\text{vertices} + \text{faces} = \text{edges} + 2 - 2g.$$

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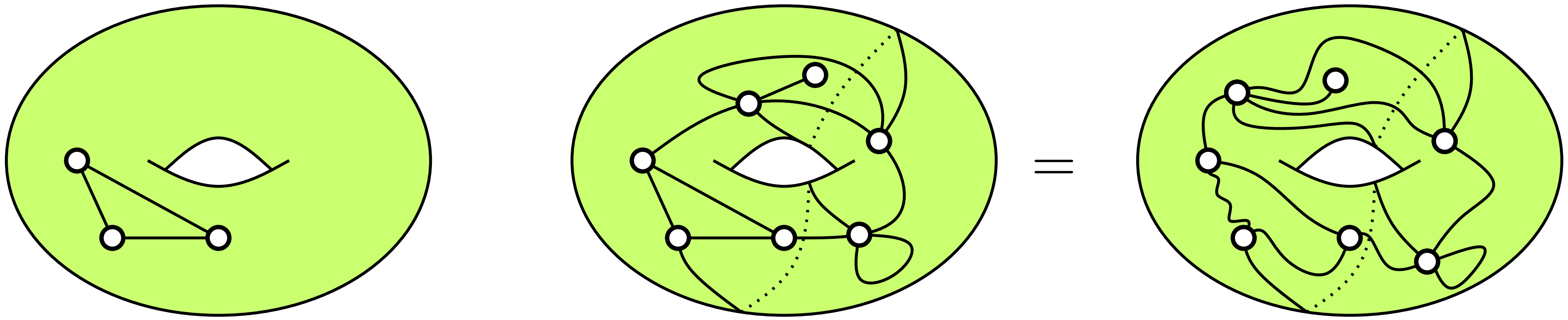

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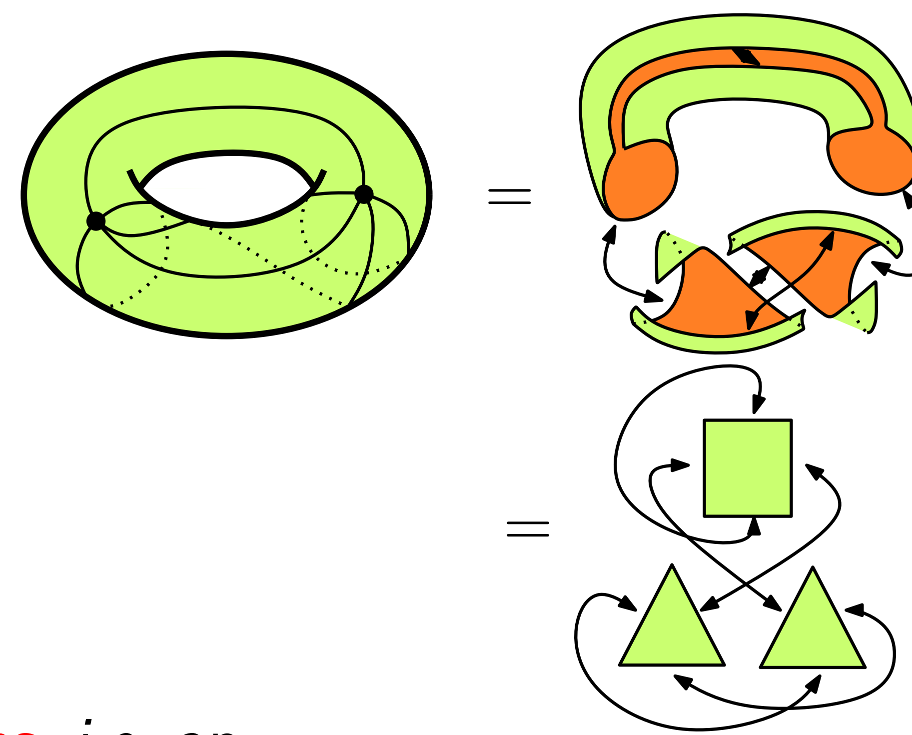

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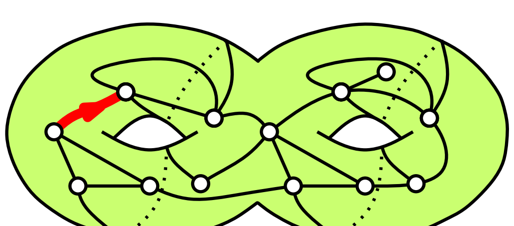

For enumeration: rooted maps, *i.e.* an edge is distinguished and oriented.

Counting maps by edges and genus – some classical results

[Tutte 1960] Planar maps with n edges:

$$Q_0^n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

$n \backslash g$	0	1	2	3	4	$\cdots$
0						
1						
2						
3						
4						
$\vdots$						

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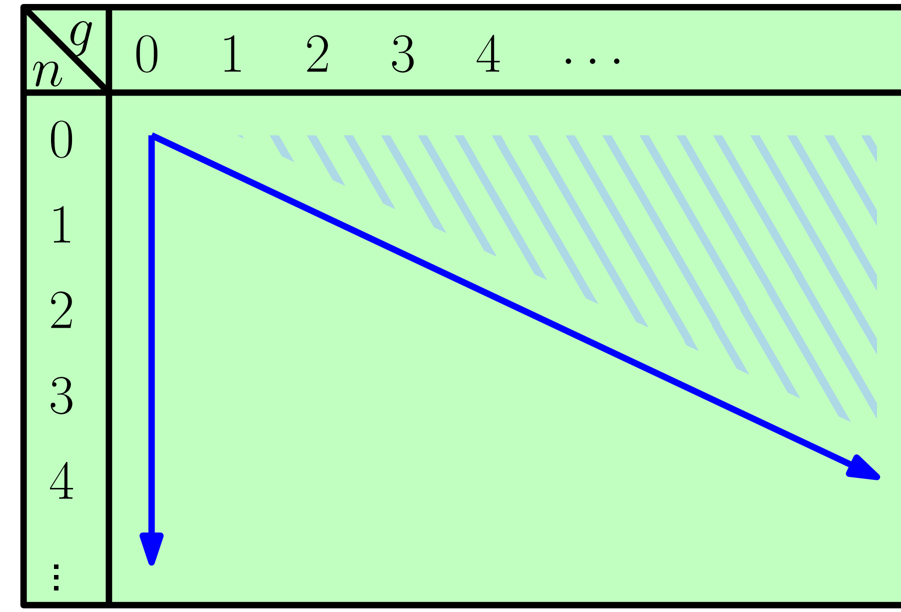
$$Q_g^{2g} = \frac{(4g-1)!!}{2g+1}$$



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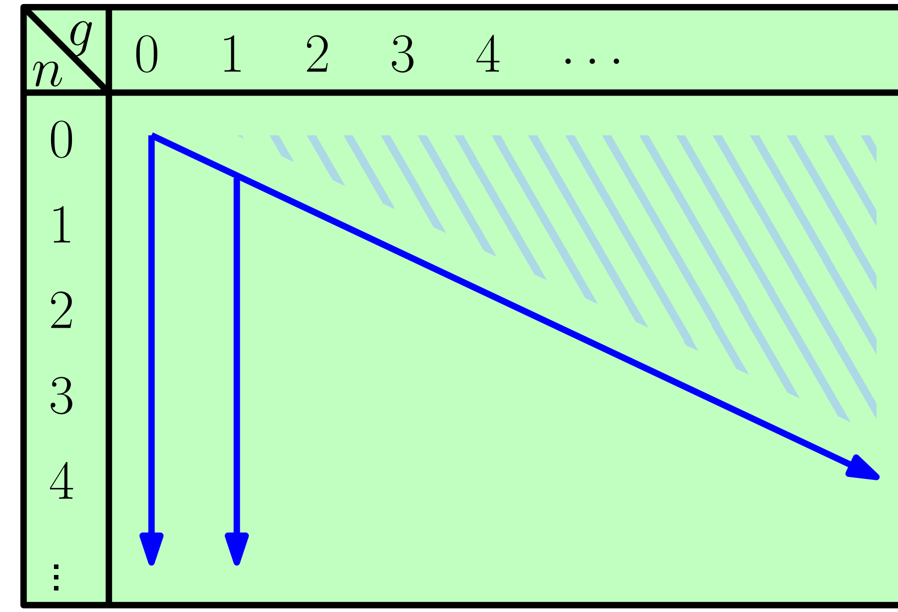

[Bender et Canfield 1986:] Maps of genus g with n edges:

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$$R_1(\rho) = \frac{(\rho-1)^2}{12(\rho+2)\rho^2}, \quad R_2(\rho) = \frac{1}{2304} \frac{\left(49\,\rho^4+122\,\rho^3+225\,\rho^2+248\,\rho+112\right)(\rho+1)^2(\rho-1)^4}{\rho^7(\rho+2)^4}, \ldots$$

Very complicated to compute R_g with this method.

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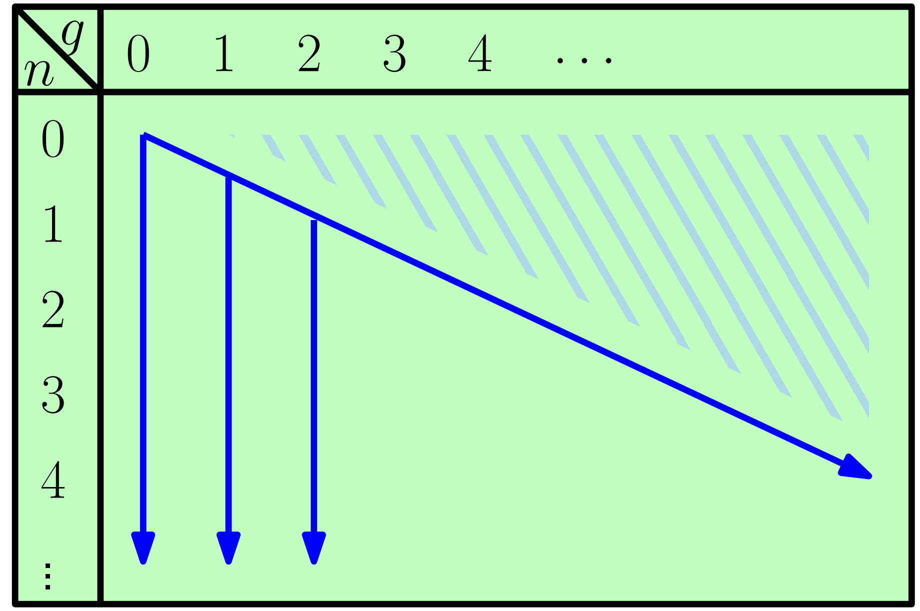

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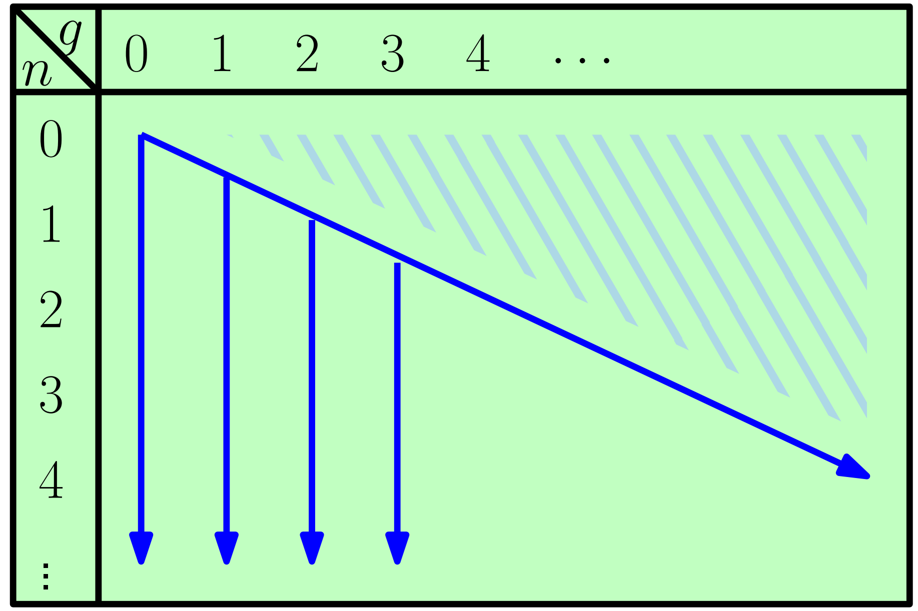

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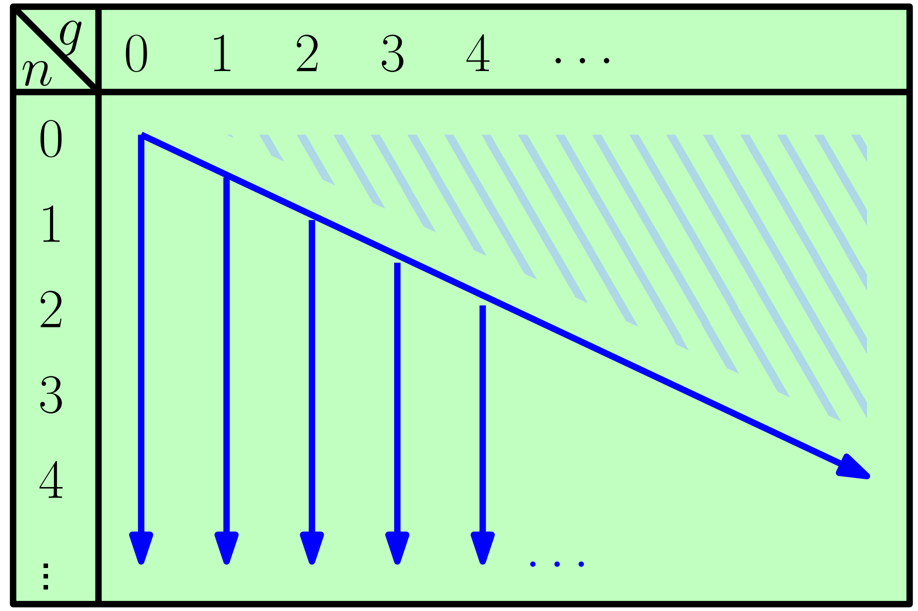

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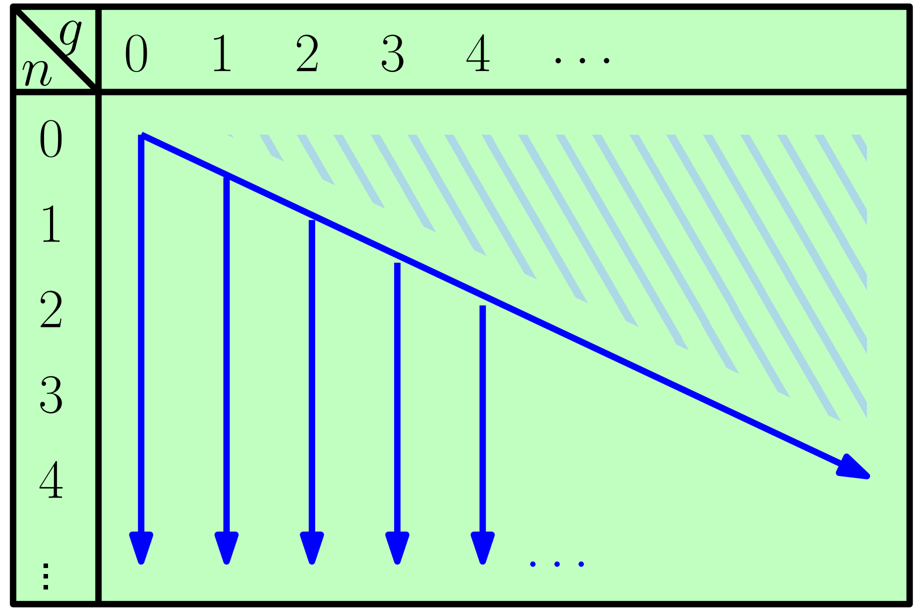

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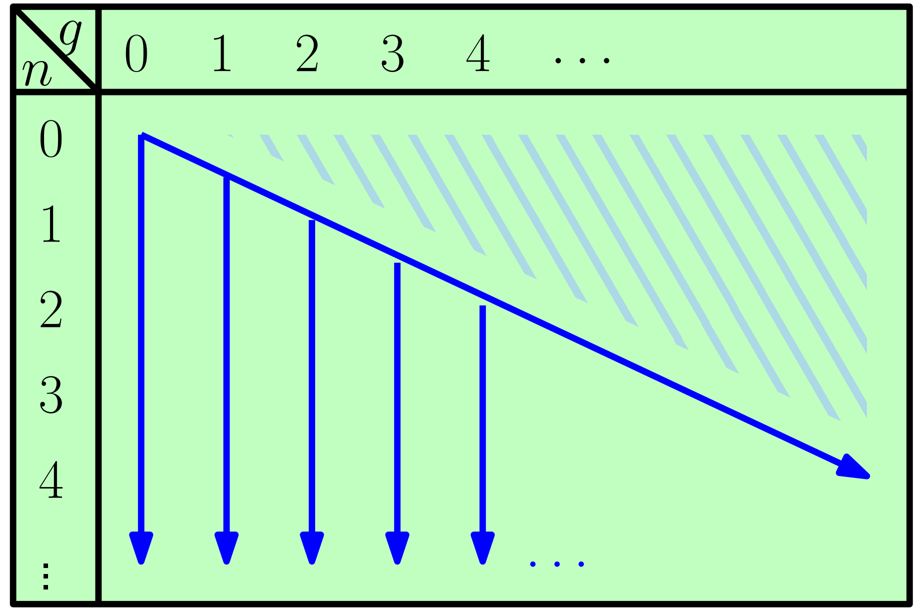

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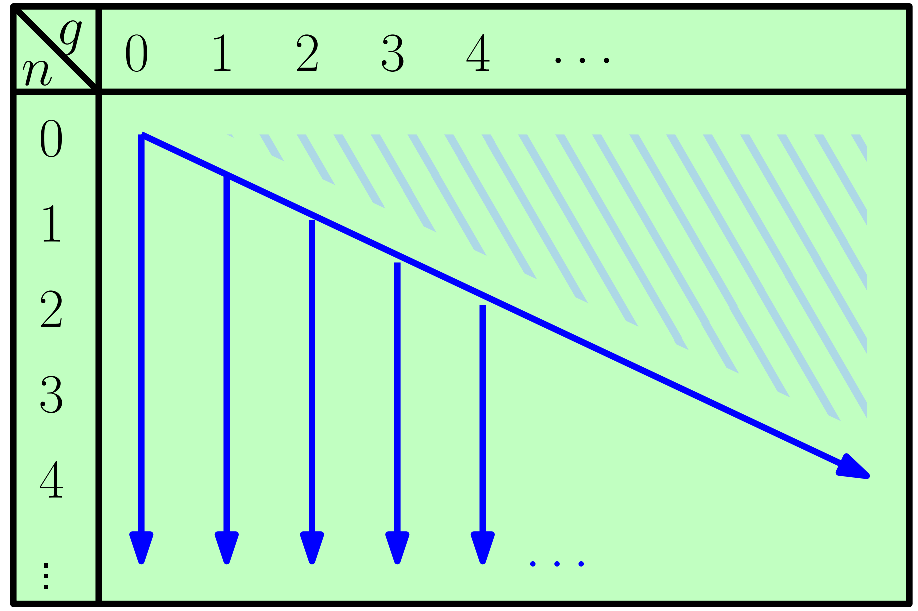
bijections [Schaeffer 97, Poulalhon, Fusy, Bouttier, Di Francesco, Guitter...]



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New – a (very) simple recurrence formula

•**Theorem** [Carrell-C. 14] The numbers Q_g^n of rooted maps of genus g with n edges satisfy the simple recurrence formula:

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• here "simple" $=$ only two variables (just n and g, or z and g).

• To compute F_g the methods based on Tutte/loop equations require to compute multivariate generating functions $F_h(z; x_0, x_1, x_2, \ldots, x_k)$ for $0 \leq h + k \leq g$ where $(x_0, x_1, x_2, \ldots, x_k)$ mark the degrees of $(k+1)$ marked faces.

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• "simple" also means that it looks very combinatorial!!!

For example with naive Maple programming my computer tells me in three minutes that:

$$
\begin{aligned}
Q_{80}^{250} = {} & 344811714404247900735775401395842999405980384512925989150734\,02 \\
& 736870775947694032578369254212911430222826730913008786493509\,68 \\
& 864789774249378175000827233563700475719709018108842812617251\,78 \\
& 478811894995466100812732975799258111674933128992693989184565\,43 \\
& 493766171493383302857620792068690155741646749016423531268273\,94 \\
& 622786018003279508960655197063869577688620805374995801347379\,07 \\
& 277581104860107848707216218686416378058225870347348986274319\,90 \\
& 784854519351902533854780260085365770076664351920276726246142\,37 \\
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& 6227860180032795089606551970638695776886208053749958013473790 7\\
& 2775811048601078487072162186864163780582258703473489862743199 0\\
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& 2454484540639888 0
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$$

which is an interesting statement (?)

New – a (very) simple recurrence formula (II)

We also have a three parameter version (genus/vertices/faces)

•**Theorem** [Carrell-C. 14] The number $M_g^{i,j}$ of maps of genus g with i vertices and f faces satisfies:

$$\frac{n+1}{6} M_g^{i,j} = \frac{(2n-1)}{3}\left(M_g^{i-1,j} + M_g^{i,j-1} + \frac{(2n-3)(2n-2)}{4} M_{g-1}^{i,j} \right)$$

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• For one-face maps ($j = 1$) this is the Harer Zagier recurrence formula!.

$$(n+1)\epsilon_g^n = 2(2n-1)\epsilon_g^{n-1} + (2n-1)(n-1)(2n-3)\epsilon_{g-1}^{n-2}$$

• Note: - ∃ lots of proofs of the HZ-formula, all very specific to the one-face situation. So this is surprising that such a simple generalization exists.

- this one of the simplest (non linear) extensions of HZ one could imagine.

II. Simple map manipulations
(KP as a black box)

Combinatorial warmup: Tutte's classical bijection

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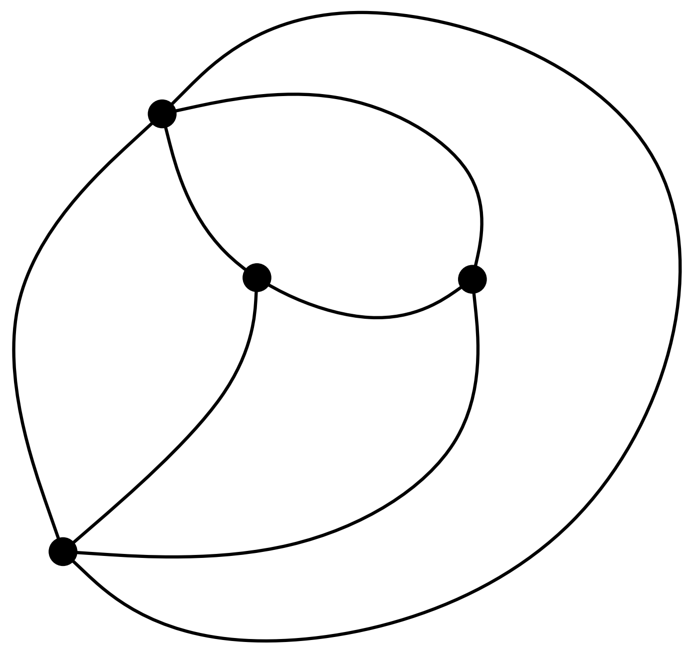

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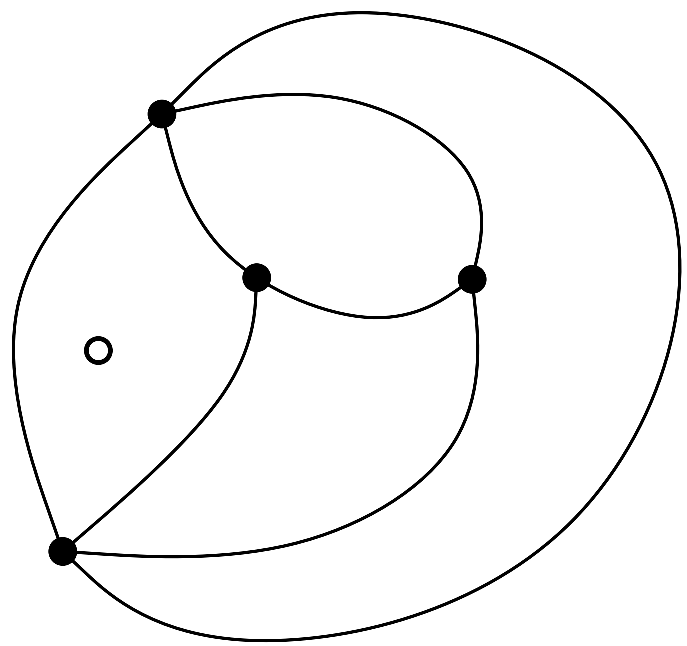

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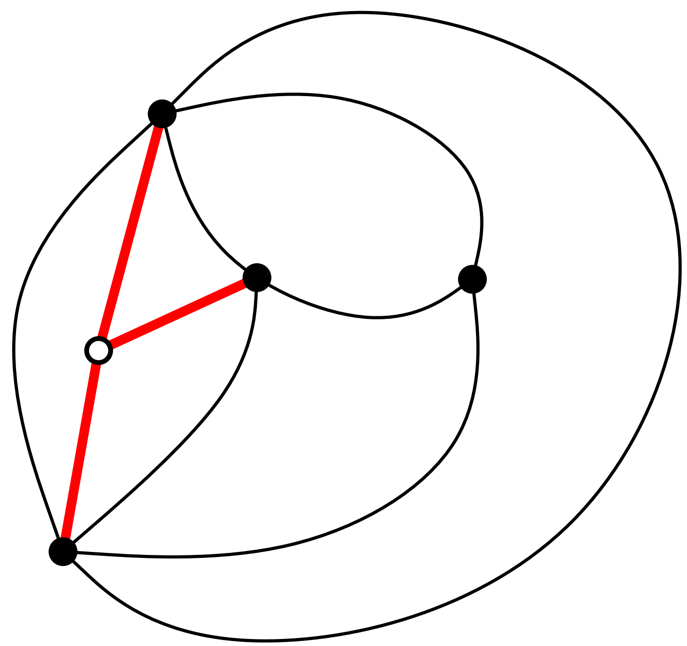

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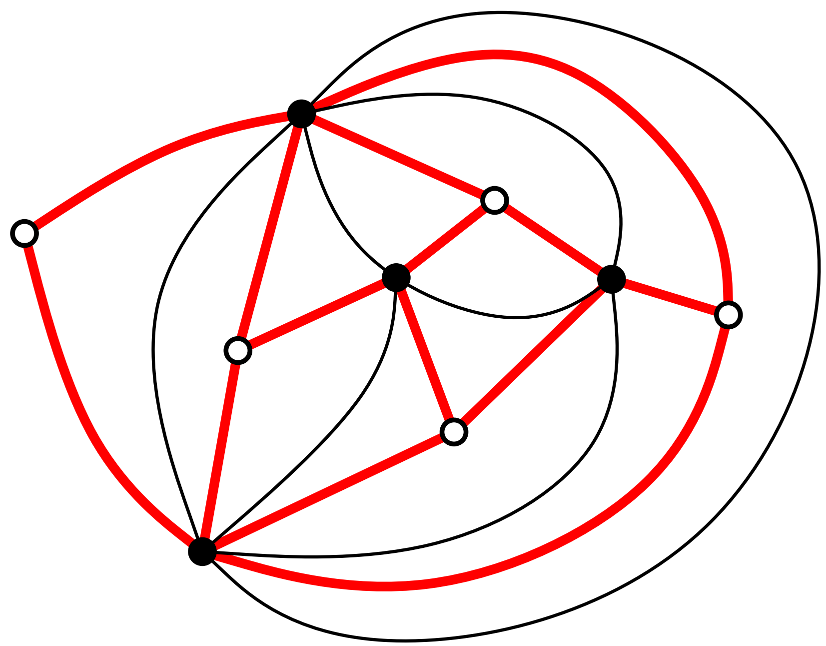
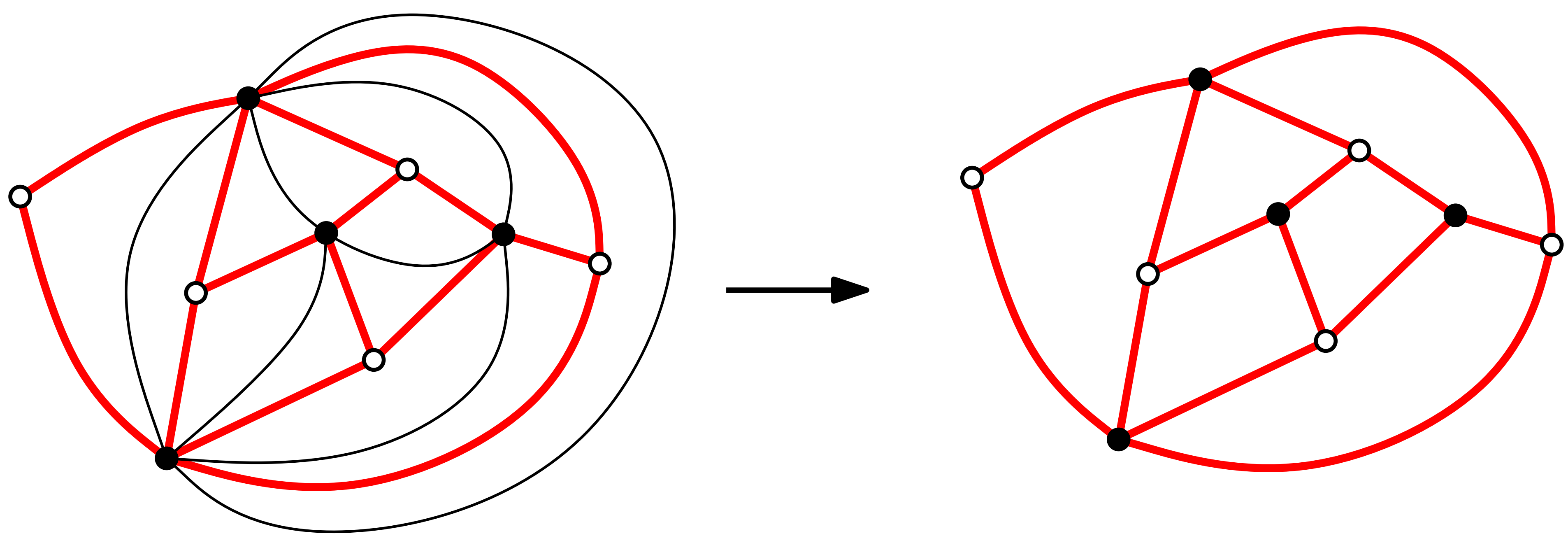


Combinatorial warmup: Tutte's classical bijection

$\{$maps with n edges $\}$

$\leftrightarrow$

$\{$bipartite quadrangulations with n faces (and $2n$ edges)$\}$



A “black box” statement (the magic of algebraic combinatorics)

- Let $F(z,w;\mathbf{p}) = F(z,w;p_1,p_2,p_3,\dots)$ be the generating function of all rooted bipartite maps where: - z marks edges, w marks vertices
  - p_i marks faces of degree $2i$

$$F(z,w;\mathbf{p}) := \sum_{\text{bip. maps}} \frac{z^{\#edges}}{\#\text{edges}} w^{\#\text{vertices}} \prod_i p_i^{\#\text{faces of deg. } 2i}$$

- **Theorem** [Classical physics stuff from 90's - Goulden-Jackson'08]

The function $F(z,w;\mathbf{p})$ satisfies an infinite set of partial differential equations in the variables p_i called the KP hierarchy.

A few small ones are: $F_{2,2} - F_{3,1} + \frac{1}{12}F_{1^4} + \frac{1}{2}F_{1^2}^2 = 0$ (KP)

$$F_{3,2} - F_{4,1} + \frac{1}{6}F_{2,1^3} + F_{1,1}F_{2,1} = 0$$

$$F_{4,2} - F_{5,1} + \frac{1}{4}F_{3,1^3} - \frac{1}{120}F_{1^6} + F_{1^2}F_{3,1} + \frac{1}{2}F_{2,1}^2 - \frac{1}{8}F_{3,1}^2 - \frac{1}{12}F_{1^2}F_{1^4} = 0$$

$$\dots$$

Our formula is an almost trivial (but previously unnoticed) consequence of that fact.

Our (new) formula is straightforward from the KP equation

$$F_{2,2} - F_{3,1} + \frac{1}{12}F_{1^4} + \frac{1}{2}\left(F_{1^2}\right)^2 = 0$$

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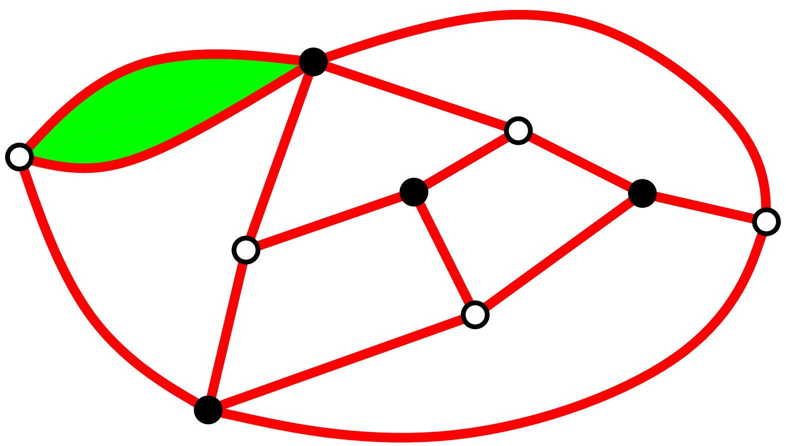
But a marked face of degree 2 is the same as a marked edge:



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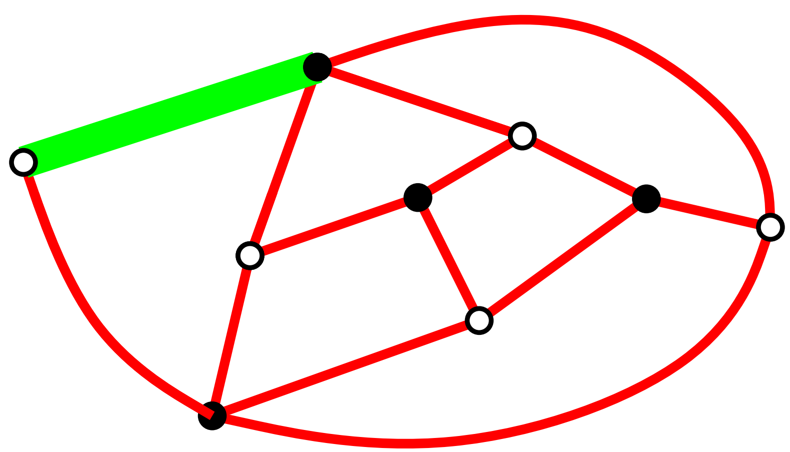

$$[z^{2n}w^{n+2-2g}]F_{1,1}\Big|_{\substack{p_2=1\\p_i=0}} = \frac{2n(2n-1)}{2n}Q_g^{n-1}$$

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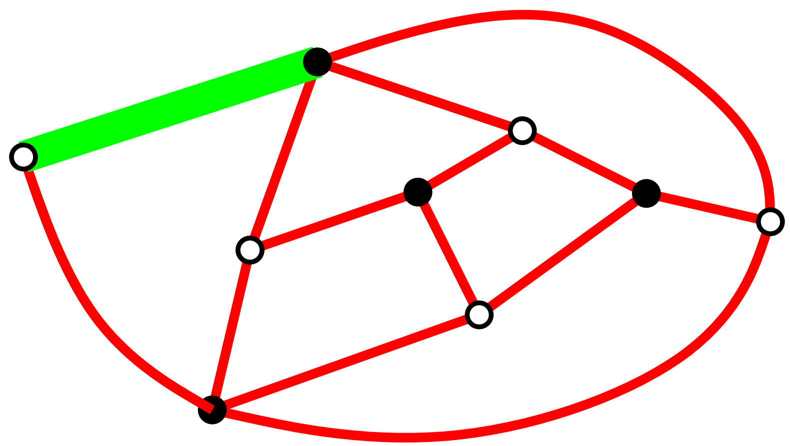

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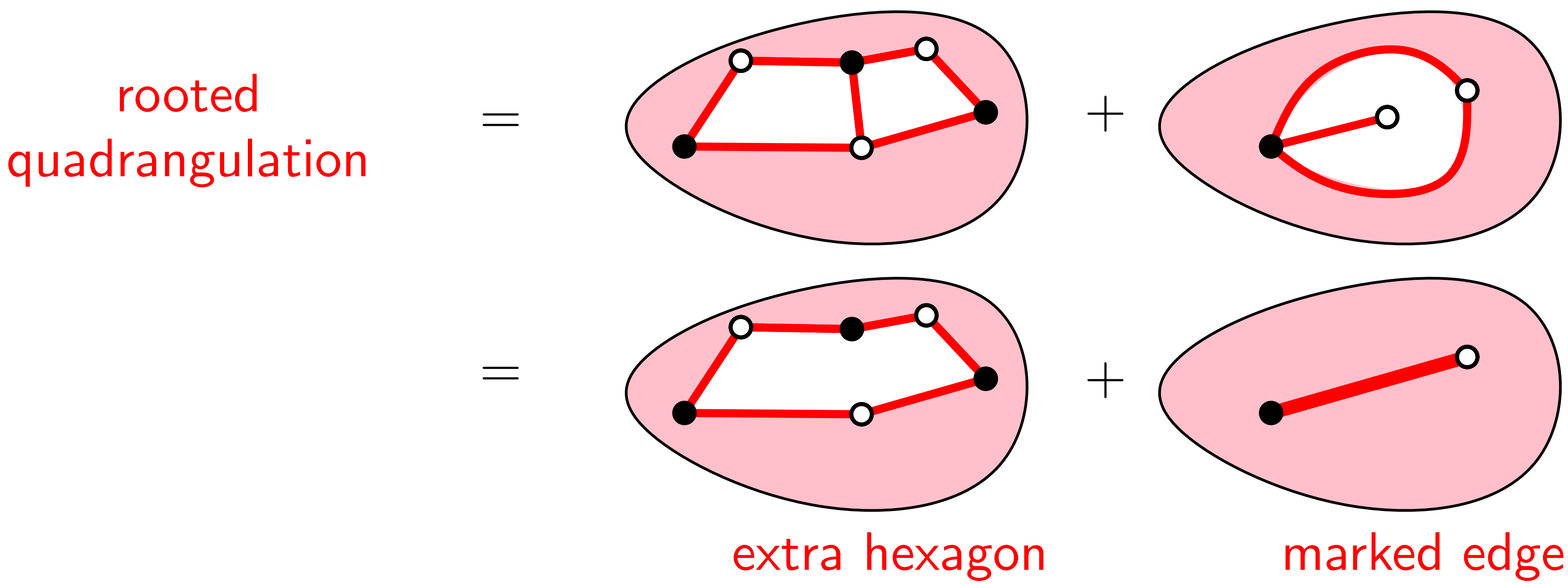
Our (new) formula is straightforward from the KP equation(II)

$$F_{2,2} - F_{3,1} + \tfrac{1}{12}F_{1^4} + \tfrac{1}{2}\left(F_{1^2}\right)^2 = 0$$

- $F_3\Big|_{\substack{p_2=1\\p_i=0}}$ counts quadrangulations with an extra hexagon.

By chance (?) this is also expressible in terms of quadrangulations:



$$[z^{2n}w^{n+2-2g}]F_{3,1}\Big|_{\substack{p_2=1\\p_i=0}} = \tfrac{(2n)2n-1}{3\cdot 2n}\left(Q_g^n - 2Q_g^{n-1}\right).$$

Our (new) formula is straightforward from the KP equation(III)

$$F_{2,2} - F_{3,1} + \tfrac{1}{12} F_{1^4} + \tfrac{1}{2}\left(F_{1^2}\right)^2 = 0$$

$$[z^{2n}w^{n+2-2g}]F_{2,2}\Big|_{\substack{p_2=1\\ p_i=0}} = \tfrac{n(n-1)}{2n} Q_g^n. \qquad [z^{2n}w^{n+2-2g}]F_{1,1}\Big|_{\substack{p_2=1\\ p_i=0}} = \tfrac{2n(2n-1)}{2n} Q_g^{n-1}$$

$$[z^{2n}w^{n+2-2g}]F_{1^4}\Big|_{\substack{p_2=1\\ p_i=0}} = \frac{2n(2n-1)(2n-2)(2n-3)}{2n} Q_g^{n-1}$$

$$[z^{2n}w^{n+2-2g}]F_{3,1}\Big|_{\substack{p_2=1\\ p_i=0}} = \tfrac{2n-1}{3}\left(Q_g^n - 2Q_g^{n-1}\right).$$

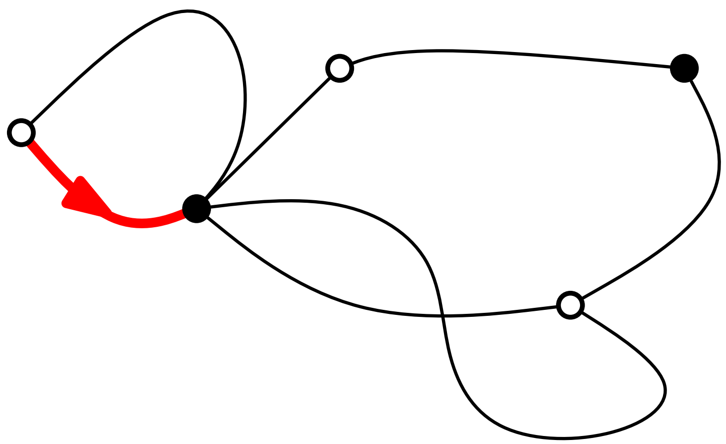
•**Theorem** [Carrell-C. 14] The numbers Q_g^n of rooted maps of genus g with n edges satisfy the simple recurrence formula:

$$\frac{n+1}{6} Q_g^n = \frac{4n-2}{3} Q_g^{n-1} + \frac{(2n-3)(2n-2)(2n-1)}{12} Q_{g-1}^{n-2}$$
$$+ \frac{1}{2} \sum_{\substack{k+\ell=n\\ k,\ell\geq 1}} \sum_{\substack{i+j=g\\ i,j\geq 0}} (2k-1)(2\ell-1) Q_i^{k-1} Q_j^{\ell-1}.$$

III. Where does this KP equation come from? (sketch)

Bipartite maps and permutations

• there is a 1 to $(n-1)!$ correspondence between bipartite maps with n edges and pairs of permutations $(\sigma_\circ, \sigma_\bullet)$ such that the subgroup $\langle \sigma_\circ, \sigma_\bullet \rangle \subset \mathfrak{S}_n$ acts transitively on $[n]$.



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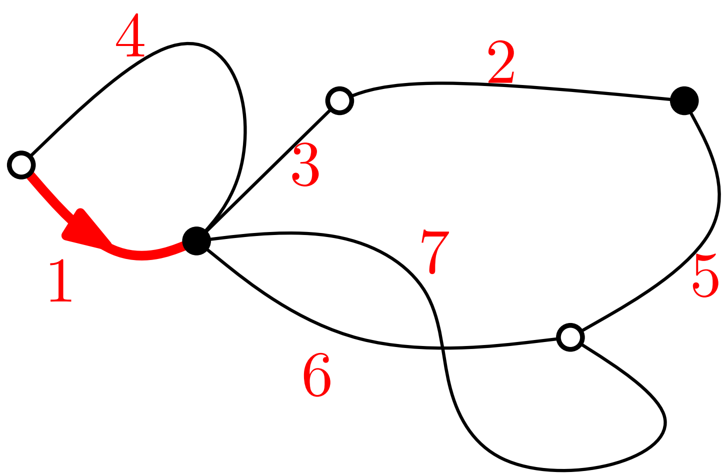

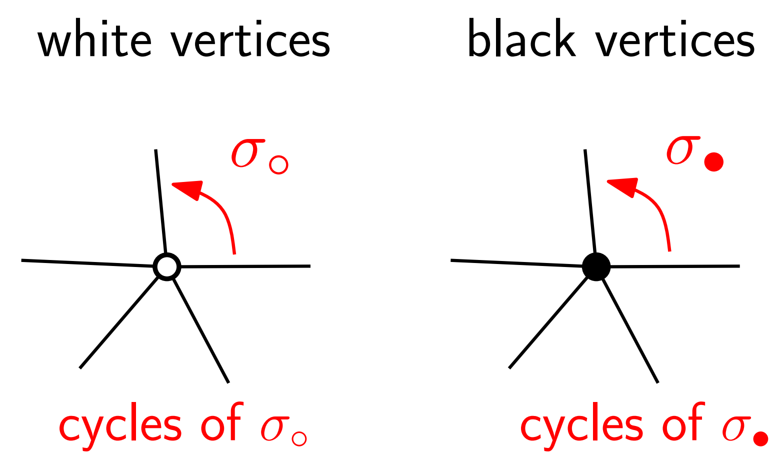

$\sigma_\circ = (1,4)(2,3)(5,6,7)$

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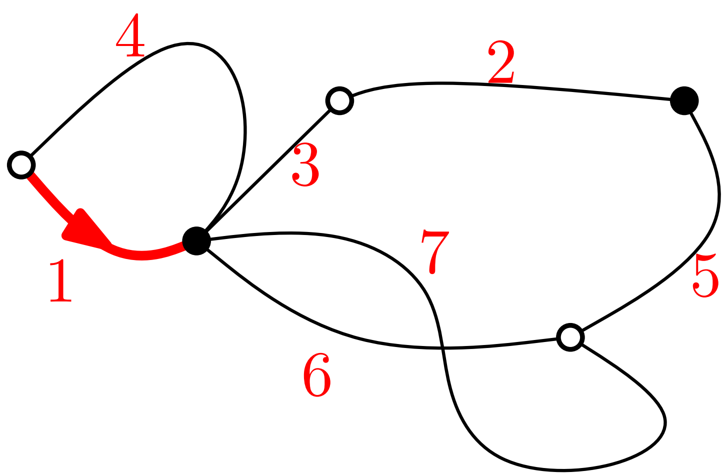

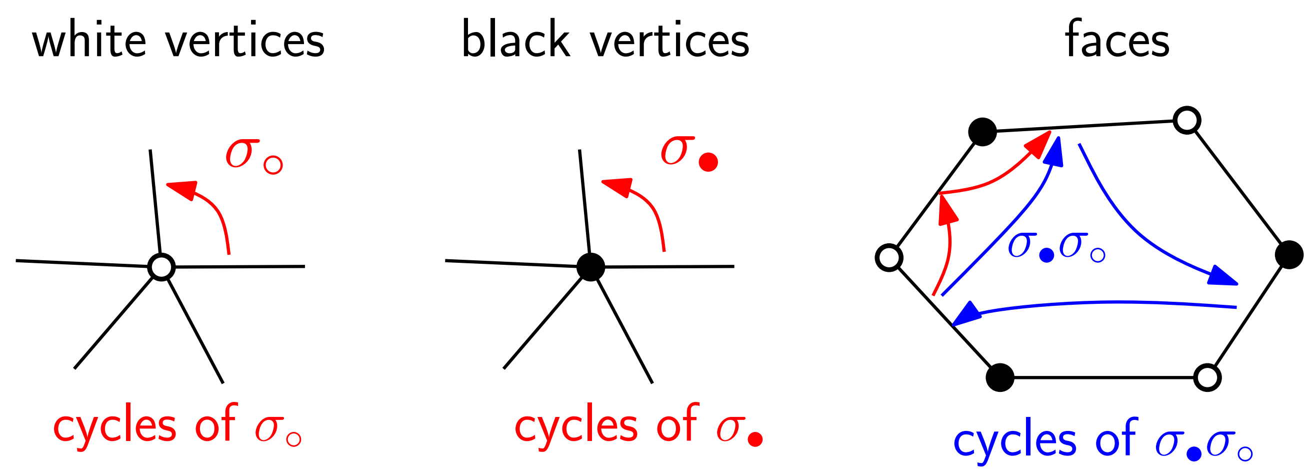

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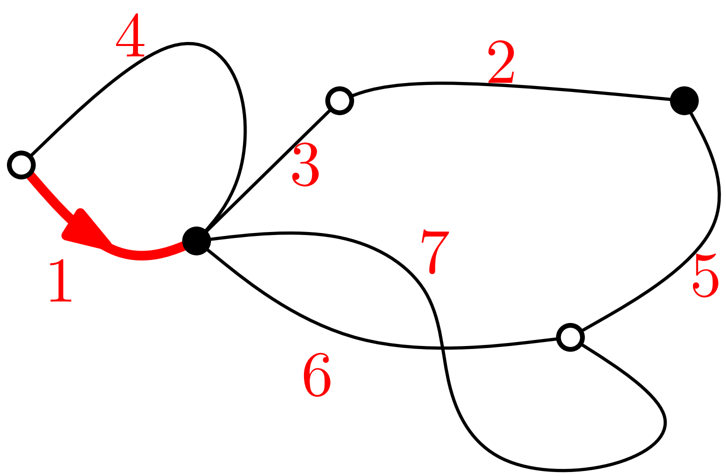
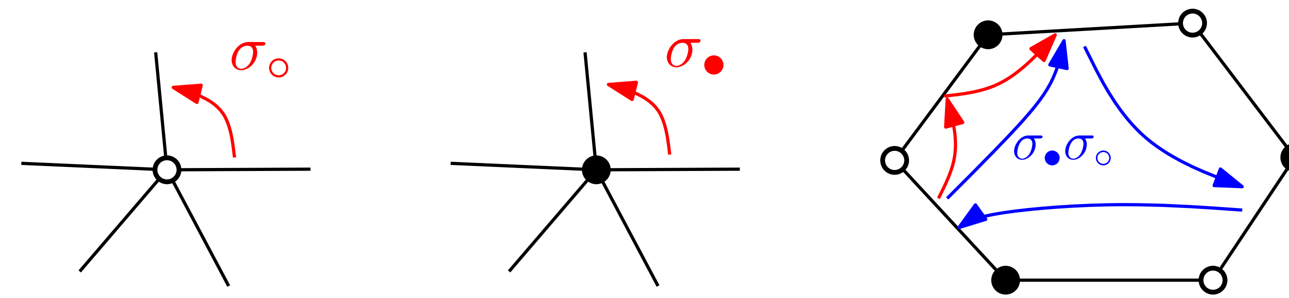


white vertices   black vertices   faces



cycles of $\sigma_\circ$ cycles of $\sigma_\bullet$ cycles of $\sigma_\bullet\sigma_\circ$

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Euler's formula gives the genus:

$$c(\sigma_\circ) + c(\sigma_\bullet) + c(\sigma_\circ\sigma_\bullet) = n + 2 - 2g$$

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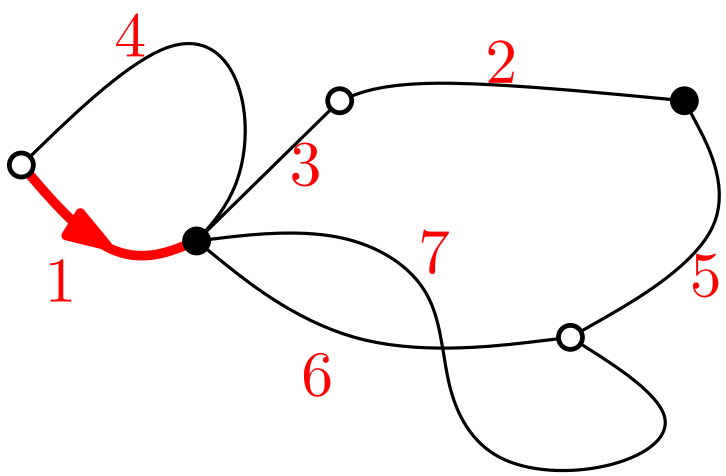

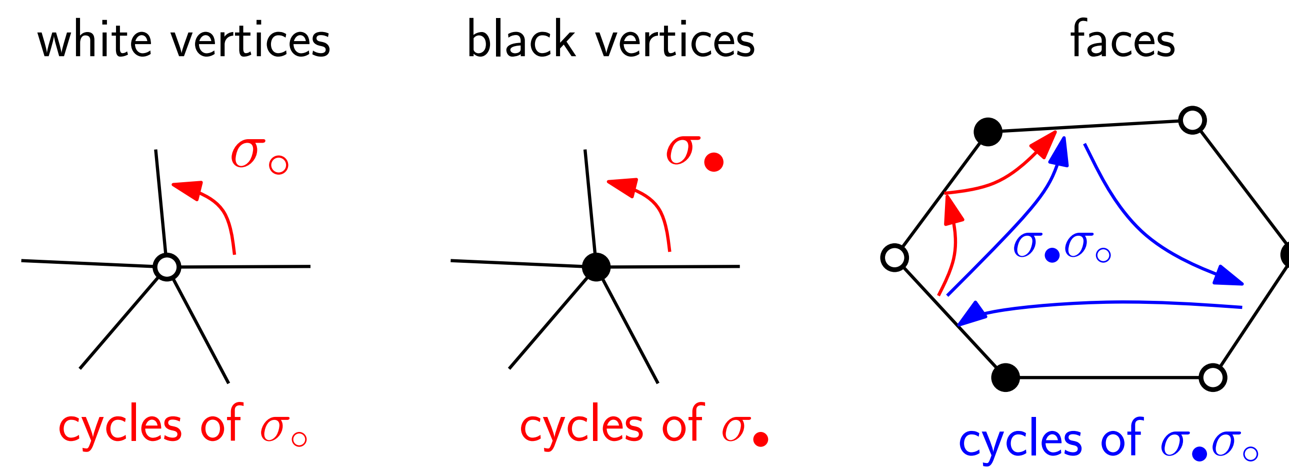

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- Standard rep. theory + standard symmetric function theory leads to

$$F(z,w;\mathbf{p}) = \log \sum_{\lambda \in \mathcal{P}} \frac{z^{|\lambda|}}{|\lambda|!} \left(\sum_{\lambda^\circ} \frac{|K_{\lambda^\circ}|\chi^\lambda(\lambda^\circ) w^{l(\lambda^\circ)}}{\dim\lambda} \right) s_\lambda(w,w,\dots) s_\lambda(\mathbf{p})$$

("character formula") where s_λ = Schur function in terms of power sums.

(An approximation of) the KP hierarchy in one slide

• Using standard symmetric function stuff (such as the Jacobi-Trudi identity and/or Jucys-Murphy elements) the character formula can be put in the form:

$$F(z, w; \mathbf{p}) = \log \sum_{\lambda \in \mathcal{P}} \det\big(M_{i-\lambda_i, j}\big) s_\lambda(\mathbf{p})$$

where $M =$ some explicit infinite (upper triangular) matrix (Jacobi-Trudi like)

• such linear combinations of Schur functions with "determinant" coefficients have lots of properties.

1. The determinants $\det(M_{\lambda_i - i + j})$ for different λ satisfy quadratic relations called the Plücker relations (being minors of a common matrix).
2. if one make the substitution $p_i \to p_i + q_i$ then the new function is still of the same form (with a different matrix M). This follows from the Jacobi-Trudi identity and the Cauchy-Binet formula.
3. By applying 1. to the shifted function $F(p_i + q_i)$, each Plücker relation gives rise to a partial differential equation for $F(z, w; q_i)$. This is what the KP hierarchy is.

Conclusion

- We prove a very simple formula with non combinatorial tools (signs, characters, etc...)

- Hopefully the simplicity of the formula can help us make progress on the bijective side.

- I should have mentioned before that there is another case where something like that works. Indeed Goulden and Jackson found a similar recurrence for triangulations.

- We still don't know whether the miracle that happened here works more generally. For example can one write a closed recurrence formula for hexangulations? pentagulations?

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$Q_{150}^{450} =$ **5689974521343167919979168252107344605562138818742703411**
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THANK YOU!