Tamari lattice, Intervals, and Enumeration

pour le GT-ALEA,
Journées du GDR-IM, Lyon. 2013
Introduction
Some classical combinatorial objects

- There are $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$ such objects (Catalan numbers – proof later)
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The Tamari lattice

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This partial order is a lattice (i.e. there is a notion of sup and inf)

- The Tamari lattice was born and had a great future ahead of it...
The Tamari lattice (pictures)
About the Tamari lattice...

- The Hasse diagram of the Tamari lattice is the graph of a polytope called the associahedron. It is studied by combinatorial geometers.

- In algebraic combinatorics the Tamari lattice is an example of Cambrian lattice underlying the combinatorial structure of Coxeter groups.

- More recently the Tamari lattice was studied in enumerative combinatorics. It has extraordinary enumerative properties...
Enumeration in the Tamari lattice

- We have seen that the number of Dyck paths is $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$
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**Theorem [Chapoton 06]** The number of intervals, i.e. pairs $[P, Q]$ such that $P \preceq Q$ is:

$$I_n = \frac{2}{n(n+1)} \binom{4n+1}{n-1}.$$
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Plan of the talk...

1. I will explain where this comes from (**non-linear catalytic equation**)
2. I’ll mention our **new results** and the kind of **new equations** we solved
3. Give some **comments and perspectives**
Part I: An equation with a catalytic variable

[Chapoton 06]
[Bousquet-Mélou, Fusy, Préville-Ratelle 12]
Crash-course on generating functions I – example

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Better: the generating function $T(t) = \sum_{n=0}^{\infty} a_n t^n$ is solution of

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$$

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a_0 = 1, \quad a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}.
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Better: the generating function $T(t) = \sum_{n=0}^{\infty} a_n t^n$ is solution of

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Recursive specification of the set of binary trees using $\cup$ and $\times$

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Operators on sets map to operators on generating functions
Recursive specification of the set of binary trees using $\sqcup$ and $\times$

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Operators on sets map to operators on generating functions

\[ \sqcup \longrightarrow + \]
\[ \times \longrightarrow \times \]
\[ T(t) = 1 + tT(t)^2 \]
Crash-course on generating functions II – abstraction

- **Recursive specification** of the set of binary trees using $\uplus$ and $\times$

  \[ T = \{\emptyset\} \uplus \{\bullet\} \times T \times T \]

- **Operators on sets** map to **operators on generating functions**

  \[
  \begin{align*}
  \uplus & \longrightarrow + \\
  \times & \longrightarrow \times 
  \end{align*}
  \]

  \[ T(t) = 1 + tT(t)^2 \]

- This is a **polynomial equation**. This is a well known class of equations and from there one can prove that $a_n = \frac{1}{n+1} \binom{2n}{n}$ in various ways.
Recursive specification of the set of binary trees using $\uplus$ and $\times$

$$\mathcal{T} = \{\emptyset\} \uplus (\{\bullet\} \times \mathcal{T} \times \mathcal{T})$$

Operators on sets map to operators on generating functions

$$\uplus \rightarrow + \quad \times \rightarrow \times \quad T(t) = 1 + tT(t)^2$$

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Main point of the talk and active subject of research:
In combinatorics there are other operators than $\uplus$ and $\times$ that lead to other classes of equations. We would like to be as good with them as we are with polynomial equations.

In this talk: equations with catalytic variables.
Fact: We have a recursive decomposition of Tamari intervals.
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Writing an equation for Tamari intervals (I)

**Fact:** We have a **recursive decomposition** of Tamari intervals.
Writing an equation for Tamari intervals (I)

**Fact:** We have a *recursive decomposition* of Tamari intervals.

... this is a *bijection*!
Writing an equation for Tamari intervals (II)

Generating functions

\[ F_i(t) := \sum_{n \geq 0} a_{n,i} t^n \]

\[ F(t; x) =: \sum_{i \geq 1} F_i(t) x^i \]

where \( a_{n,i} = \text{nb of intervals of size } n \text{ with } i \text{ zeros} \) in the lower path.
Writing an equation for Tamari intervals (II)

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\[ F(t; x) = x + t \sum_{i \geq 1} \left( x + x^2 + \cdots + x^i \right) F_i(t) F(t, x) \]
Writing an equation for Tamari intervals (II)

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\[ F(t; x) = x + t \sum_{i \geq 1} \left( x + x^2 + \cdots + x^i \right) F_i(t) F(t, x) \]

\[ = x + tx \sum_{i \geq 1} \frac{x^i - 1}{x - 1} F_i(t) F(t, x) \]

\[ = x + tx \frac{F(t, x) - F(t, 1)}{x - 1} F(t, x) \]
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Generating functions

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Writing an equation for Tamari intervals (II)

$$F(t, x) = x + tx \frac{F(t, x) - F(t, 1)}{x - 1} F(t, x)$$

- This is a polynomial equation with one catalytic variable, i.e. it involves the operators $+, \times$ and $\Delta : A \mapsto \frac{A - A|_{x=1}}{x - 1}$. 
Writing an equation for Tamari intervals (II)

\[ F(t, x) = x + tx \frac{F(t, x) - F(t, 1)}{x - 1} F(t, x) \]

- This is a polynomial equation with one catalytic variable, i.e. it involves the operators +, × and \( \Delta : A \mapsto A - A |_{x=1} \).
- There is a theory for that coming from map enumeration, going back to Knuth and Tutte.
- Exemples of solving techniques:
  - prehistory (Tutte): guess \( F(t, 1) \), solve for \( F(t, x) \), and check the value at \( x = 1 \).
  - 21st century [Bousquet-Mélou/Jehanne]: general theorem, the solution is an algebraic function, and there is an algorithm to find it that you can run on (say) Maple.
An version of the algorithm [Brown, Tutte, 1960’s]

\[ F(t, x) = x + tx \frac{F(t, x) - F(t, 1)}{x - 1} F(t, x) \]

- Write this equation \( P(F, f, x, t) = 0 \) with \( f = F(t, 1) \) and \( F = F(t, x) \)
An version of the algorithm [Brown, Tutte, 1960’s]

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• Write this equation \( P(F, f, x, t) = 0 \) with \( f = F(t, 1) \) and \( F = F(t, x) \)

• Force \( x \) to live on a special ”curve” \( x = x(t) \) by adding the equation \( P_F'(F, f, x, t) = 0 \).

• Then we also have that \( P_x'(F, f, x, t) = 0 \).

• Solve the system

\[
\begin{align*}
P(F, f, x, t) &= 0 \\
P_F'(F, f, x, t) &= 0 \\
P_x'(F, f, x, t) &= 0
\end{align*}
\]

for the 3 unknowns \( F = F(t, x), f = F(t, 1), x = x(t) \).

[Bousquet-Mélou-Jehanne 04] say that this always works (actually a far reaching generalization of this…)

Part II: Labelled Dyck paths and intervals
Labelled Dyck paths

A labelled Dyck path

up steps labelled from 1 to \( n \) and increasing along rises
Labelled Dyck paths

A labelled Dyck path

- Number of labelled Dyck paths $= (n + 1)^{n-1}$
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- **Refinement:** Let $\sigma \in S_n$ be a permutation. Then the number of labelled Dyck paths whose rise-partition is stable by $\sigma$ is $(n + 1)^{k-1}$ where $k = \#cycles(\sigma)$. 
A labelled Tamari interval is a pair $[P, Q]$ where
- $P$ is a Dyck path
- $Q$ is a labelled Dyck path
- $P \preceq Q$ for Tamari
A labelled Tamari interval is a pair \([P, Q]\) where
- \(P\) is a Dyck path
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The number of labelled Tamari intervals is \(2^n(n + 1)^{n-2}\)

**Theorem** [Bousquet-Mélou,C., Préville-Ratelle 2011]

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Refinement: Let \(\sigma \in S_n\) be a permutation. Then the number of labelled Tamari intervals whose rise-partition is stable by \(\sigma\) is

\[
(n + 1)^{k-2} \prod_{i \geq 1} \binom{2i}{i}^{\alpha_i}
\]

if \(\sigma\) has \(\alpha_i\) cycles of length \(i\) for \(i \geq 1\) and \(k\) cycles in total
The decomposition for LABELLED intervals

- The number of labellings of a Dyck path depends on the lengths of the rises.
- Our recursive decomposition does not change the lengths of rises... except for the first one!
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The decomposition for LABELLED intervals

- The number of labellings of a Dyck path depends on the lengths of the rises.
- Our recursive decomposition does not change the lengths of rises... except for the first one!
- We introduce a new variable $y$ for first rise of $Q$.

\[
\frac{\partial}{\partial y} F(t, x, y) = x + tx \frac{F(t, x; y) - F(t, 1; y)}{x - 1} F(t, x; 1)
\]

since: \[
\frac{\partial}{\partial y} y^k = ky^{k-1}
\]
→ the factor $k = \frac{k!}{(k-1)!}$ compensates the change of the first rise
What about LABELLED intervals (II)

\[
\frac{\partial}{\partial y} F(t, x, y) = x + tx \frac{F(t, x; y) - F(t, 1; y)}{x - 1} F(t, x; 1)
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- Never seen such an equation (two catalytic variables, one “standard”, one “differential”).
What about LABELLED intervals (II)

\[ \frac{\partial}{\partial y} F(t, x, y) = x + tx \frac{F(t, x; y) - F(t, 1; y)}{x - 1} F(t, x; 1) \]

- Never seen such an equation (two catalytic variables, one “standard”, one “differential”).

- Go back to prehistory:
  
  1. guess \( F(t, x, 1) \) (“only” 2 variables).
  2. use the symmetries of the equation to eliminate \( F(t, 1; y) \)
  3. solve the differential equation
  4. reconstitute \( F(t, x, y) \) and check the value at \( y = 1 \)
Part III: comments
Why we are interested in all this

**Theorem** [Bousquet-Mélou, C., Prévile-Ratelle 2011]

The number of labelled Tamari intervals is $2^n (n + 1)^{n-2}$

Refinement: Let $\sigma \in S_n$ be a permutation. Then the number of labelled Tamari intervals whose rise-partition is stable by $\sigma$ is

$$(n + 1)^{k-2} \prod_{i \geq 1} \left( \begin{array}{c} 2i \\ 2i \end{array} \right)^{\alpha_i}$$

if $\sigma$ has $\alpha_i$ cycles of length $i$ for $i \geq 1$ and $k$ cycles in total

- Original motivation: algebraists believe that this formula is the character of the trivariate coinvariant module over $S_n$. (very hard conjecture!)
- Our proof is extremely technical but contains ideas hidden behind piles of details. We don’t fully understand why it worked but we hope that this will open the way to a general theory.
- There is a generalization of everything to the $m$-Tamari lattice and it is harder and even more technical.
A historical analogy with planar maps

- A planar map is a planar graph drawn on the plane.
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- A planar map is a planar graph drawn on the plane.

- 1960: the number of planar maps with $n$ edges is $\frac{2 \cdot 3^n}{n + 2} \text{Cat}(n)$.

[Tutte via the first catalytic equation solved with prehistorical techniques]
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• 1960-1990’s many variants discovered with similar techniques
  [Tutte, Brown, Bender, Canfield.... the techniques get stronger]
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- 2004 theory + algorithms for these equations.
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- 1998 and 2000’s BIJECTIVE PROOFS of these formulas
  [Schaeffer, Bouttier, Di Francesco, Guitter]

  Planar maps reveal their true structure via nice tree-decompositions
  The theory of random planar maps becomes extremely rich and active
  Many applications to theoretical physics and probability theory...
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Merci !