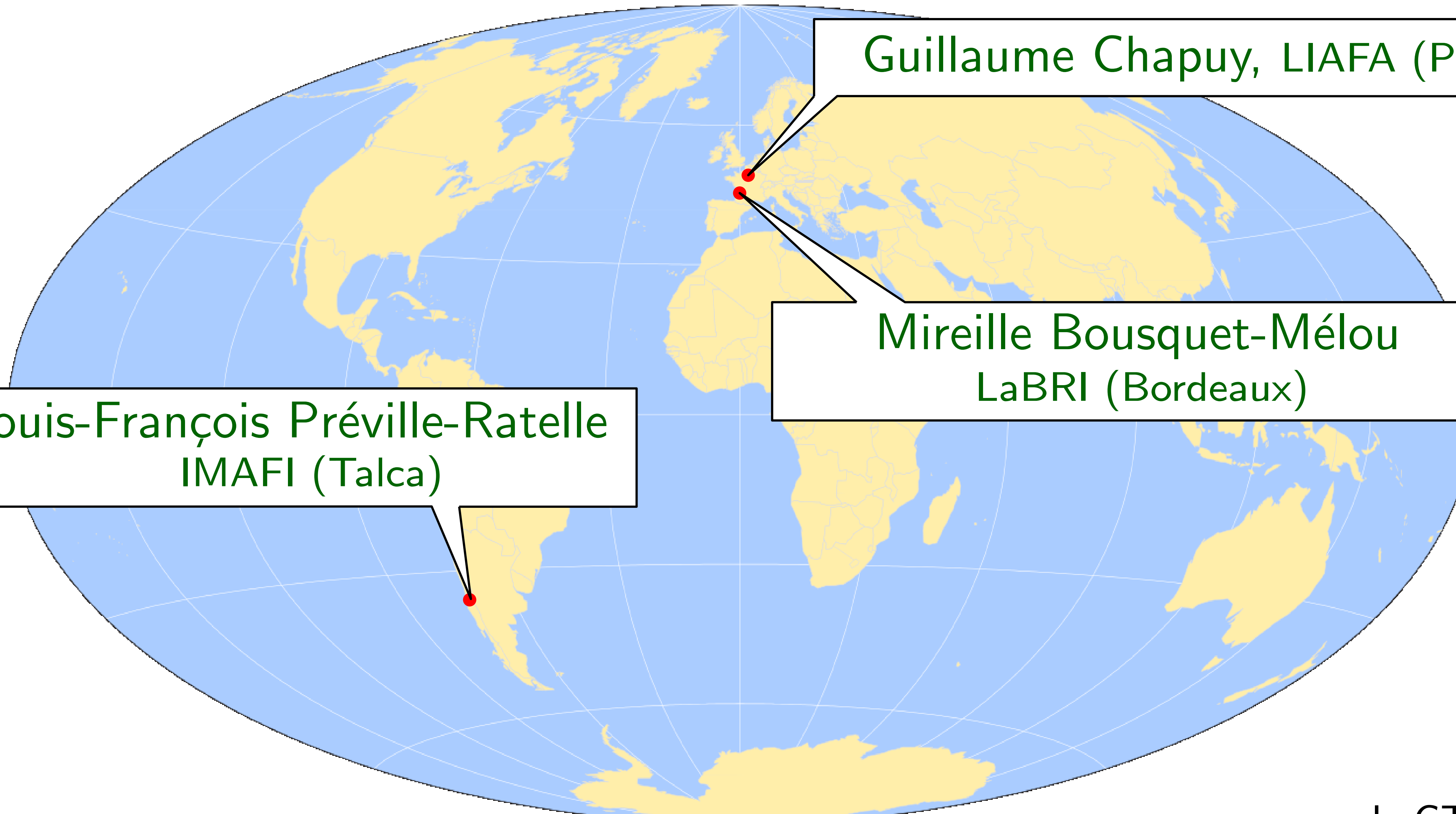
Tamari lattice, Intervals, and Enumeration



Guillaume Chapuy, LIAFA (Paris)

Mireille Bousquet-Mélou
LaBRI (Bordeaux)

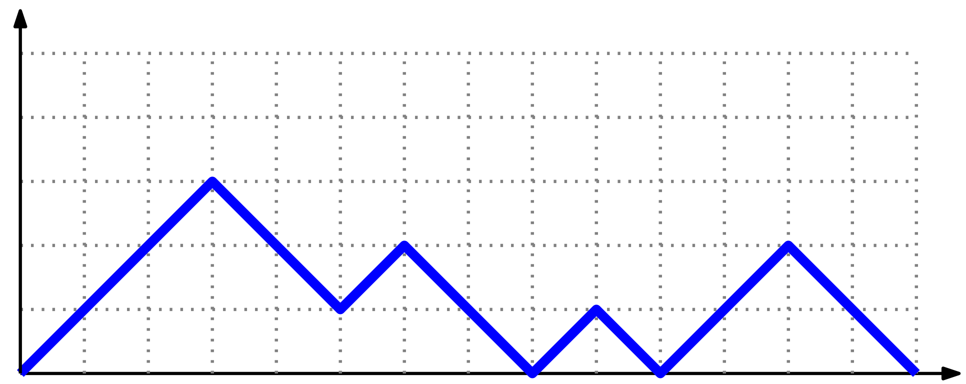
Louis-François Préville-Ratelle
IMAFI (Talca)

pour le GT-ALEA,

Journées du GDR-IM, Lyon. 2013

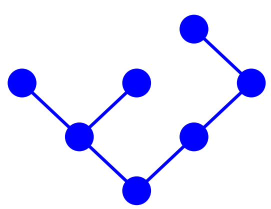
Introduction

Some classical combinatorial objects



a Dyck path of length $2n$

$=$



a binary tree with n vertices

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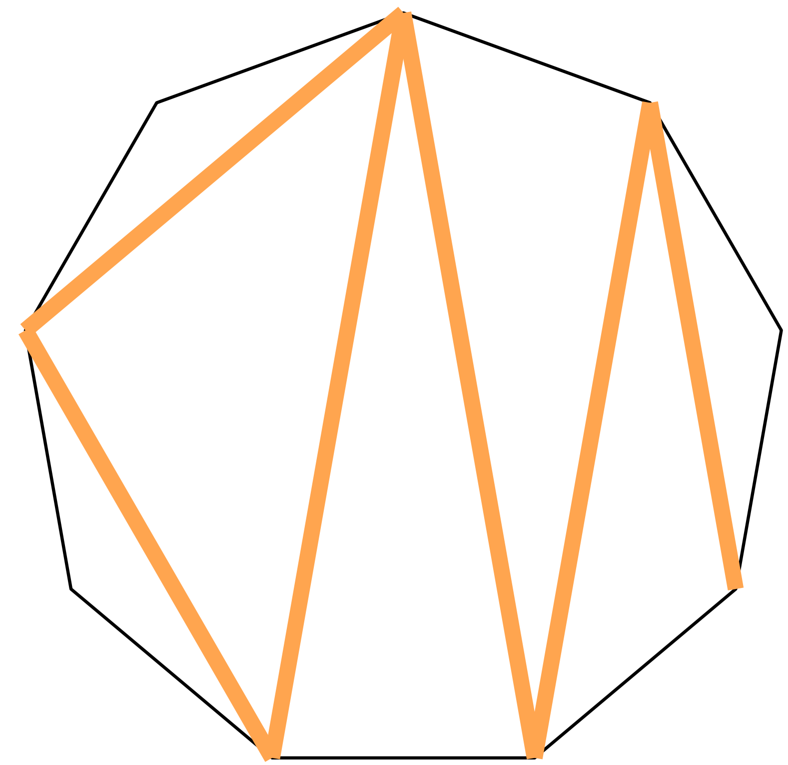

a triangulation of an $(n+2)$-gon with n triangles

- There are $\mathrm{Cat}(n) = \dfrac{1}{n+1}\dbinom{2n}{n}$ such objects (Catalan numbers – proof later)

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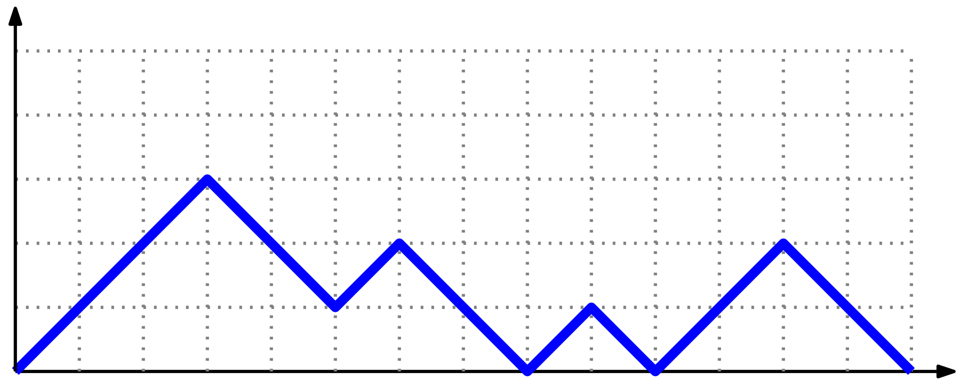

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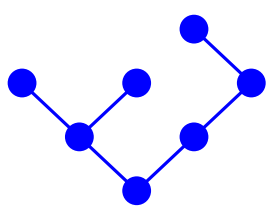

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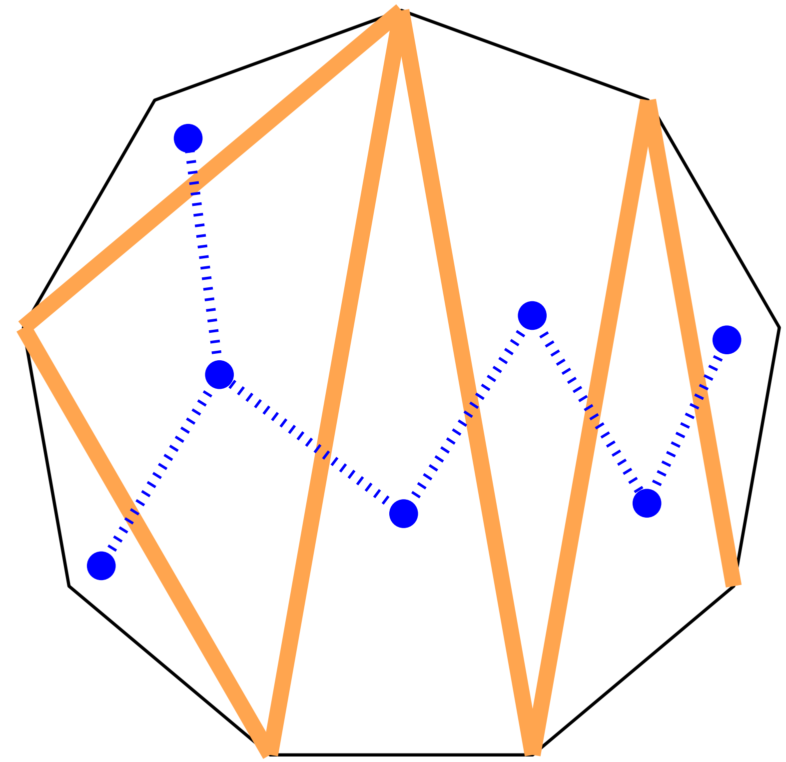

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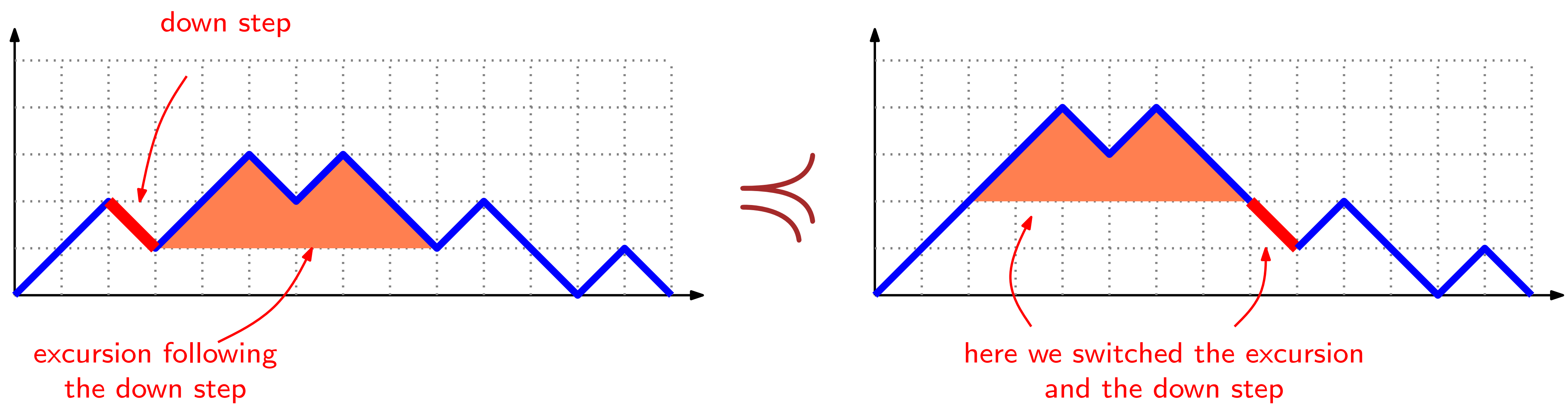

flip on Dyck paths

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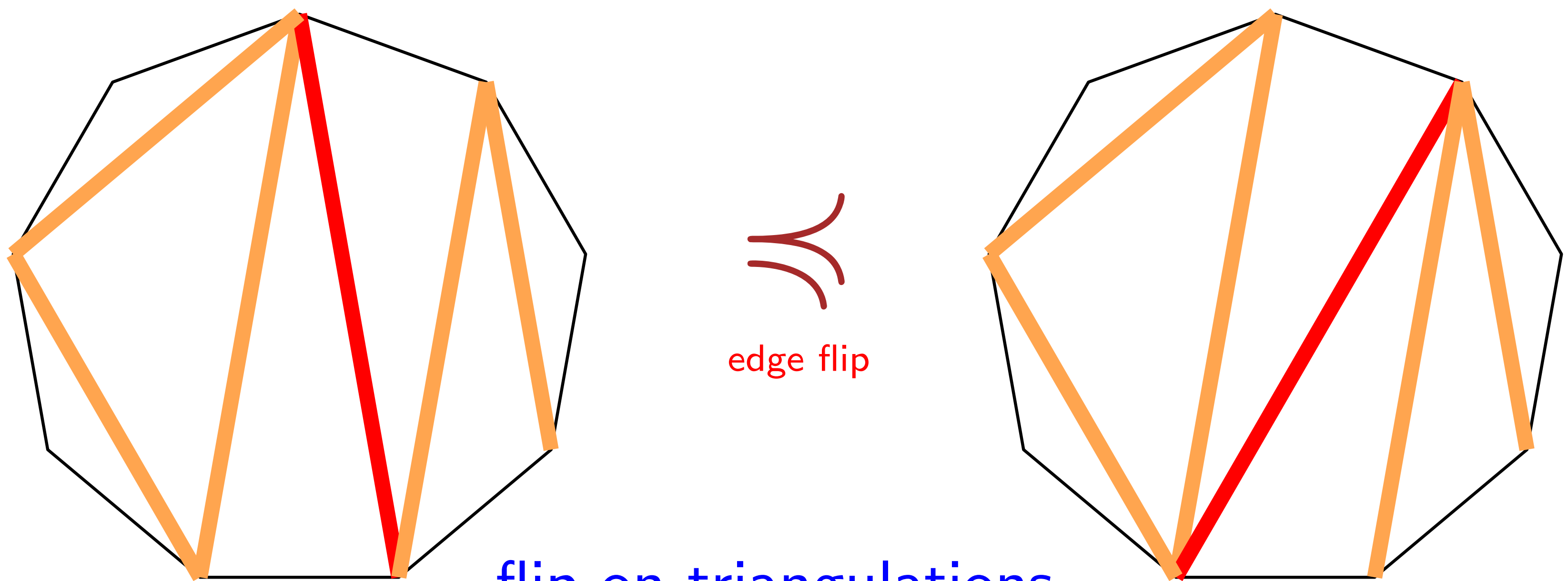

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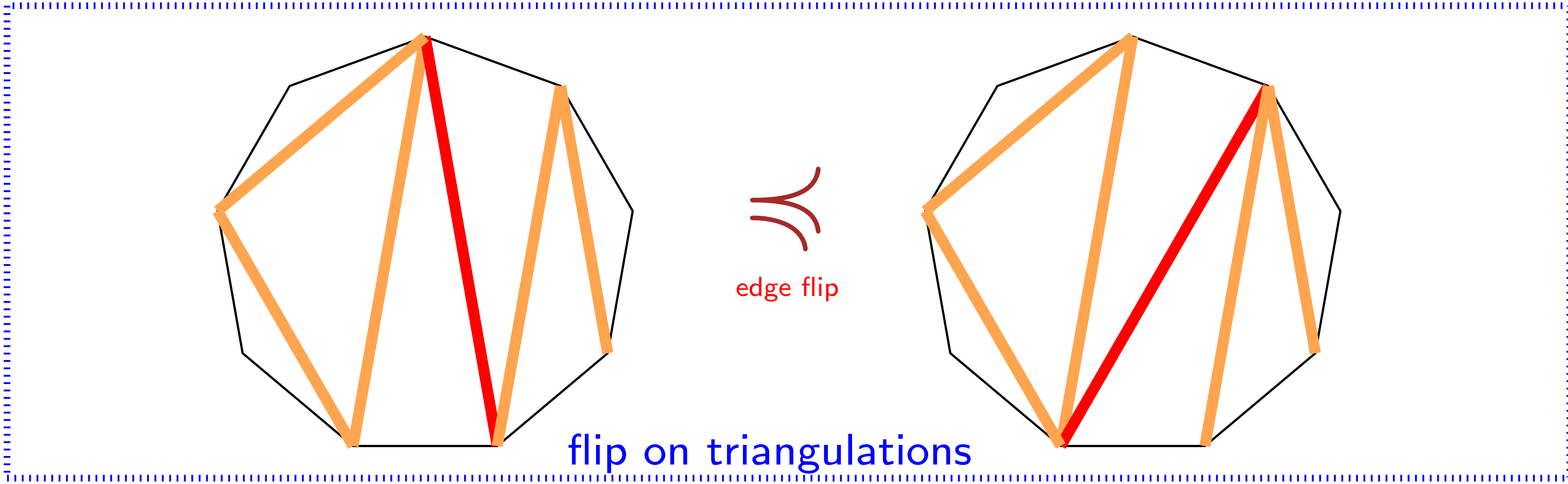
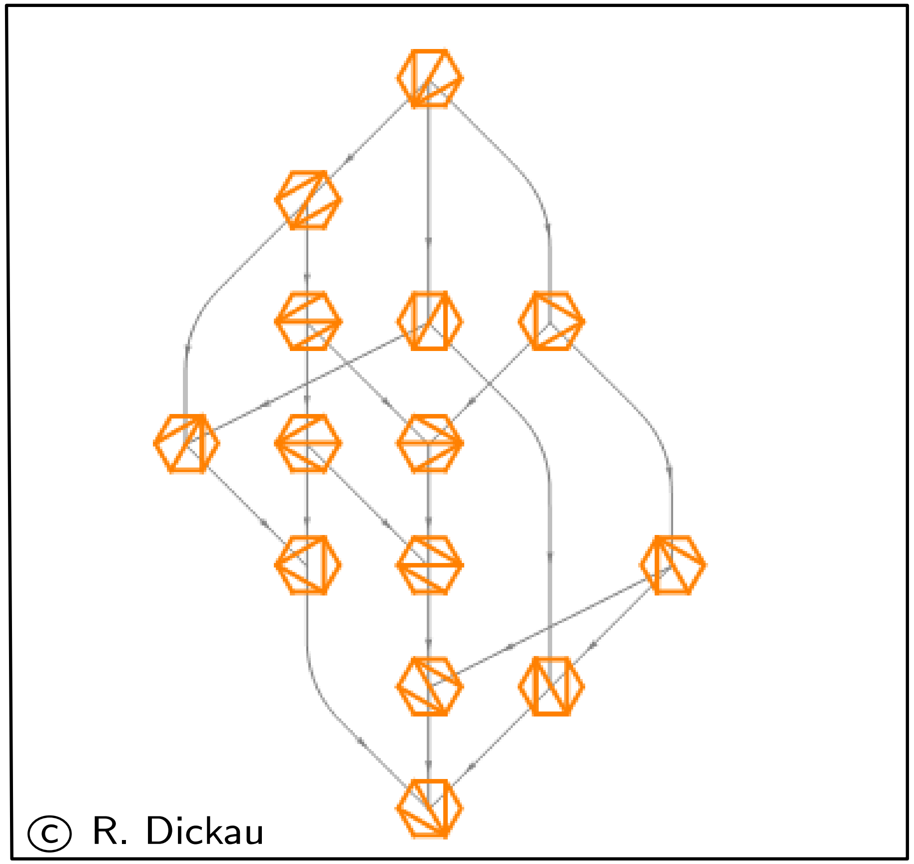
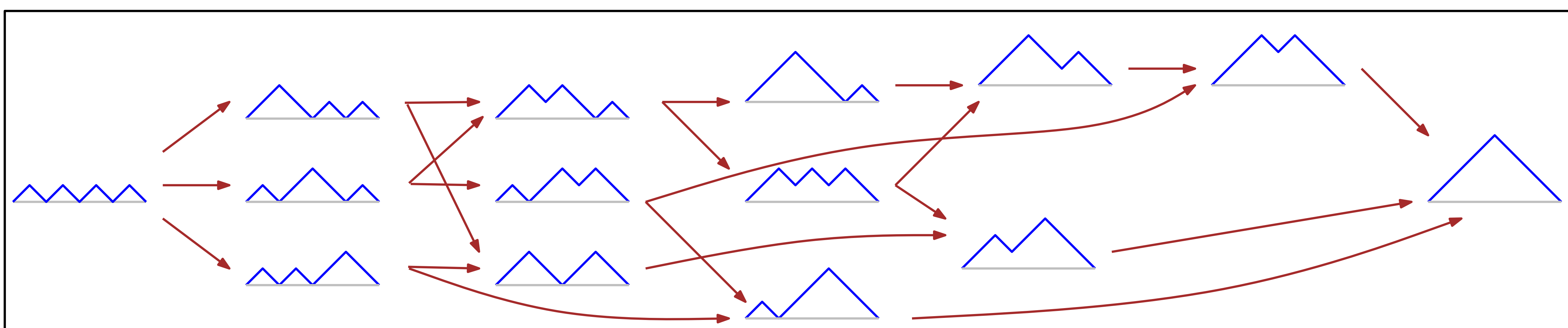


- This partial order is a lattice (i.e. there is a notion of sup and inf)

- The Tamari lattice was born and had a great future ahead of it...

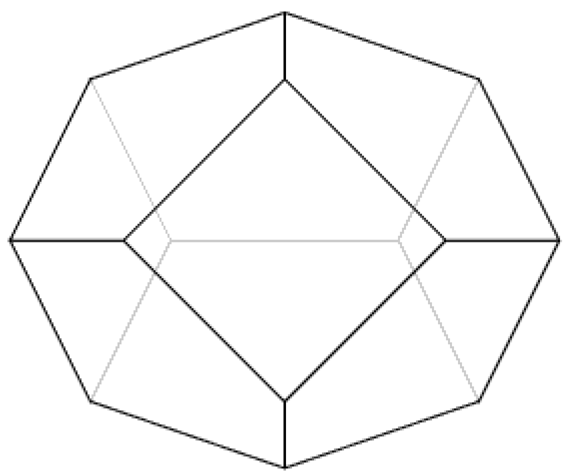
The Tamari lattice (pictures)

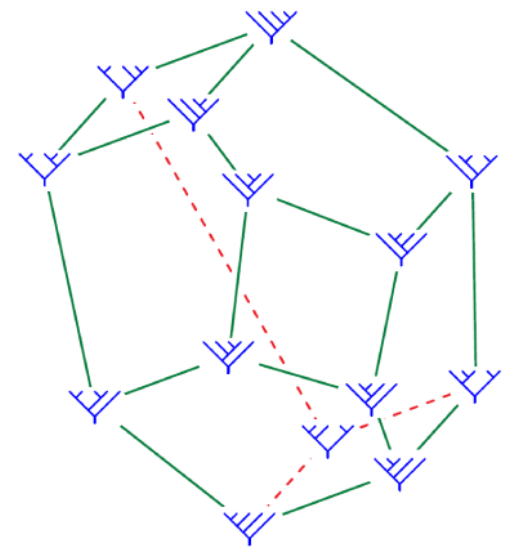




About the Tamari lattice...

• The Hasse diagram of the Tamari lattice is the graph of a polytope called the associahedron. It is studied by combinatorial geometers.





• In algebraic combinatorics the Tamari lattice is an example of Cambrian lattice underlying the combinatorial structure of Coxeter groups.

• More recently the Tamari lattice was studied in enumerative combinatorics. It has extraordinary enumerative properties...

Enumeration in the Tamari lattice

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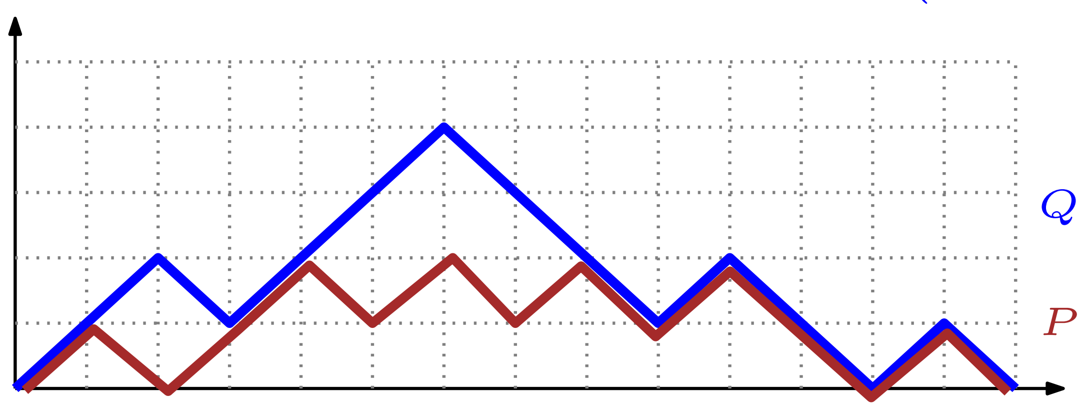
$$I_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$



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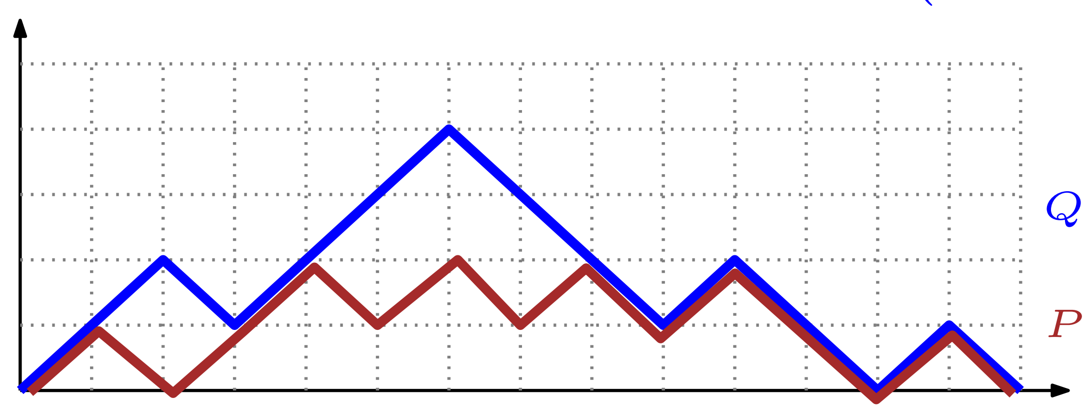


Plan of the talk...

1. I will explain where this comes from (non-linear catalytic equation)
2. I'll mention our new results and the kind of new equations we solved
3. Give some comments and perspectives

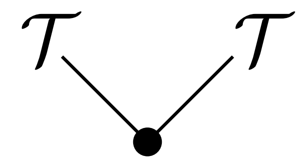
Part I: An equation with a catalytic variable

[Chapoton 06]
[Bousquet-Mélou, Fusy, Préville-Ratelle 12]

Crash-course on generating functions I – example

- The class $\mathcal{T}$ of binary trees is defined by the formula

$$\mathcal{T} = \varnothing \quad + $$



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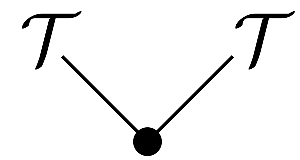

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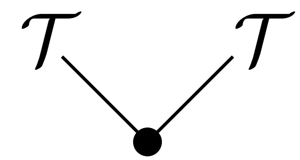

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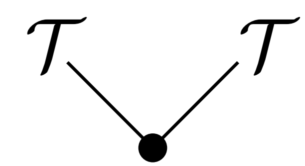

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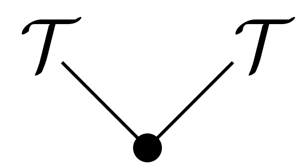
$$\mathcal{T} = \varnothing \quad + $$ 

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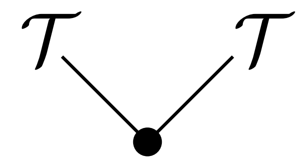

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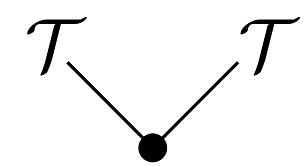

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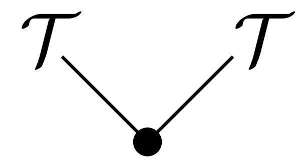

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- Recursive specification of the set of binary trees using $\uplus$ and $\times$

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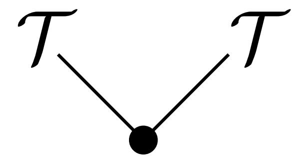

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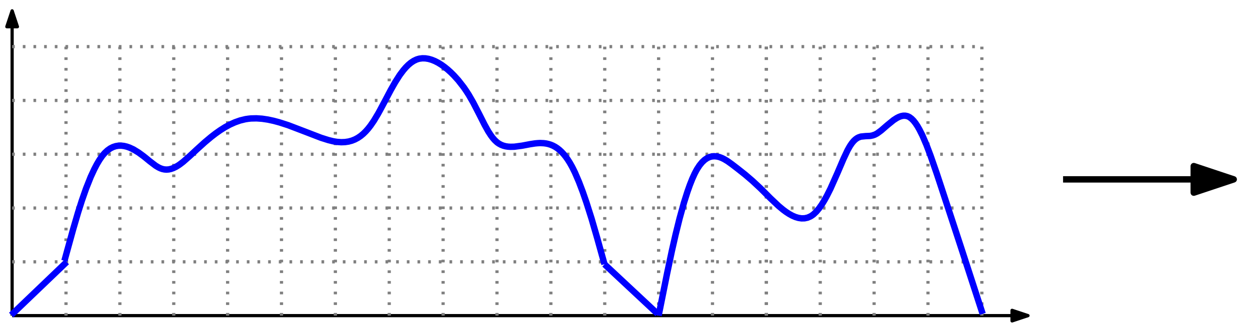
Main point of the talk and active subject of research:
In combinatorics there are other operators than $\uplus$ and $\times$ that lead to other classes of equations. We would like to be as good with them as we are with polynomial equations.

In this talk: equations with catalytic variables.

Writing an equation for Tamari intervals (I)

Fact: We have a recursive decomposition of Tamari intervals.



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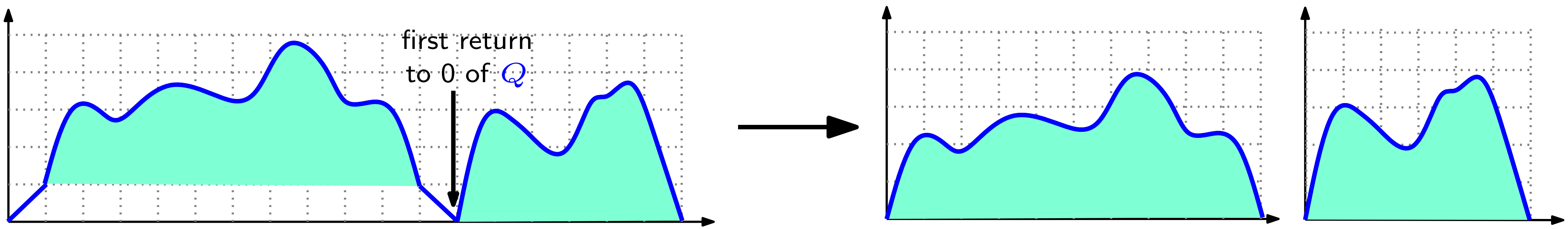

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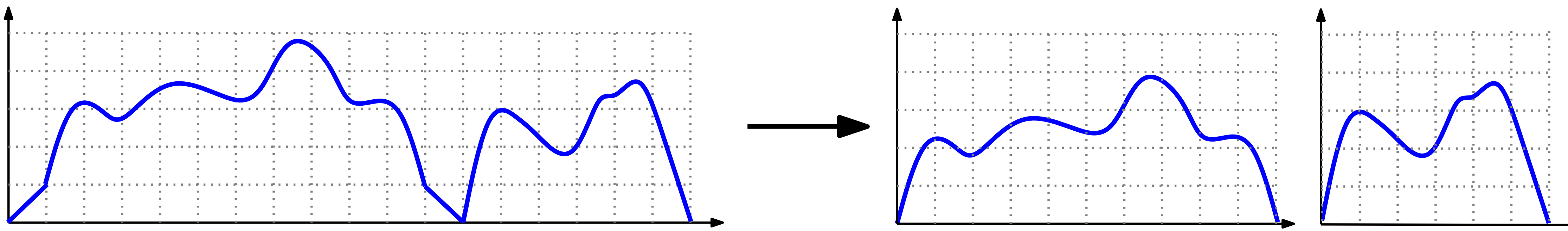

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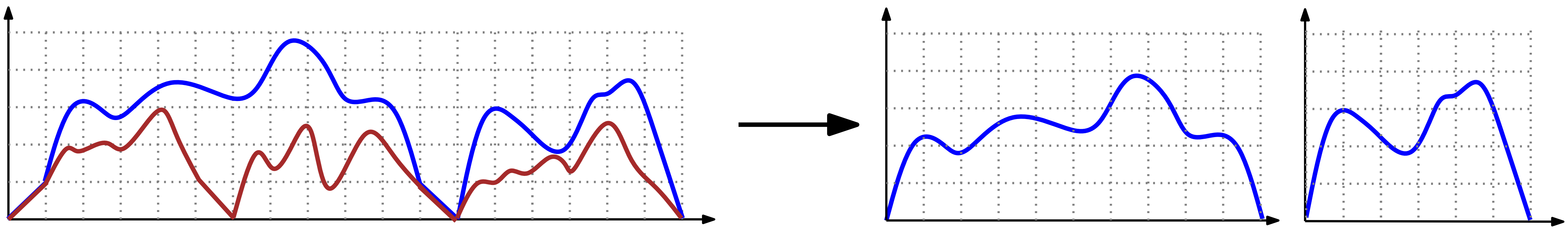

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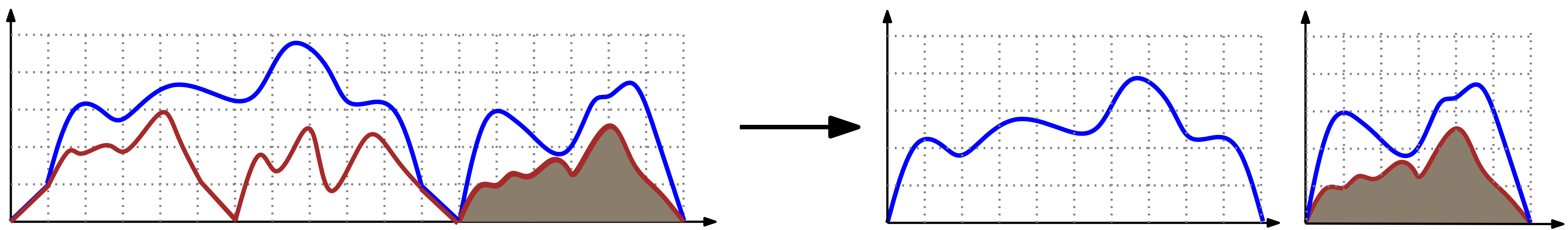

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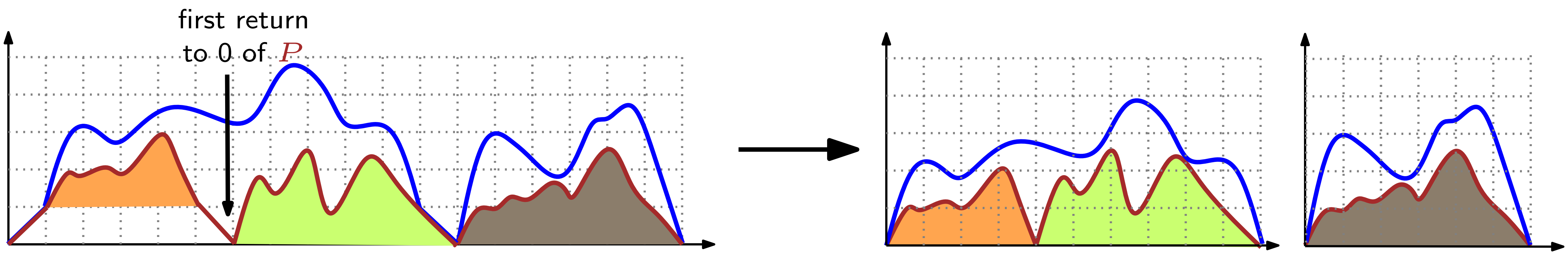

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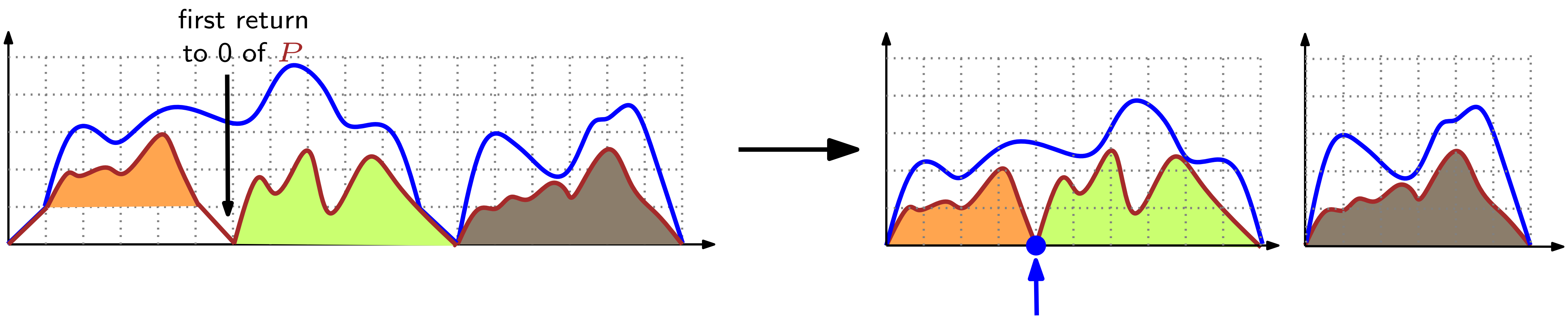
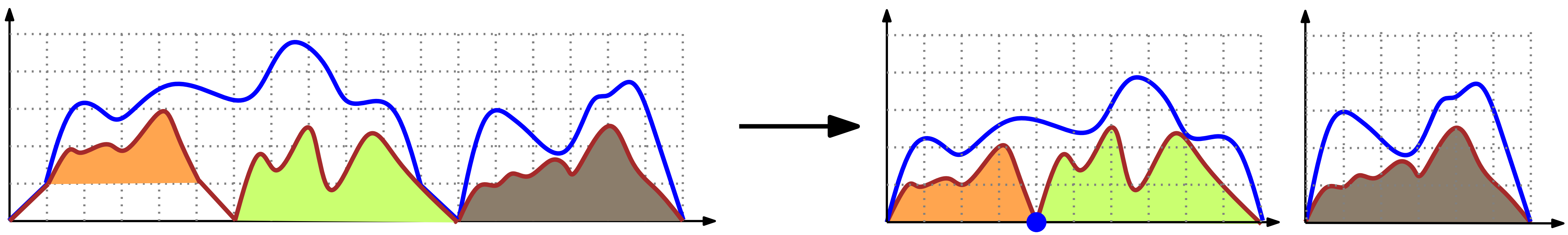


Tamari interval → Tamari interval with a pointed zero in the lower path | Tamari interval

... this is a bijection!

Writing an equation for Tamari intervals (II)



Generating functions

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$$F(t;x) =: \sum_{i \geq 1} F_i(t) x^i$$

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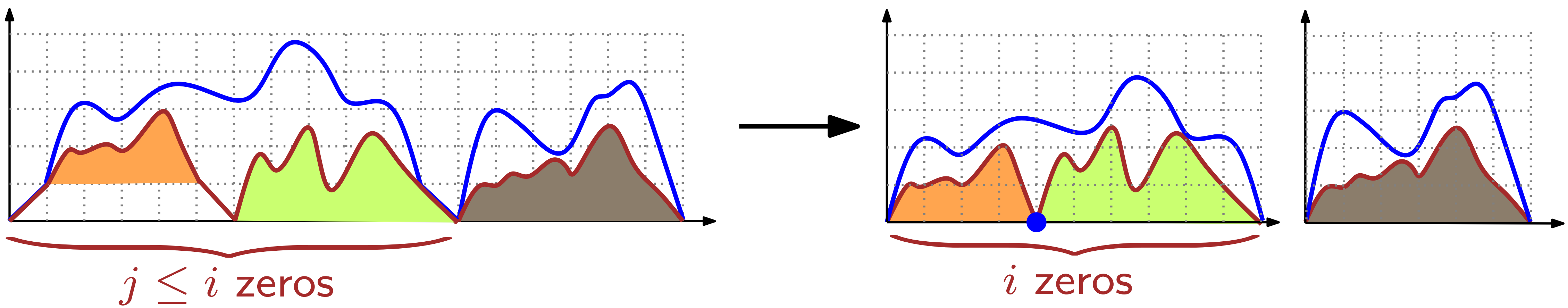

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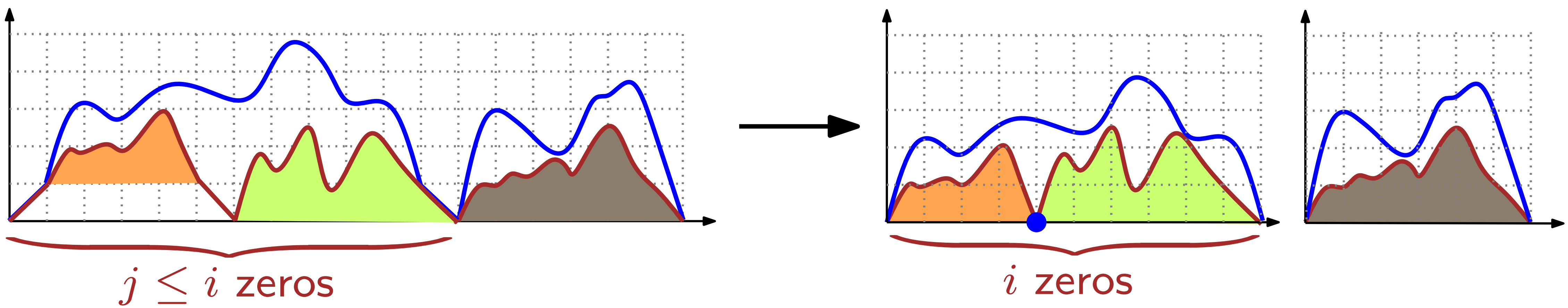

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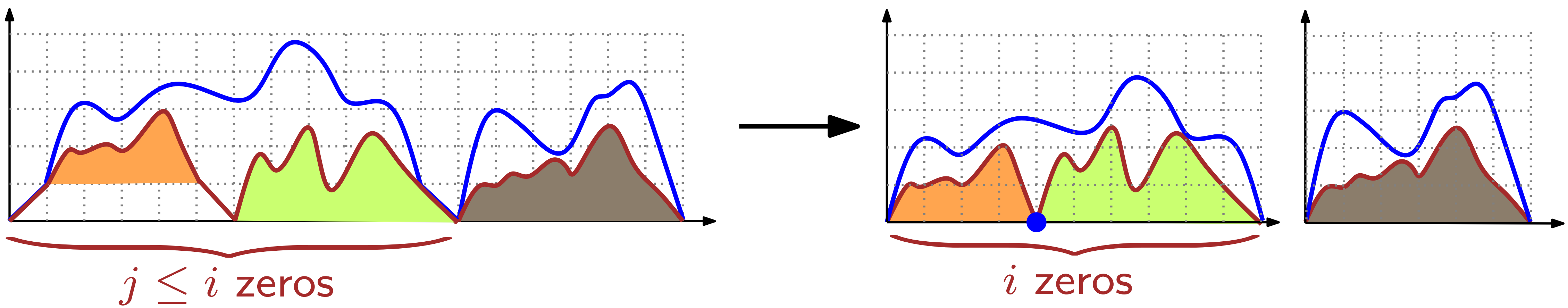

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- There is a theory for that coming from map enumeration, going back to Knuth and Tutte.

- Exemples of solving techniques:

  - prehistory (Tutte): guess $F(t,1)$, solve for $F(t,x)$, and check the value at $x=1$.

  - 21st century [Bousquet-Mélou/Jehanne]: general theorem, the solution is an algebraic function, and there is an algorithm to find it that you can run on (say) Maple.

An version of the algorithm [Brown, Tutte, 1960's]

$$\boxed{F(t,x) = x + tx\frac{F(t,x) - F(t,1)}{x-1}F(t,x)}$$

- Write this equation $P(F, f, x, t) = 0$ with $f = F(t,1)$ and $F = F(t,x)$

An version of the algorithm [Brown, Tutte, 1960's]

$$\boxed{F(t,x) = x + tx\frac{F(t,x) - F(t,1)}{x-1}F(t,x)}$$

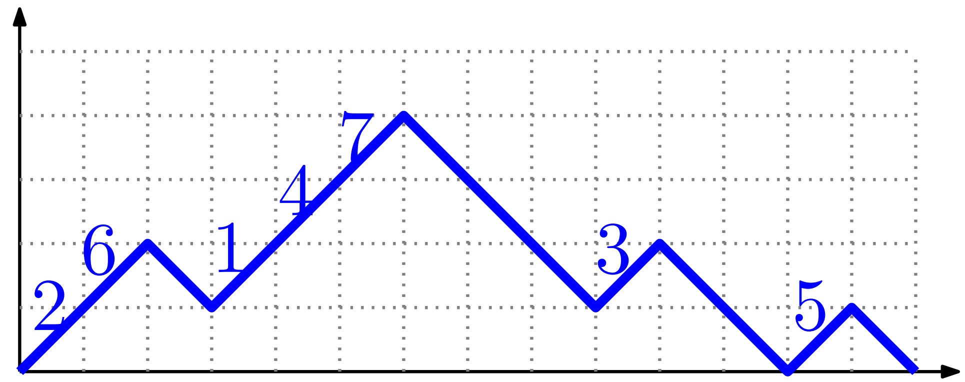
- Write this equation $P(F, f, x, t) = 0$ with $f = F(t,1)$ and $F = F(t,x)$
- Force x to live on a special "curve" $x = x(t)$ by adding the equation $P'_F(F, f, x, t) = 0$.
- Then we also have that $P'_x(F, f, x, t) = 0$.
- Solve the system $\begin{cases} P(F, f, x, t) & = 0 \\ P'_F(F, f, x, t) & = 0 \\ P'_x(F, f, x, t) & = 0 \end{cases}$

for the 3 unknowns $F = F(t,x), f = F(t,1), x = x(t)$.

[Bousquet-Mélou-Jehanne 04] say that this always works
(actually a far reaching generalization of this...)

Part II: Labelled Dyck paths and intervals

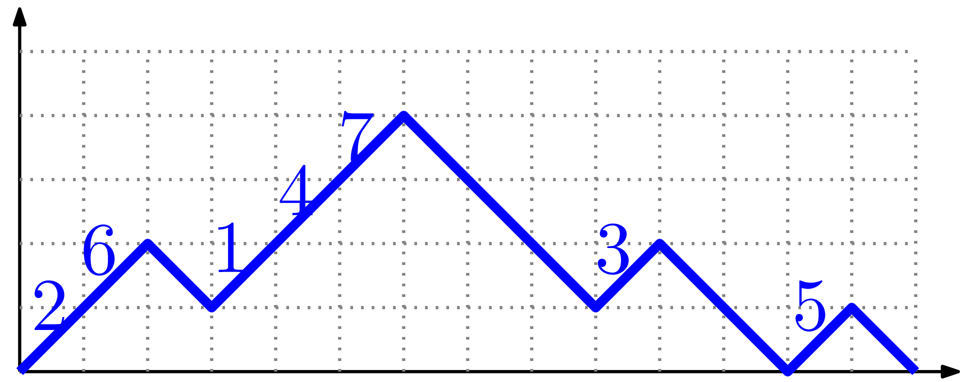
Labelled Dyck paths



up steps labelled from 1 to n and increasing along rises

A labelled Dyck path

Labelled Dyck paths



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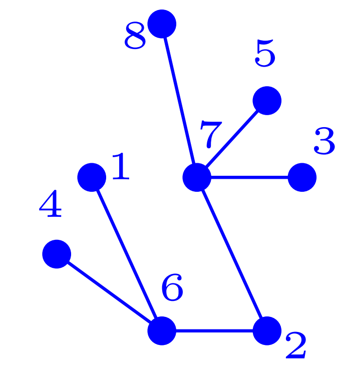
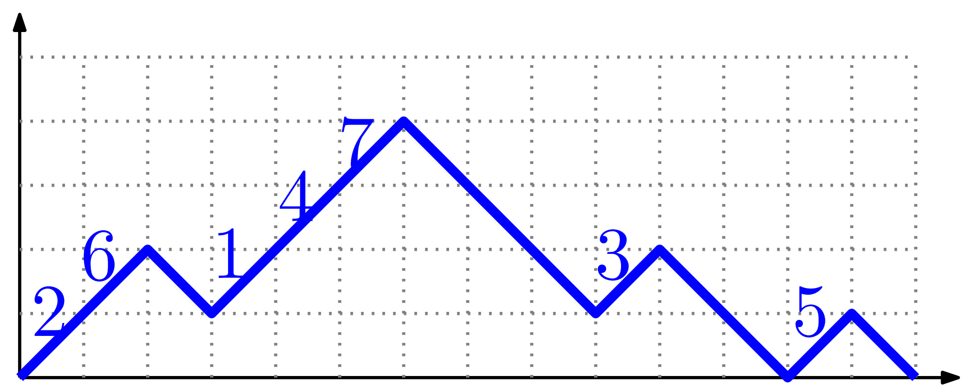


spanning tree of K_{n+1}

- Number of labelled Dyck paths $= (n+1)^{n-1}$

Labelled Dyck paths



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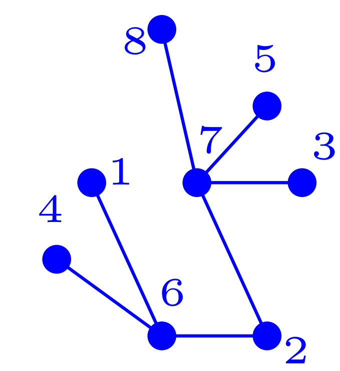

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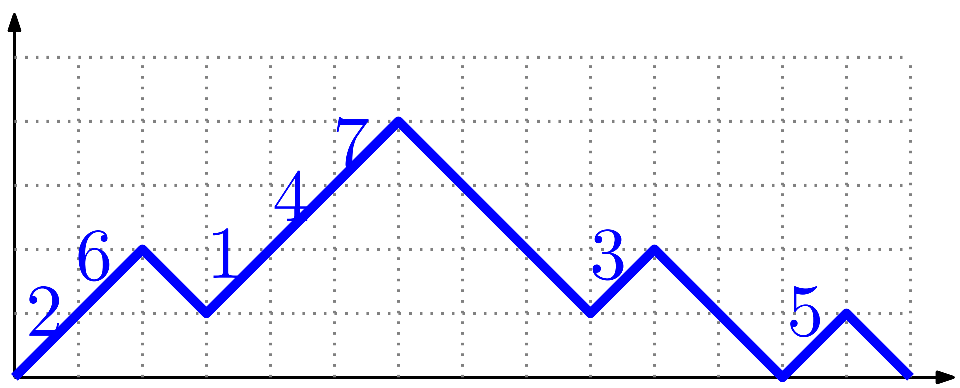
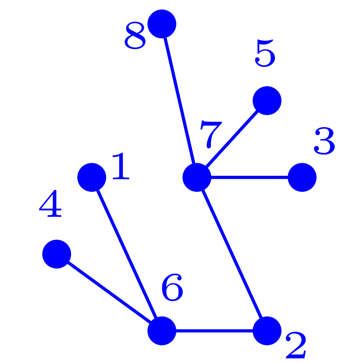


up steps labelled from 1 to n and increasing along rises

rise-partition $\{2,6\}$ $\{1,4,7\}$ $\{3\}$ $\{5\}$

A labelled Dyck path



spanning tree of K_{n+1}

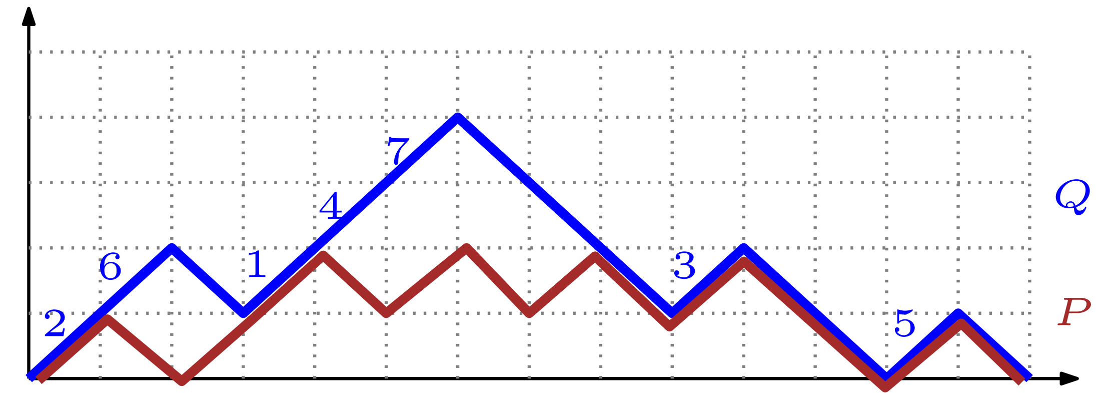
- Number of labelled Dyck paths $= (n+1)^{n-1}$

- **Refinement:** Let $\sigma \in \mathfrak{S}_n$ be a permutation. Then the number of labelled Dyck paths whose rise-partition is stable by σ is $(n+1)^{k-1}$ where $k = \#\text{cycles}(\sigma)$.

Labelled Tamari intervals: Bergeron's conjectures

A labelled Tamari interval is a pair $[P, Q]$ where

- P is a Dyck path
- Q is a labelled Dyck path
- $P \preccurlyeq Q$ for Tamari



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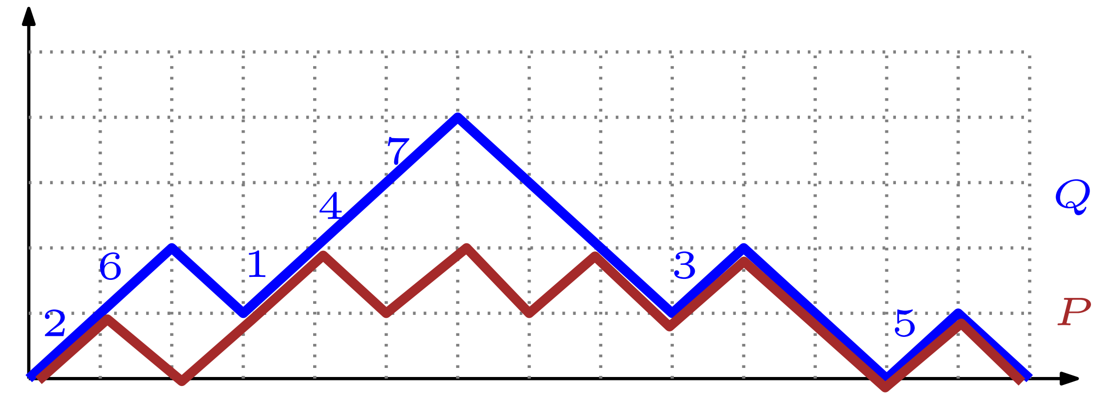


Theorem [Bousquet-Mélou,C., Préville-Ratelle 2011]

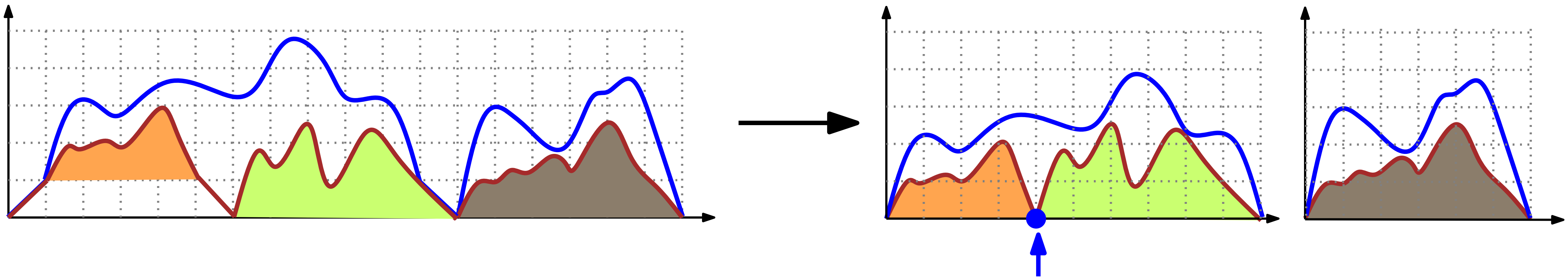
The number of labelled Tamari intervals is $2^n(n+1)^{n-2}$

Refinement: Let $\sigma \in \mathfrak{S}_n$ be a permutation. Then the number of labelled Tamari intervals whose rise-partition is stable by σ is

$$(n+1)^{k-2} \prod_{i \geq 1} \binom{2i}{i}^{\alpha_i}$$

if σ has α_i cycles of length i for $i \geq 1$ and k cycles in total

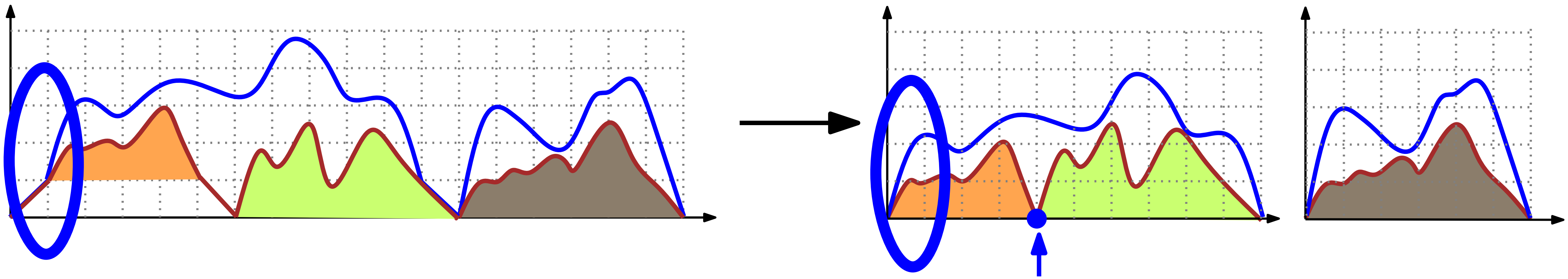
The decomposition for LABELLED intervals



- The number of labellings of a Dyck path depends on the lengths of the rises.

- Our recursive decomposition does not change the lengths of rises... except for the first one!

The decomposition for LABELLED intervals



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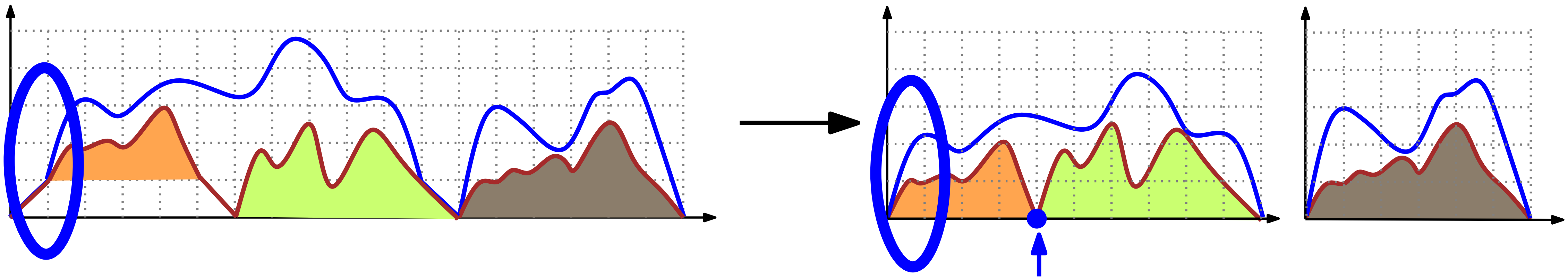

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- We introduce a new variable y for first rise of Q.

$$\boxed{\frac{\partial}{\partial y} F(t,x,y) = x + tx \frac{F(t,x;y) - F(t,1;y)}{x-1} F(t,x;1)}$$

since: $\frac{\partial}{\partial y} y^k = k y^{k-1}$

$\rightarrow$ the factor $k = \frac{k!}{(k-1)!}$ compensates the change of the first rise

What about LABELLED intervals (II)

$$\boxed{\frac{\partial}{\partial y} F(t,x,y) = x + tx\frac{F(t,x;y) - F(t,1;y)}{x-1} F(t,x;1)}$$

- Never seen such an equation (two catalytic variables, one "standard", one "differential").

What about LABELLED intervals (II)

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- Never seen such an equation (two catalytic variables, one "standard", one "differential").

- Go back to prehistory:

  1. guess $F(t,x,1)$ ("only" 2 variables).

  2. use the symmetries of the equation to eliminate F(t,1;y)

  3. solve the differential equation

  4. reconstitute $F(t,x,y)$ and check the value at $y=1$

Part III: comments

Why we are interested in all this

Theorem [Bousquet-Mélou,C., Préville-Ratelle 2011]

The number of labelled Tamari intervals is $2^n(n+1)^{n-2}$

Refinement: Let $\sigma \in \mathfrak{S}_n$ be a permutation. Then the number of labelled Tamari intervals whose rise-partition is stable by σ is

$$(n+1)^{k-2} \prod_{i \geq 1} \binom{2i}{i}^{\alpha_i}$$

if σ has α_i cycles of length i for $i \geq 1$ and k cycles in total

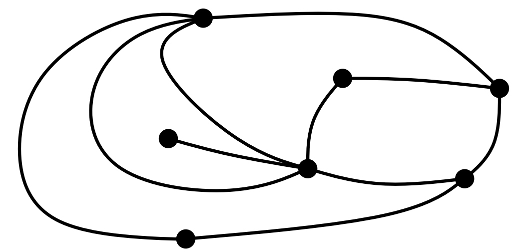
- Original motivation: algebraists believe that this formula is the character of the trivariate coinvariant module over $\mathfrak{S}_n$. (very hard conjecture!)

- Our proof is extremely technical but contains ideas hidden behind piles of details. We don't fully understand why it worked but we hope that this will open the way to a general theory.

- There is a generalization of everything to the m-Tamari lattice and it is harder and even more technical.

A historical analogy with planar maps

- A planar map is a planar graph drawn on the plane.



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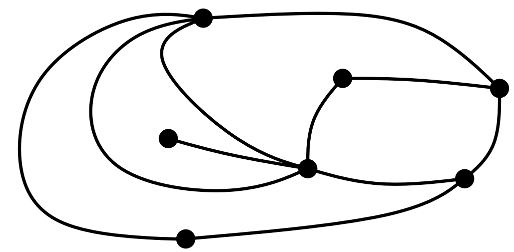

- 1960: the number of planar maps with n edges is $\dfrac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n)$.

[Tutte via the first catalytic equation solved with prehistorical techniques]

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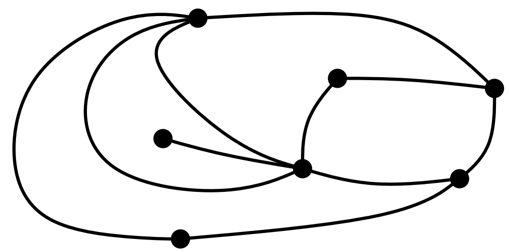

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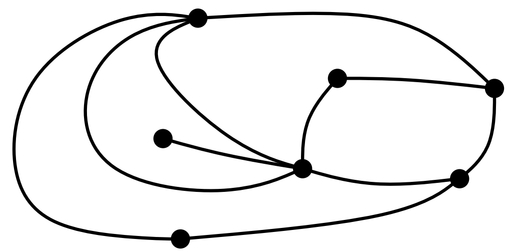

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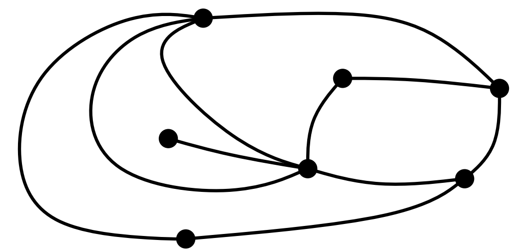

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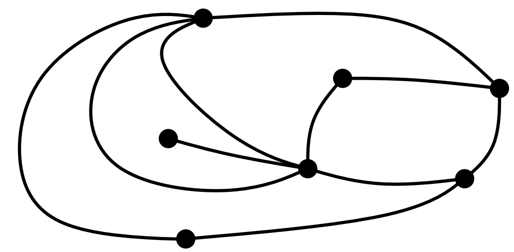

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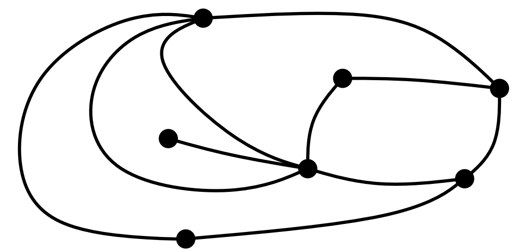

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Merci !