A bijection for covered maps on orientable surfaces.

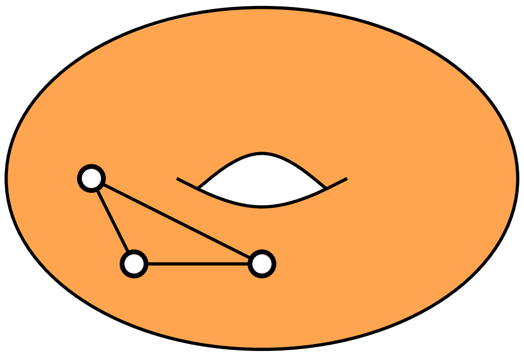
Guillaume Chapuy, LIX, École Polytechnique.
joint work with
Olivier Bernardi, CNRS, Université d'Orsay.

TGGT, May 2008.

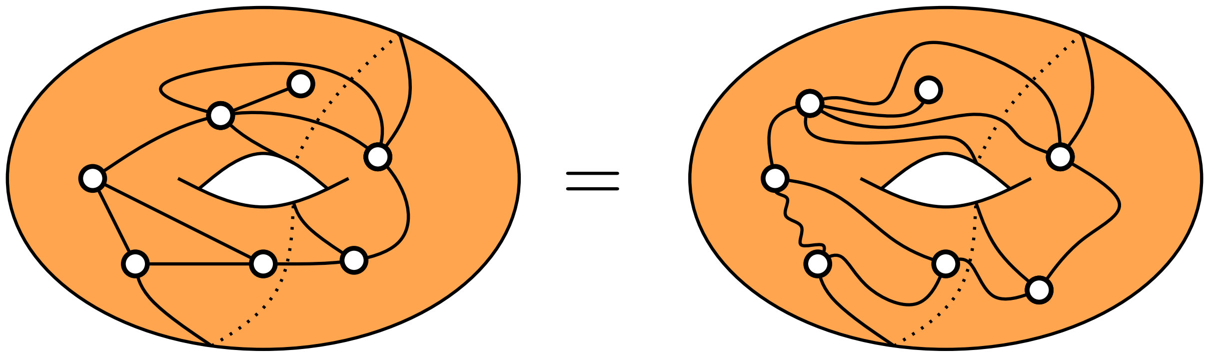
Map of genus g

$=$ drawing of a graph on the g-torus, such that the faces are simply connected.

Examples :



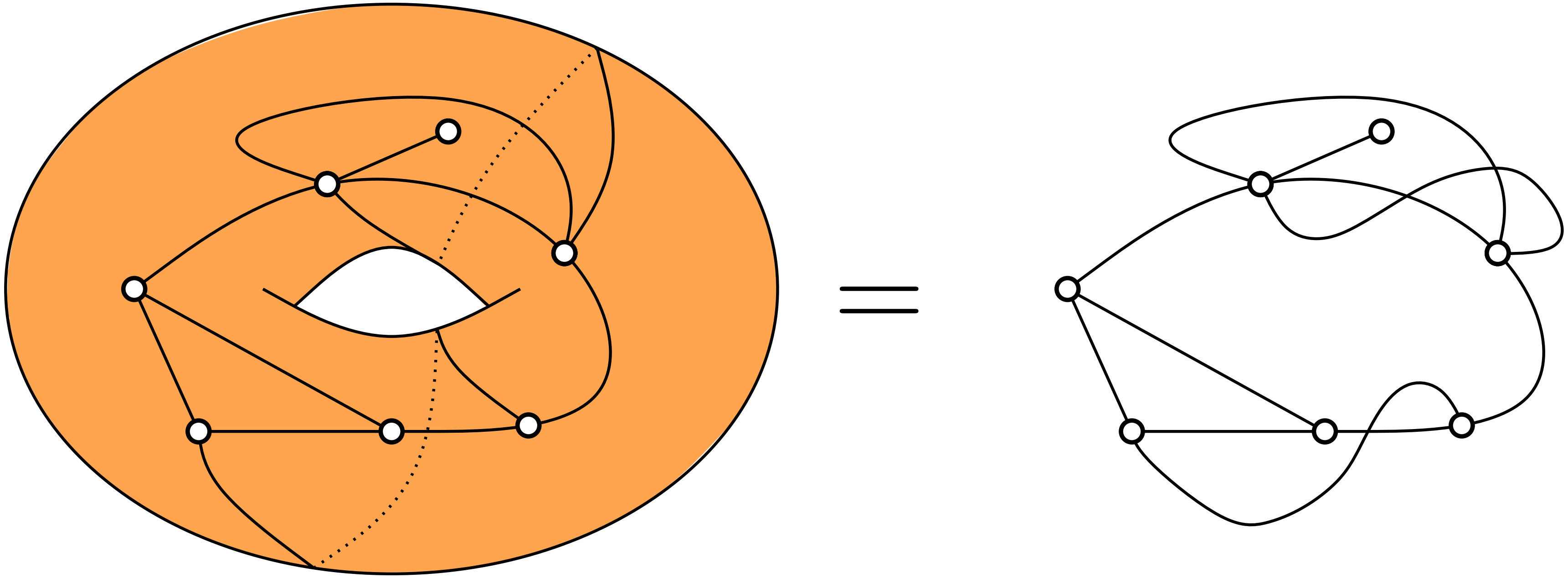
not a map



map on the torus

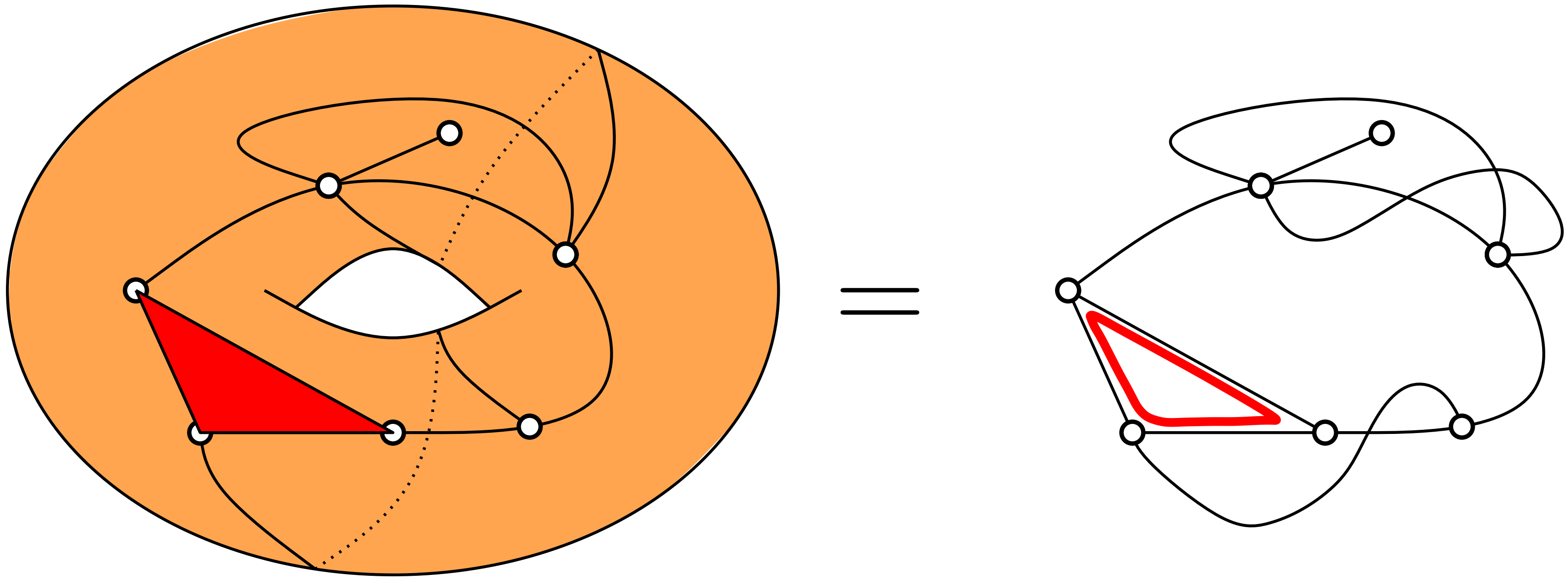
A map is the same thing as a fat graph, i.e. a graph with a cyclic ordering of half-edges around each vertex.

A map is the same thing as a fat graph, i.e. a graph with a cyclic ordering of half-edges around each vertex.



(no need to draw the surface).

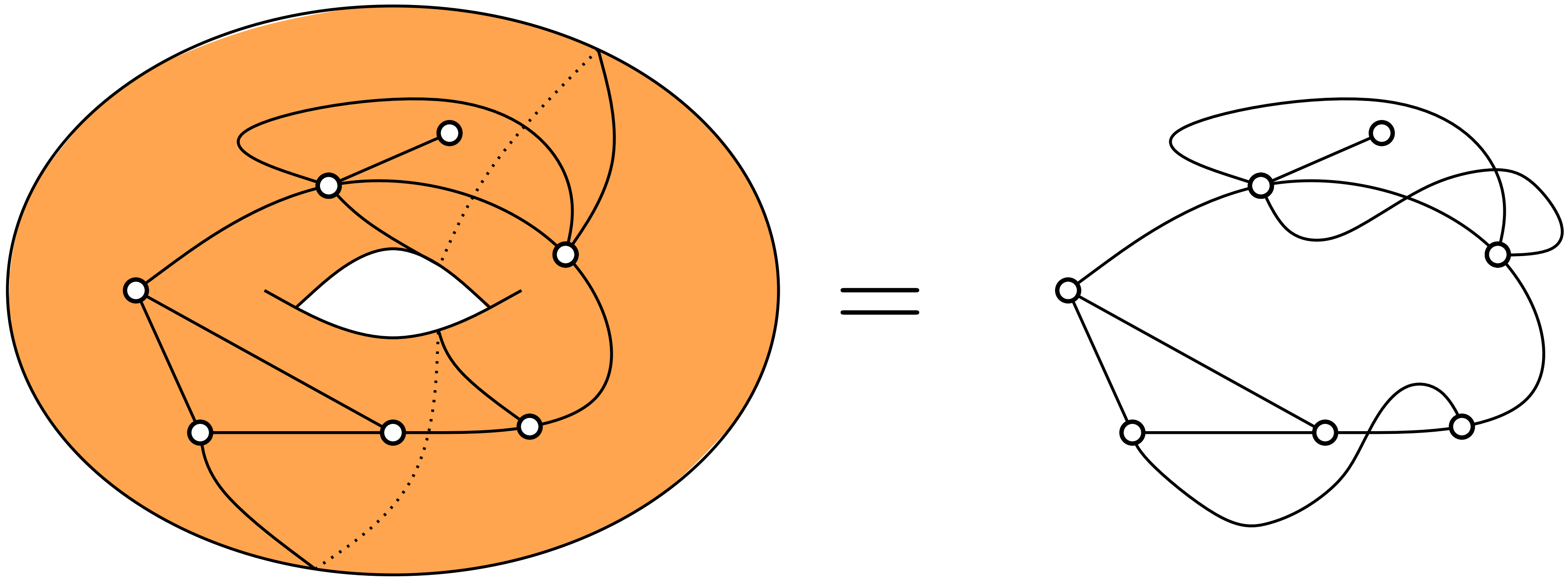
A map is the same thing as a fat graph, i.e. a graph with a cyclic ordering of half-edges around each vertex.



(no need to draw the surface).

faces of the map = borders of the fat graph.

A map is the same thing as a fat graph, i.e. a graph with a cyclic ordering of half-edges around each vertex.

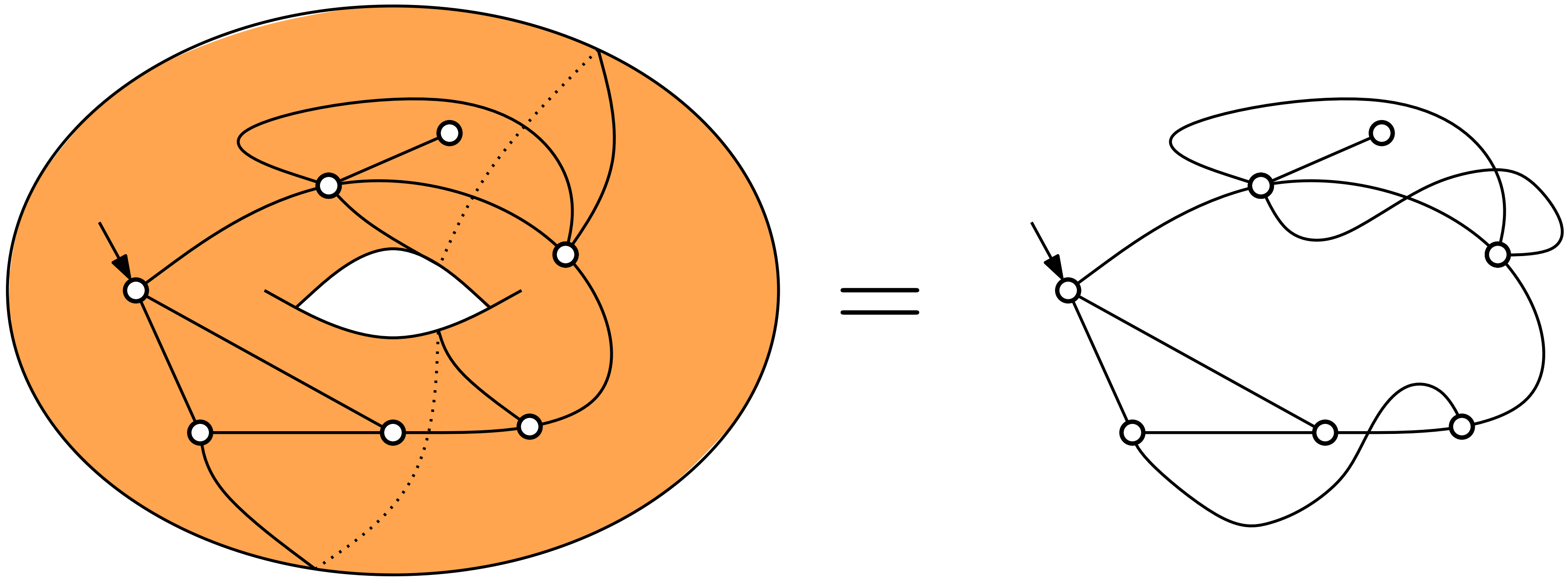


(no need to draw the surface).

faces of the map = borders of the fat graph.

Our maps are rooted, i.e. carry a distinguished corner (equivalent to classical "Tutte rooting").

A map is the same thing as a fat graph, i.e. a graph with a cyclic ordering of half-edges around each vertex.

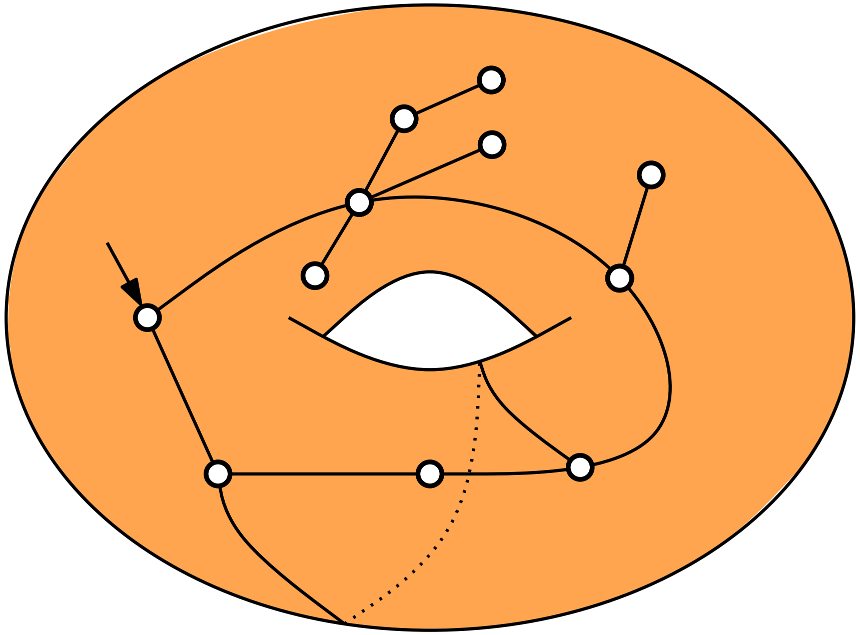


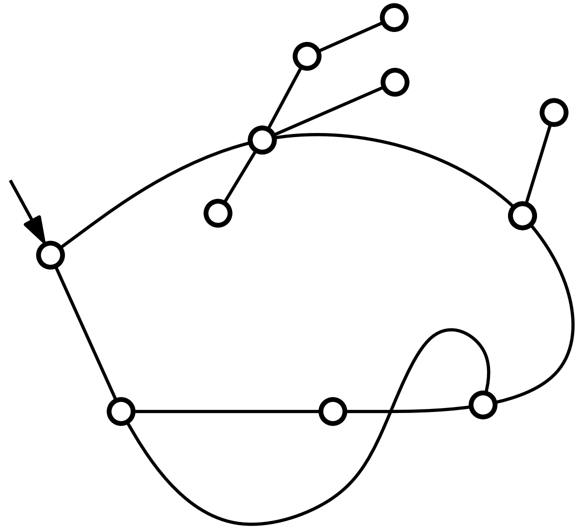
(no need to draw the surface).

faces of the map = borders of the fat graph.

Our maps are rooted, i.e. carry a distinguished corner (equivalent to classical "Tutte rooting").

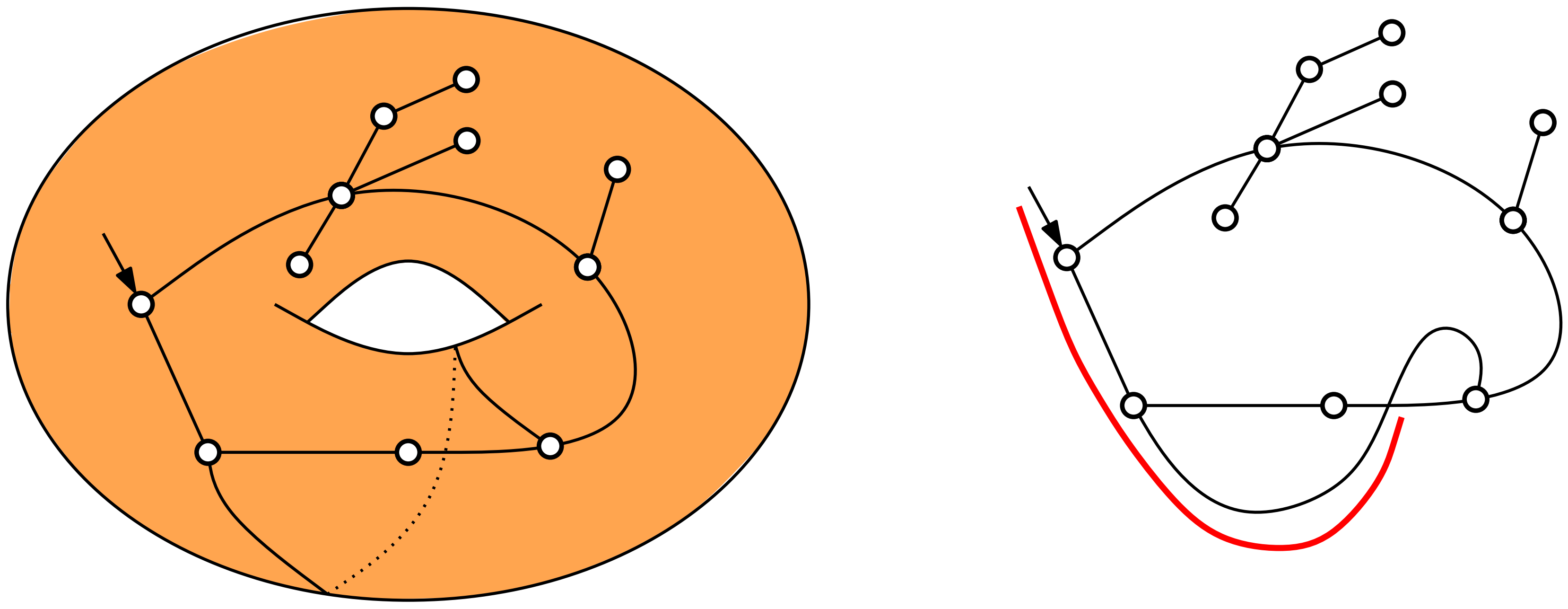
A unicellular map is a map which has only one face.





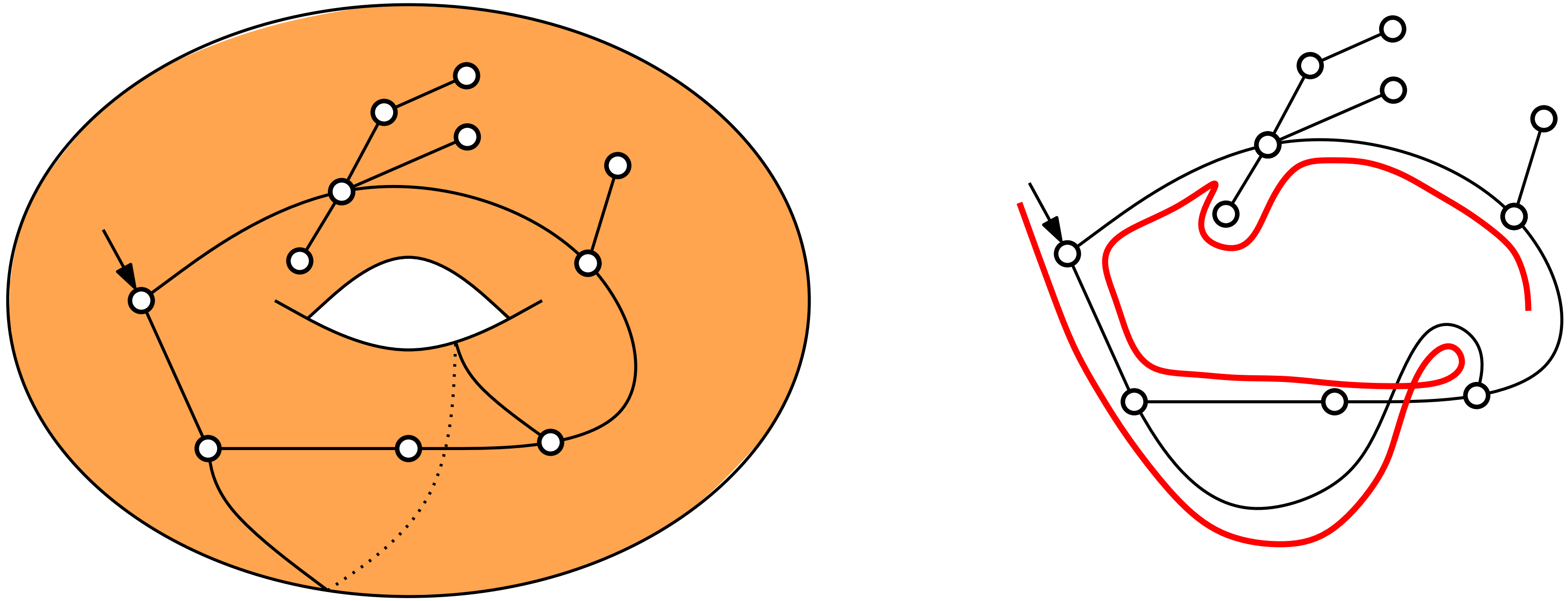
Equivalently, the fat graph has only one border.

A unicellular map is a map which has only one face.



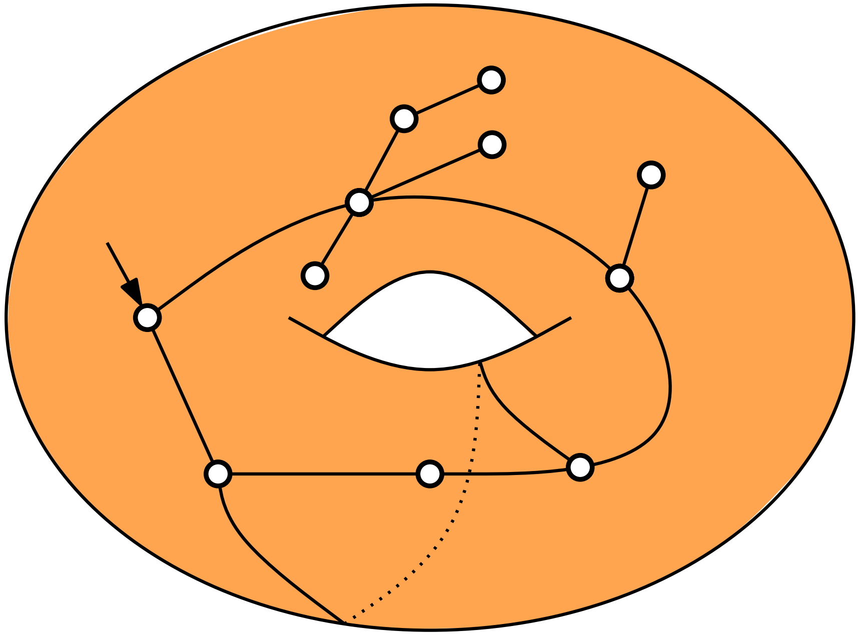
Equivalently, the fat graph has only one border.

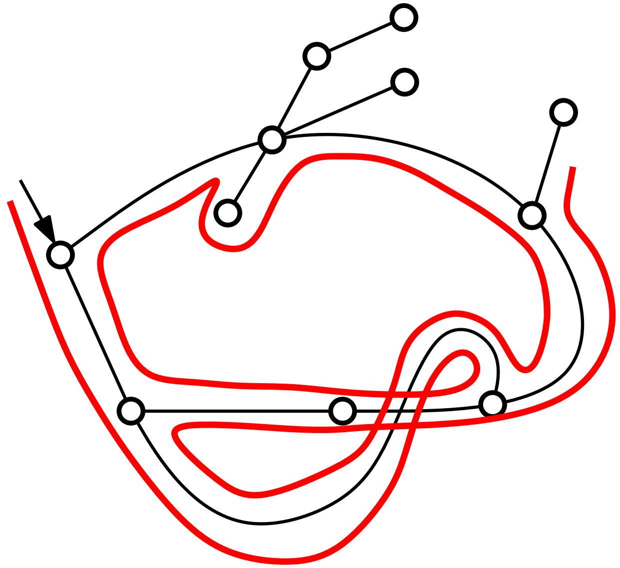
A unicellular map is a map which has only one face.



Equivalently, the fat graph has only one border.

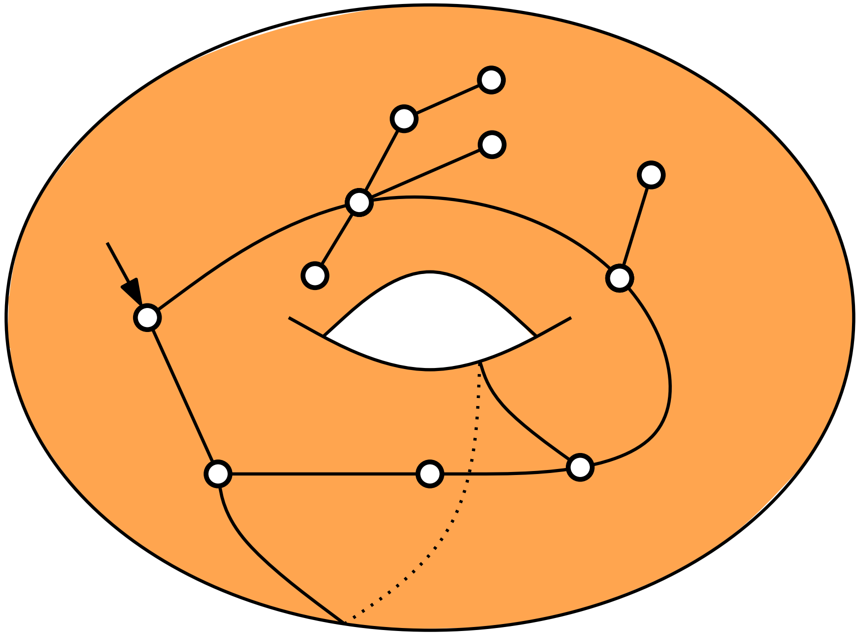
A unicellular map is a map which has only one face.

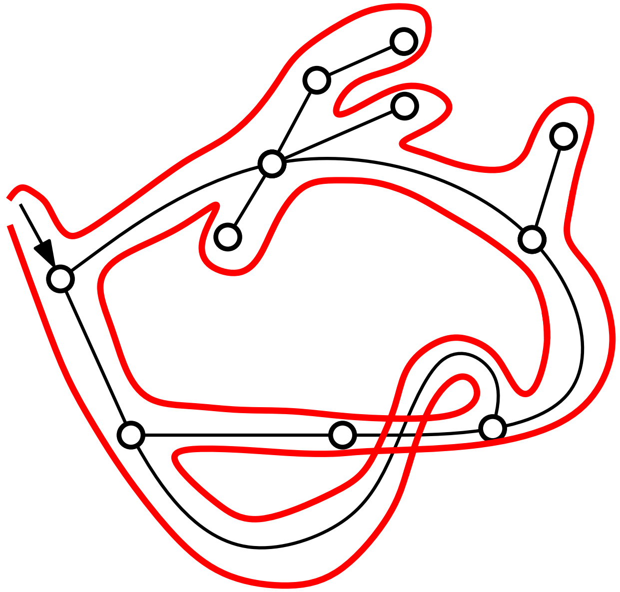




Equivalently, the fat graph has only one border.

A unicellular map is a map which has only one face.



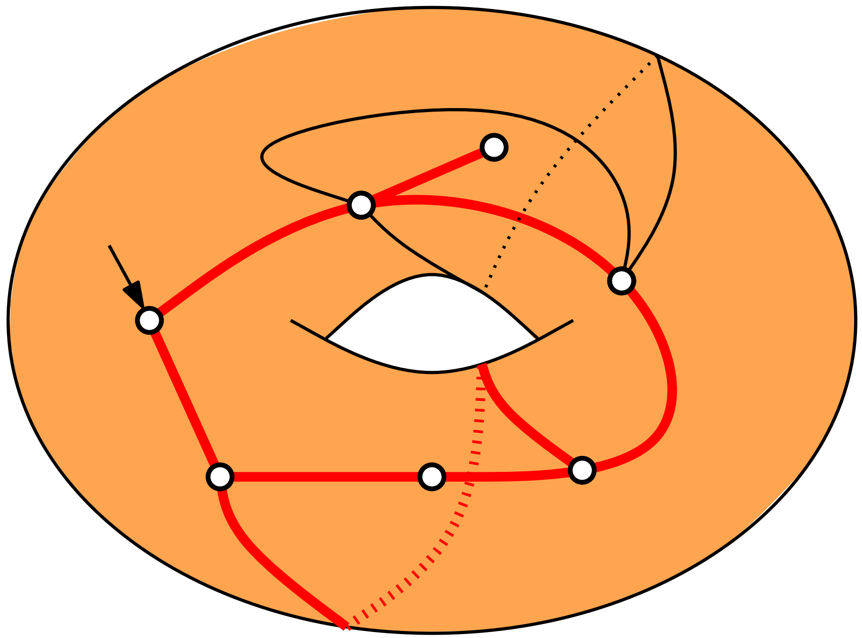


Equivalently, the fat graph has only one border.

In genus 0, unicellular maps are exactly plane trees, but in positive genus, things are more complicated.

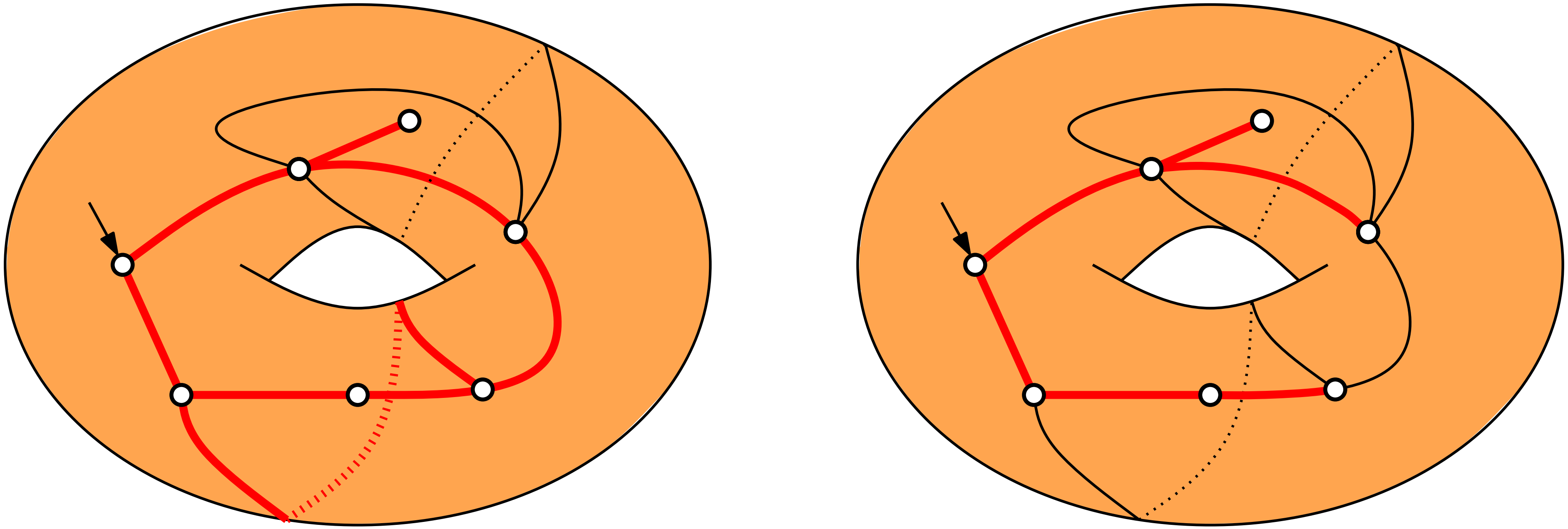
Covered maps.

A covered map is a map with a distinguished spanning unicellular submap.



Covered maps.

A covered map is a map with a distinguished spanning unicellular submap.



We do not impose that the spanning submap has genus g (it can have genus $0, 1, ..., g$).

Special case: map with a spanning tree $=$ tree-rooted map.

Tree-rooted maps were previously studied.

In the planar case, a very nice formula from [Mullin 67]:

(nb. of tree-rooted maps w. n edges) $= Cat_n \times Cat_{n+1}$

Tree-rooted maps were previously studied.

In the planar case, a very nice formula from [Mullin 67]:

(nb. of tree-rooted maps w. n edges) $= Cat_n \times Cat_{n+1}$

Bijective proofs: [Cori, Dulucq, Viennot 82], [Bernardi 06].

Tree-rooted maps were previously studied.

In the planar case, a very nice formula from [Mullin 67]:

(nb. of tree-rooted maps w. n edges) $= Cat_n \times Cat_{n+1}$

Bijective proofs: [Cori, Dulucq, Viennot 82], [Bernardi 06].

In higher genus:

[Lehman, Walsh 72]: nice formula for genus 1.
more complicated formulas for $g \geq 2$.

[Bender, Robert, Robinson 88]: asymptotics.

Theorem: There is a bijection:

$$\{\text{covered maps of genus } g \text{ with } n \text{ edges}\}$$
$$\updownarrow$$
$$\{\text{unicellular bipartite maps of genus } g\text{, } n+1 \text{ edges}\}$$
$$\times \{\text{plane trees, } n \text{ edges}\}$$

Theorem: There is a bijection:

$$\{\text{covered maps of genus } g \text{ with } n \text{ edges}\}$$
$$\updownarrow$$
$$\{\text{unicellular bipartite maps of genus } g,\ n+1 \text{ edges}\}$$
$$\times\ \{\text{plane trees, } n \text{ edges}\}$$

Corollary:

For each g, there is a closed formula for the number of covered maps.

$$C_0(n) = Cat_n \times Cat_{n+1}$$
$$C_1(n) = Cat_n \times \frac{(2n-2)!}{12(n-1)!(n-3)!}$$
$$C_2(n) = Cat_n \times \frac{(5n^2-7n+6)(2n-5)!}{720(n-3)!(n-5)!}$$

Corollary: Nice formulas for tree rooted-maps.

- $g = 0$: there is a closed formula for the number of tree-rooted planar maps:

$$T_0(n) = C_0(n)$$

Corollary: Nice formulas for tree rooted-maps.

- $g = 0$: there is a closed formula for the number of tree-rooted planar maps:

$$T_0(n) = C_0(n)$$

- $g = 1$: there is a closed formula for the number of tree-rooted toroidal maps:

$$T_1(n) = \frac{1}{2}C_1(n) \quad \text{by a duality argument.}$$

Corollary: Nice formulas for tree rooted-maps.

- $g = 0$: there is a closed formula for the number of tree-rooted planar maps:

$$T_0(n) = C_0(n)$$

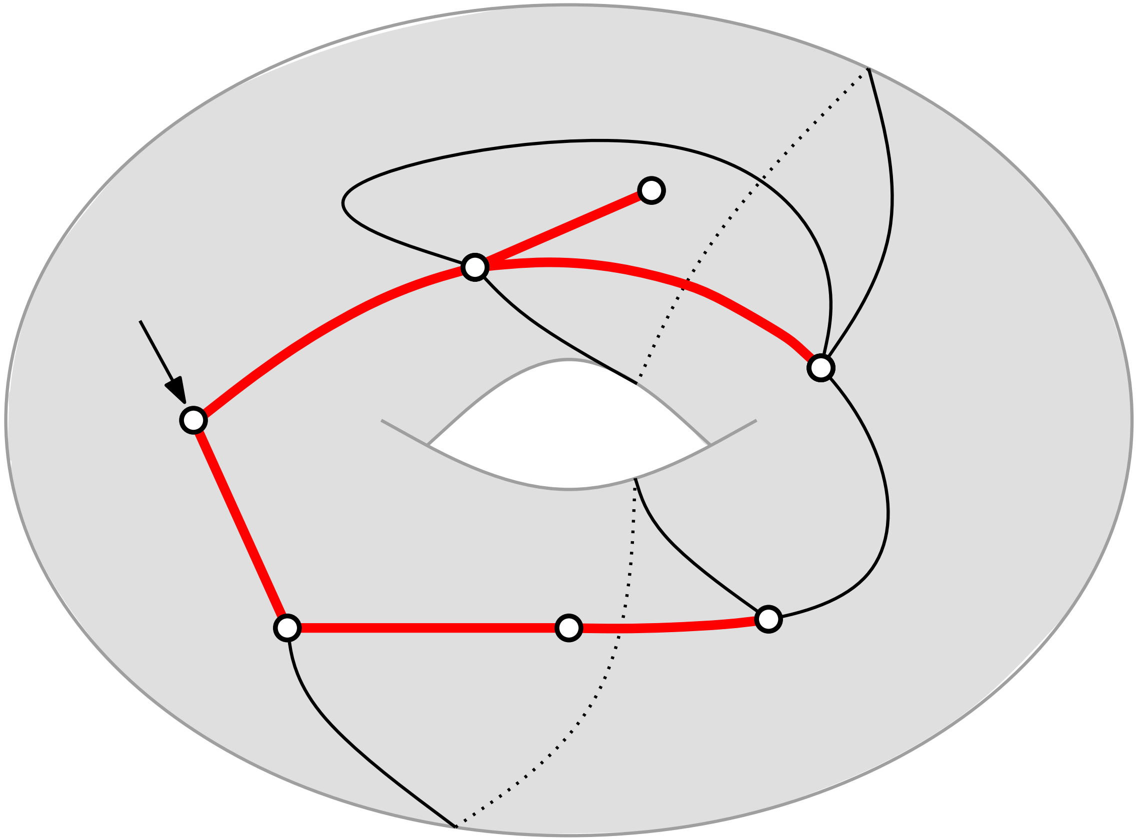
- $g = 1$: there is a closed formula for the number of tree-rooted toroidal maps:

$$T_1(n) = \frac{1}{2}C_1(n) \quad \text{by a duality argument.}$$

- No similar argument for $g \geq 2$: explains (?) why formulas for tree-rooted maps seem to be more complicated.

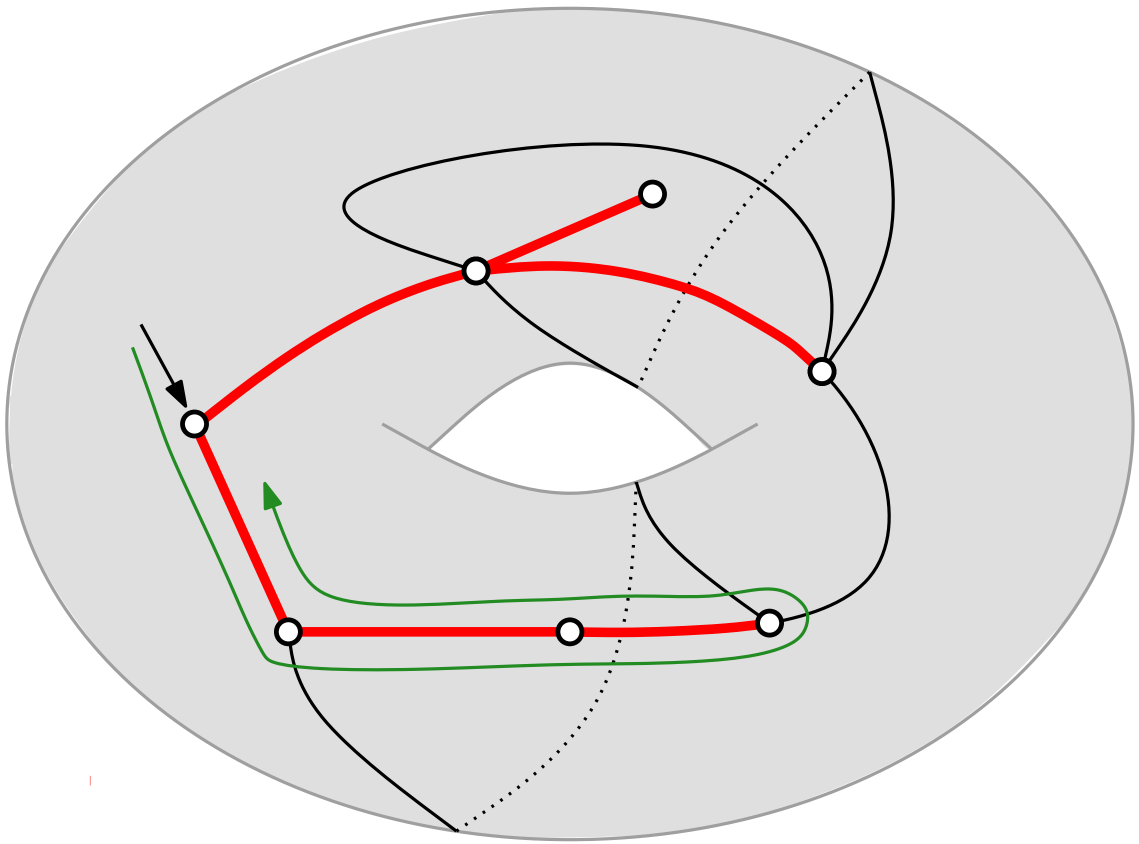
The bijection.

step 1: from covered maps to orientations.



The bijection.

step 1: from covered maps to orientations.

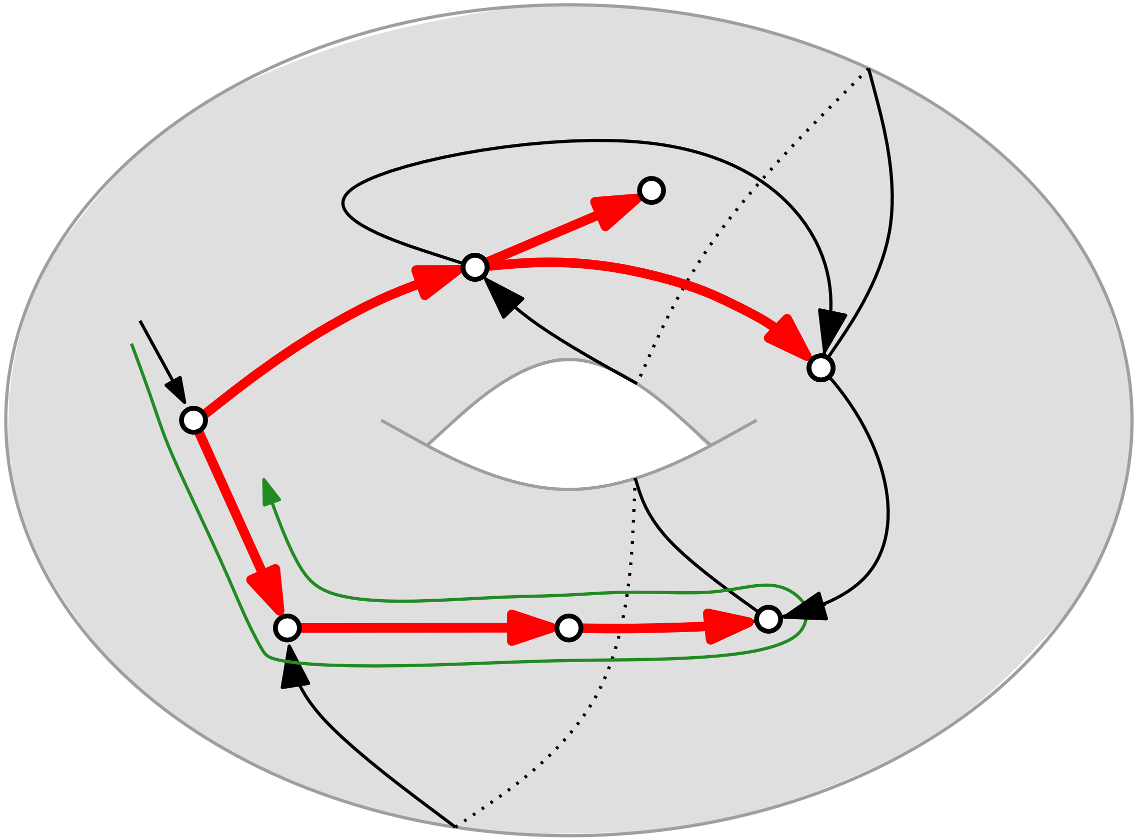


We make the tour of the submap, and orient:

- red edges as we followed them for the first time
- black edges s.t. we see their head before their tail

The bijection.

step 1: from covered maps to orientations.

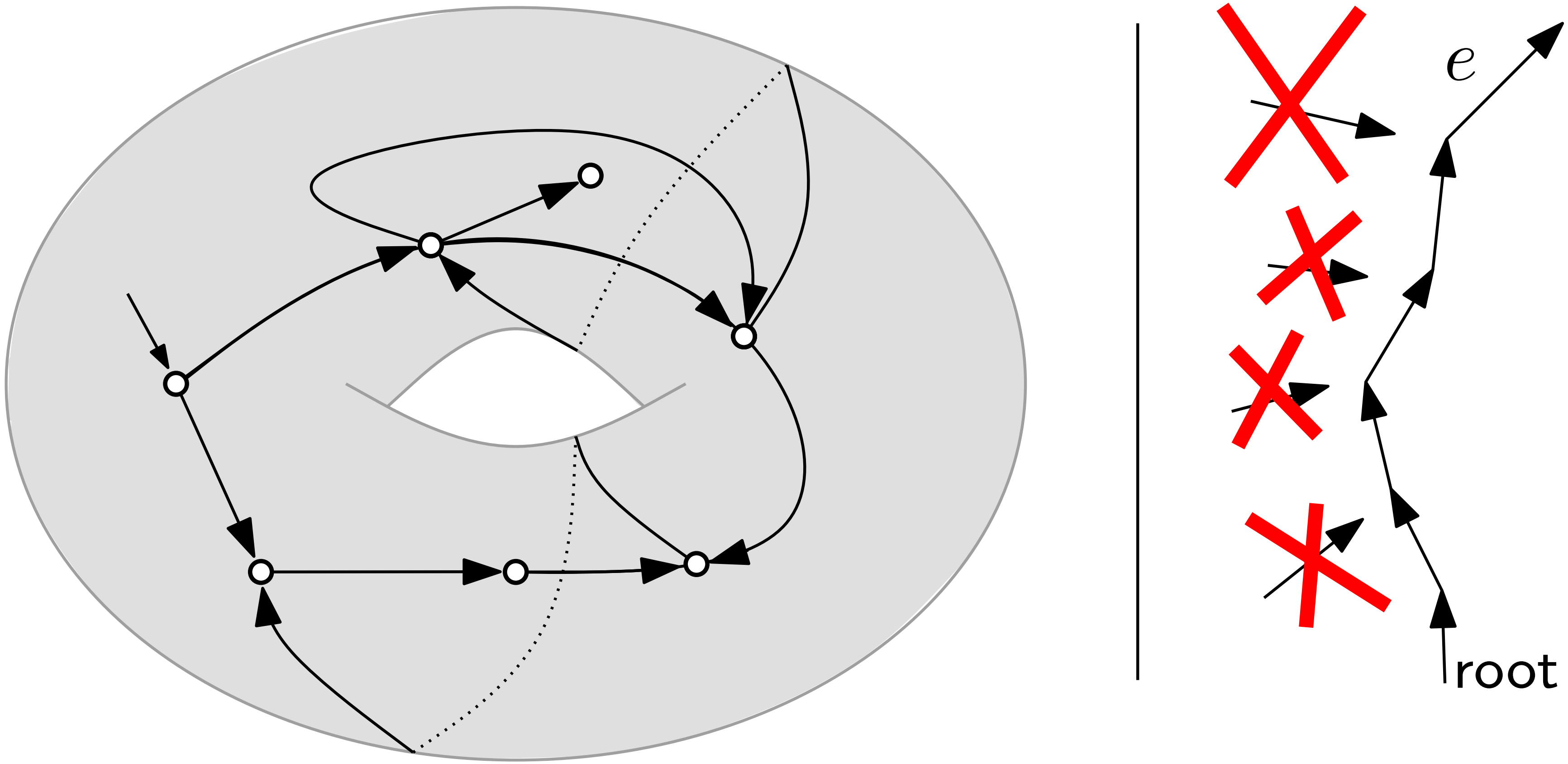


We make the tour of the submap, and orient:

- red edges as we followed them for the first time
- black edges s.t. we see their head before their tail

The bijection.

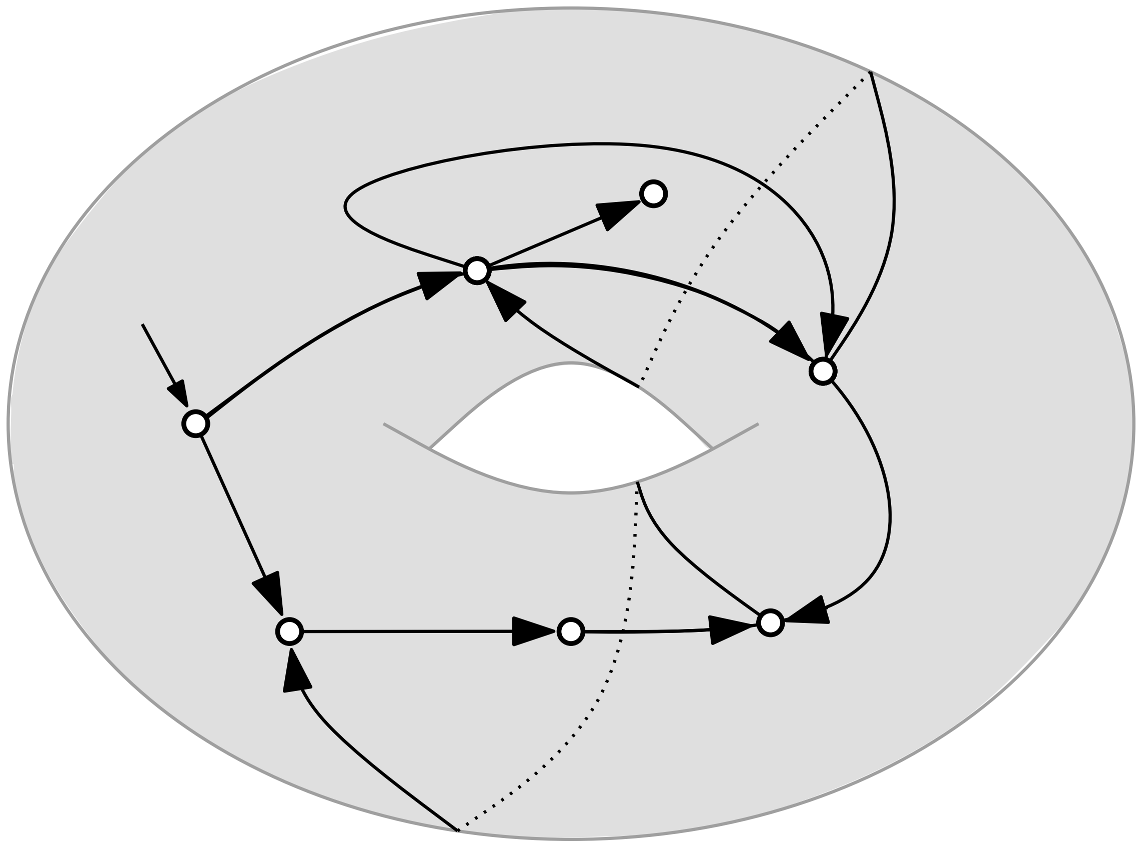
step 1: from covered maps to orientations.



We obtain a left-orientation: each edge e can be reached from the root by a left-path. The construction is bijective.

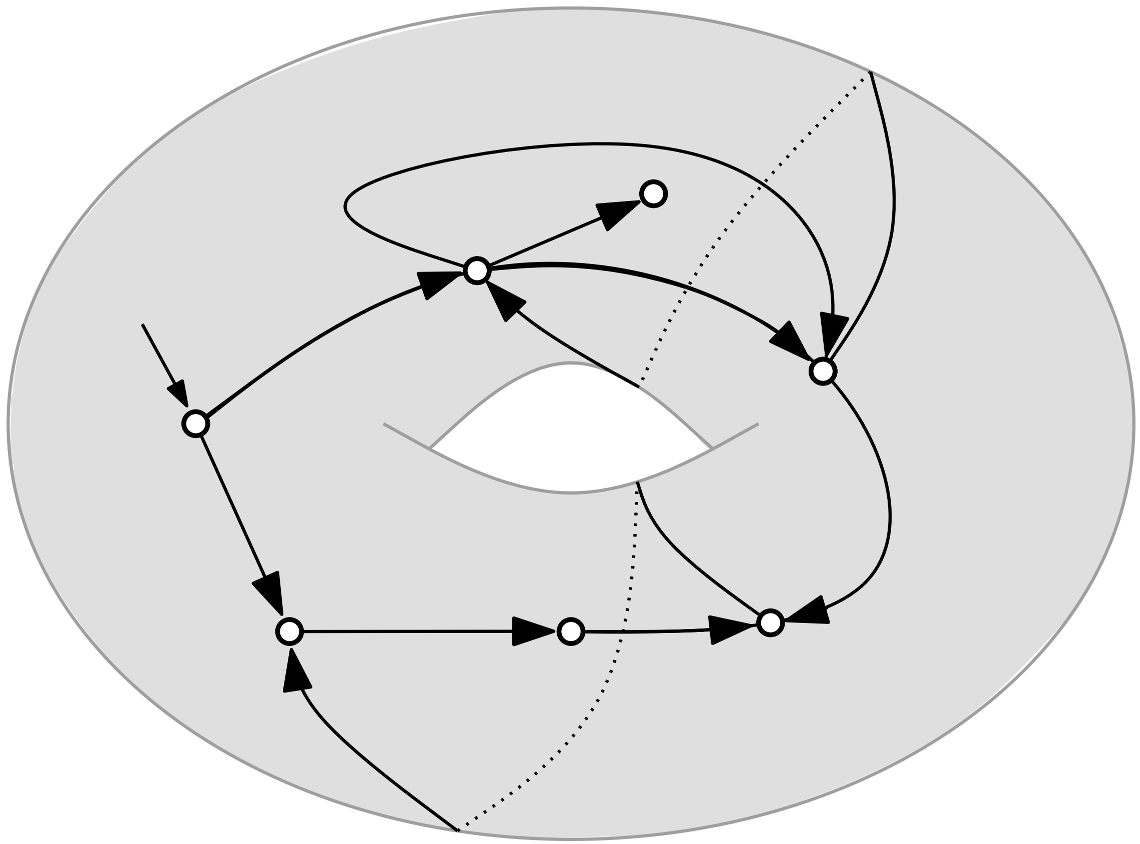
The bijection.

step 2: from left-orientations to pairs (tree, unicellular bipartite map).

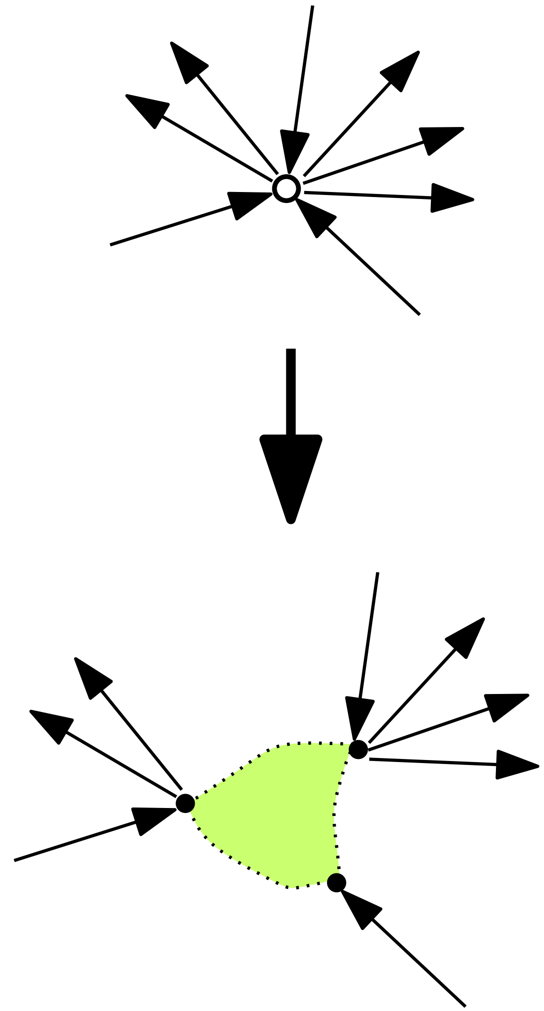


The bijection.

step 2: from left-orientations to pairs (tree, unicellular bipartite map).

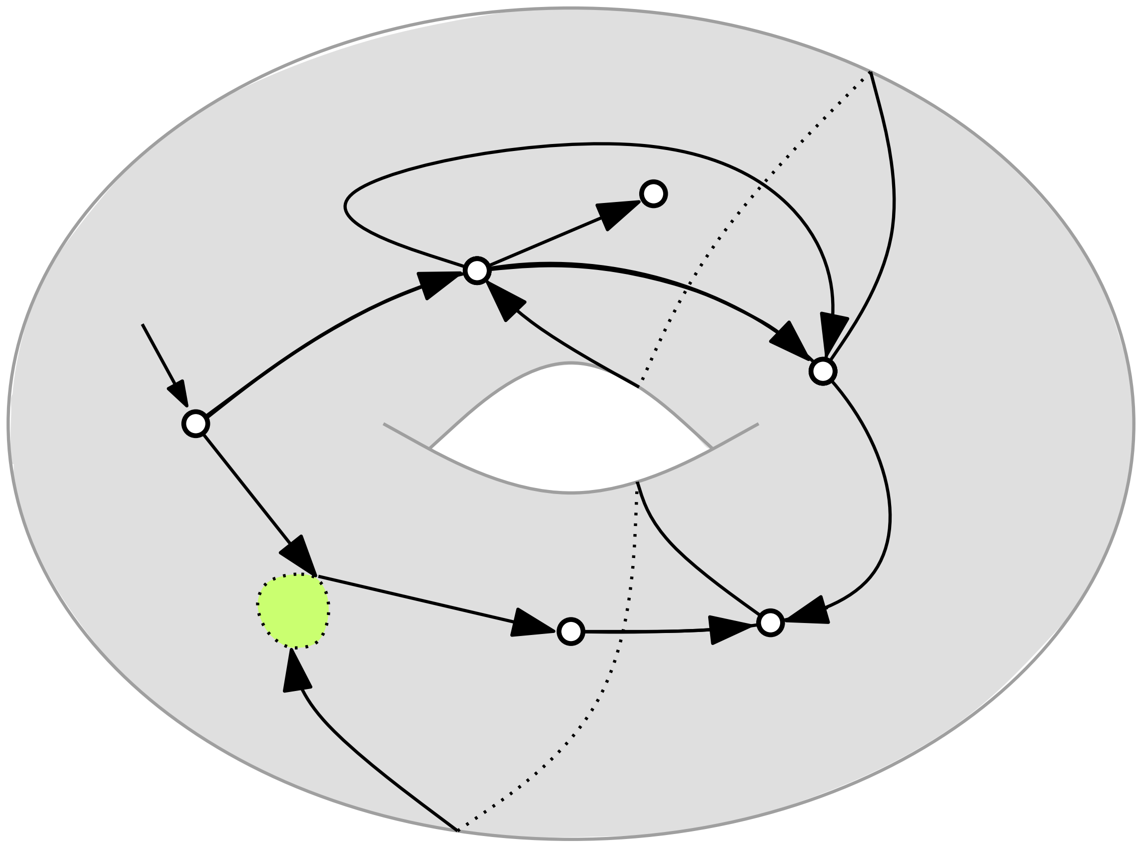


Unfolding a vertex:

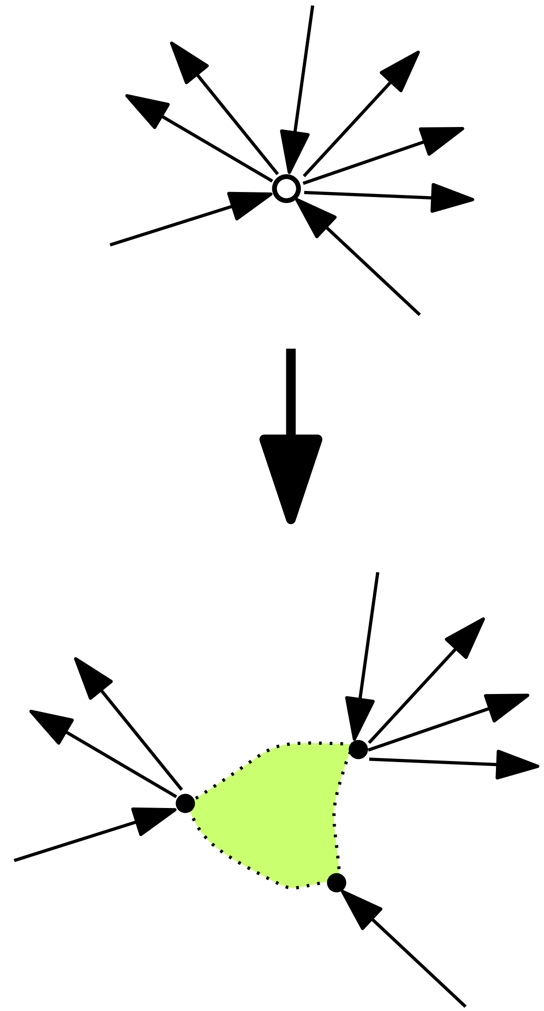


The bijection.

step 2: from left-orientations to pairs (tree, unicellular bipartite map).

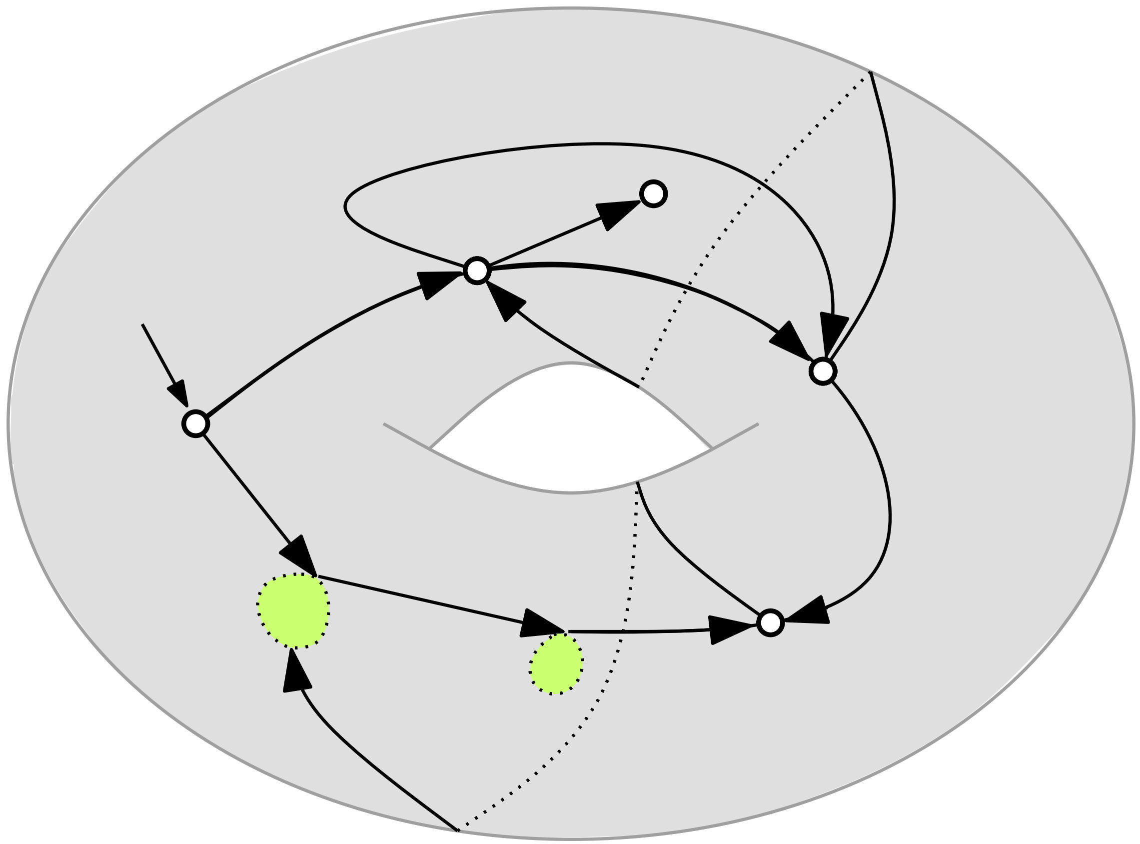


Unfolding a vertex:

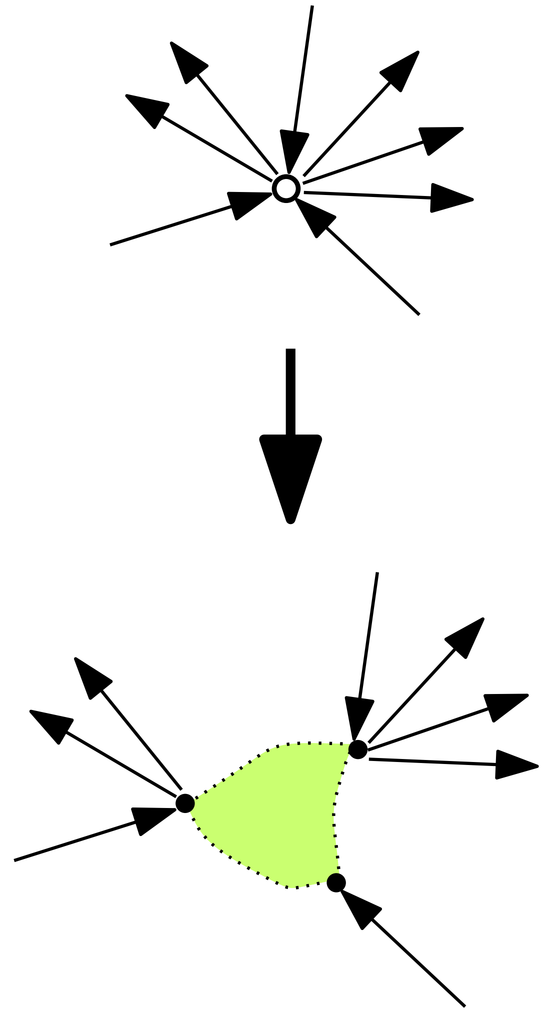


The bijection.

step 2: from left-orientations to pairs (tree, unicellular bipartite map).

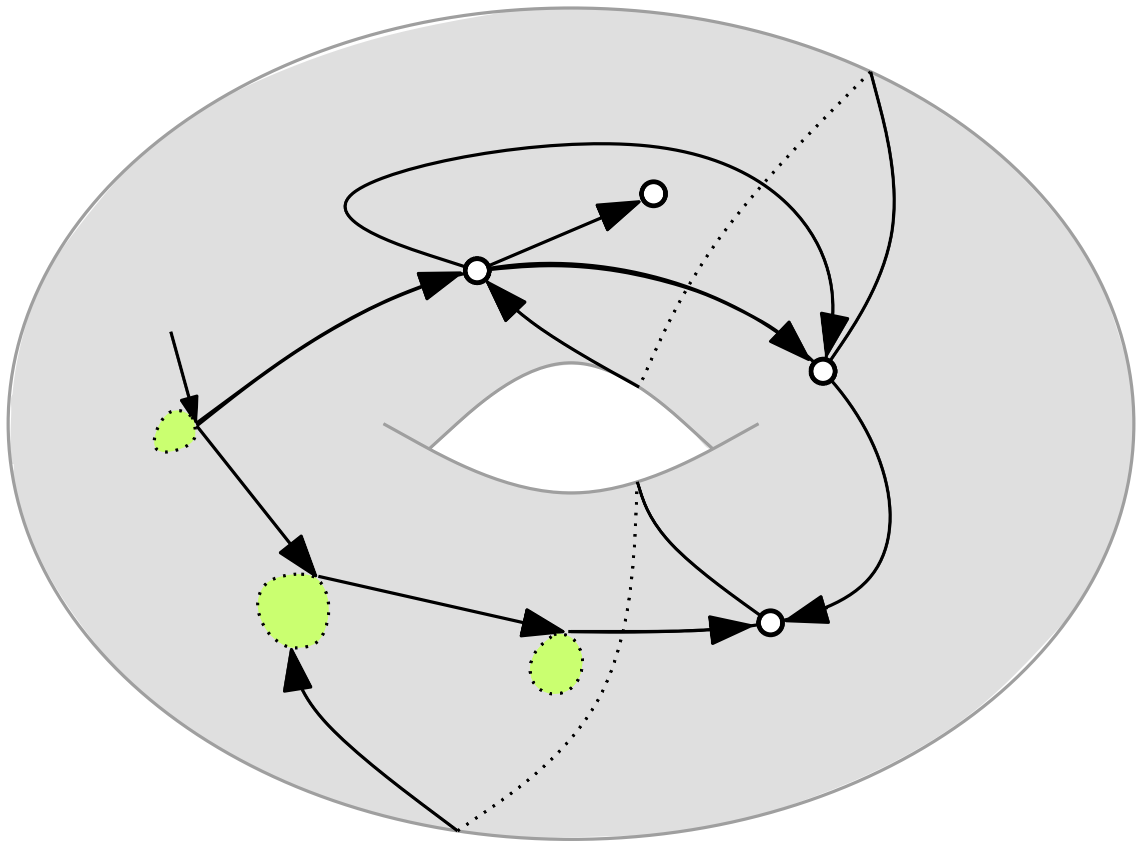


Unfolding a vertex:

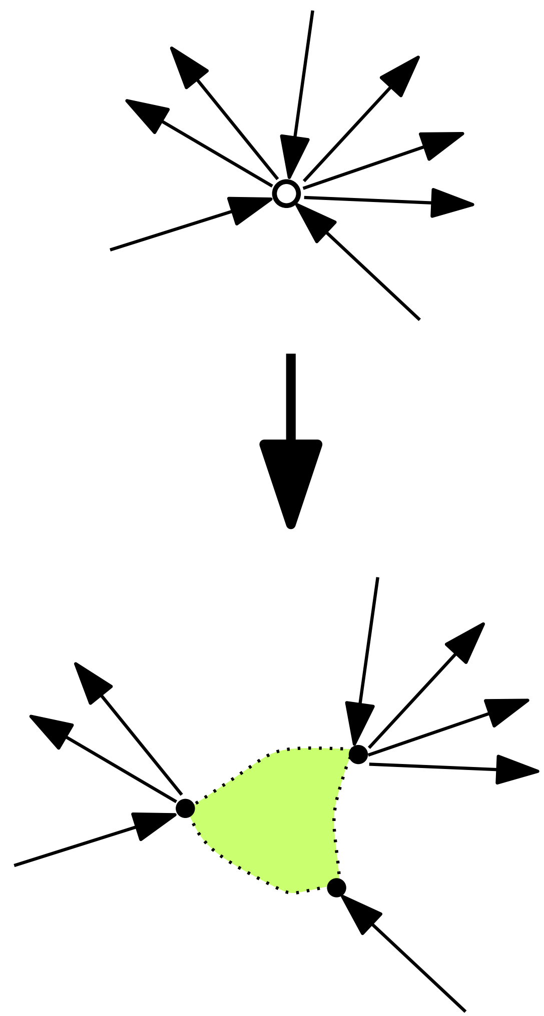


The bijection.

step 2: from left-orientations to pairs (tree, unicellular bipartite map).

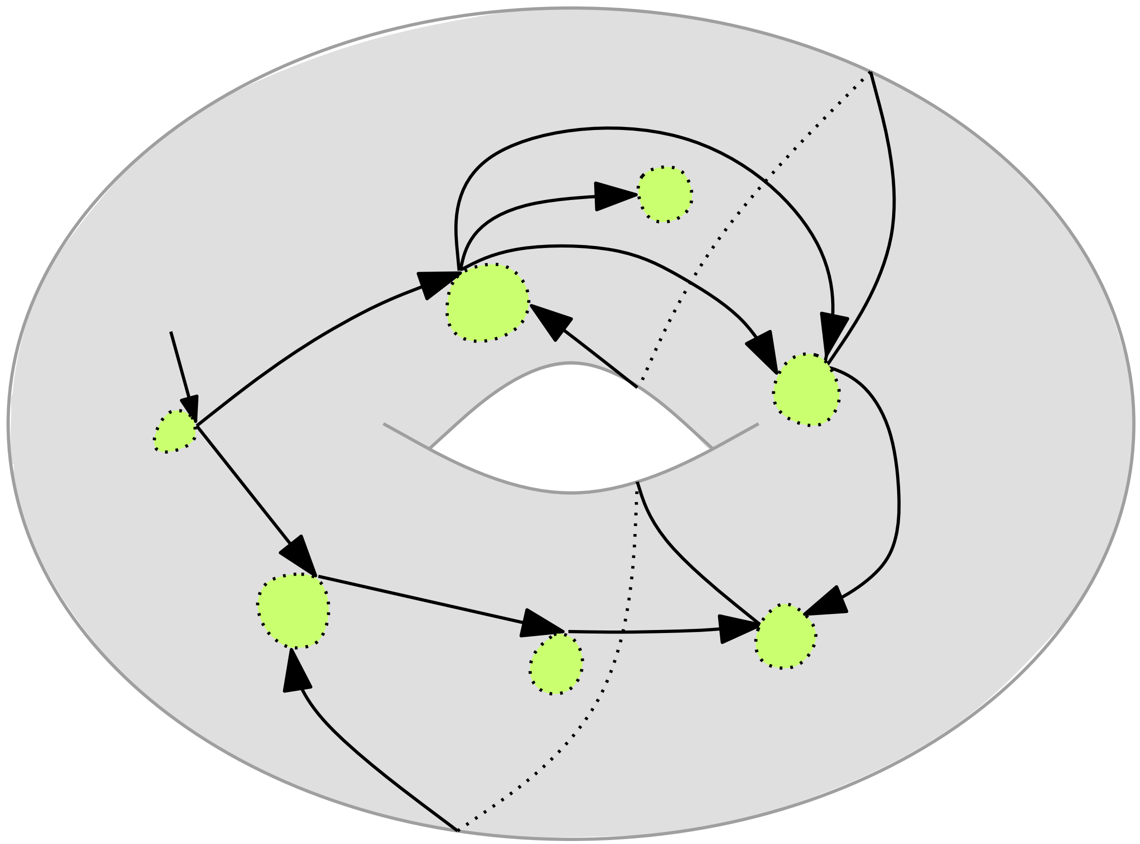


Unfolding a vertex:

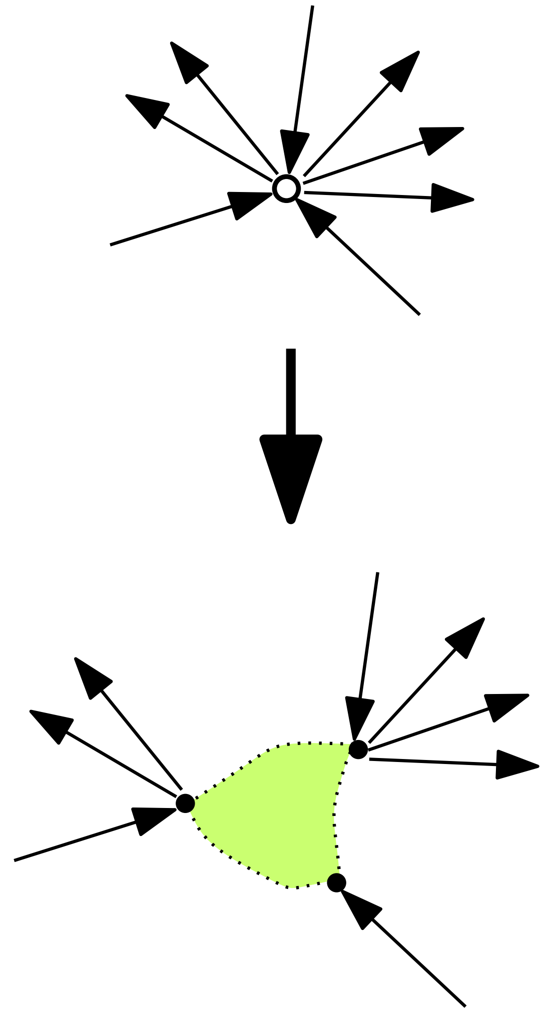


The bijection.

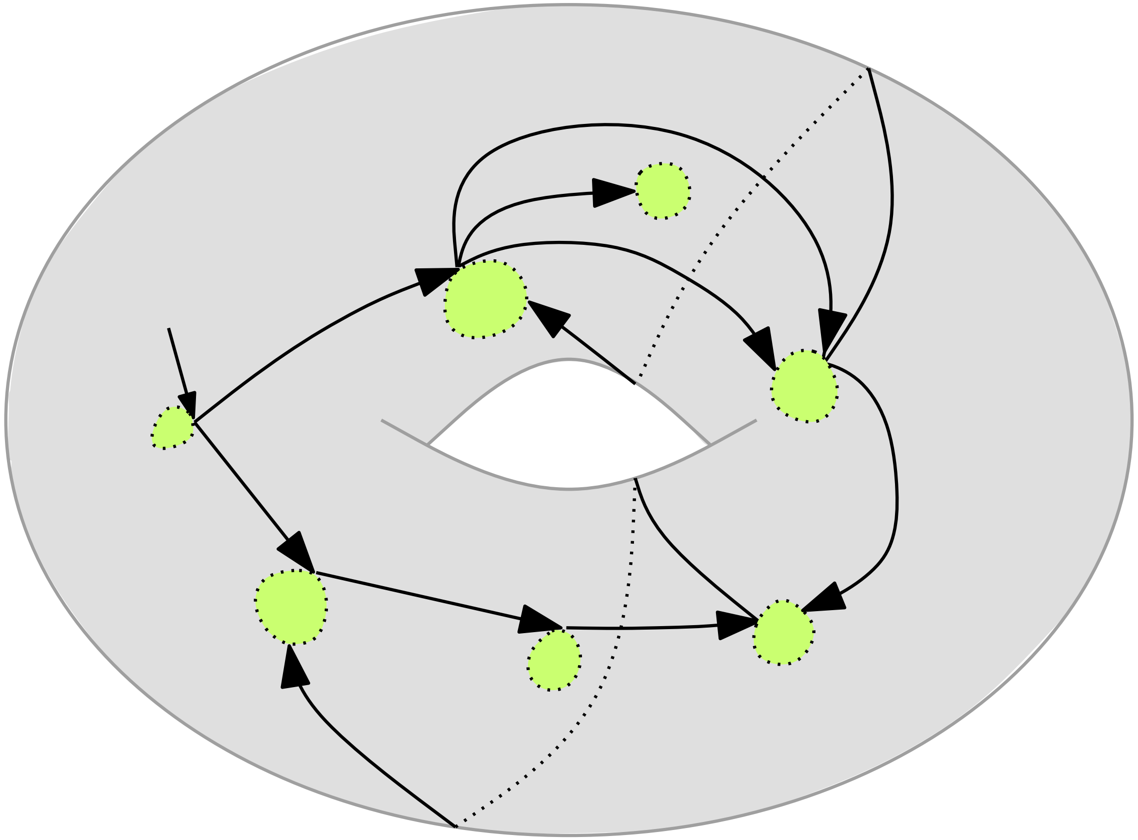
step 2: from left-orientations to pairs (tree, unicellular bipartite map).



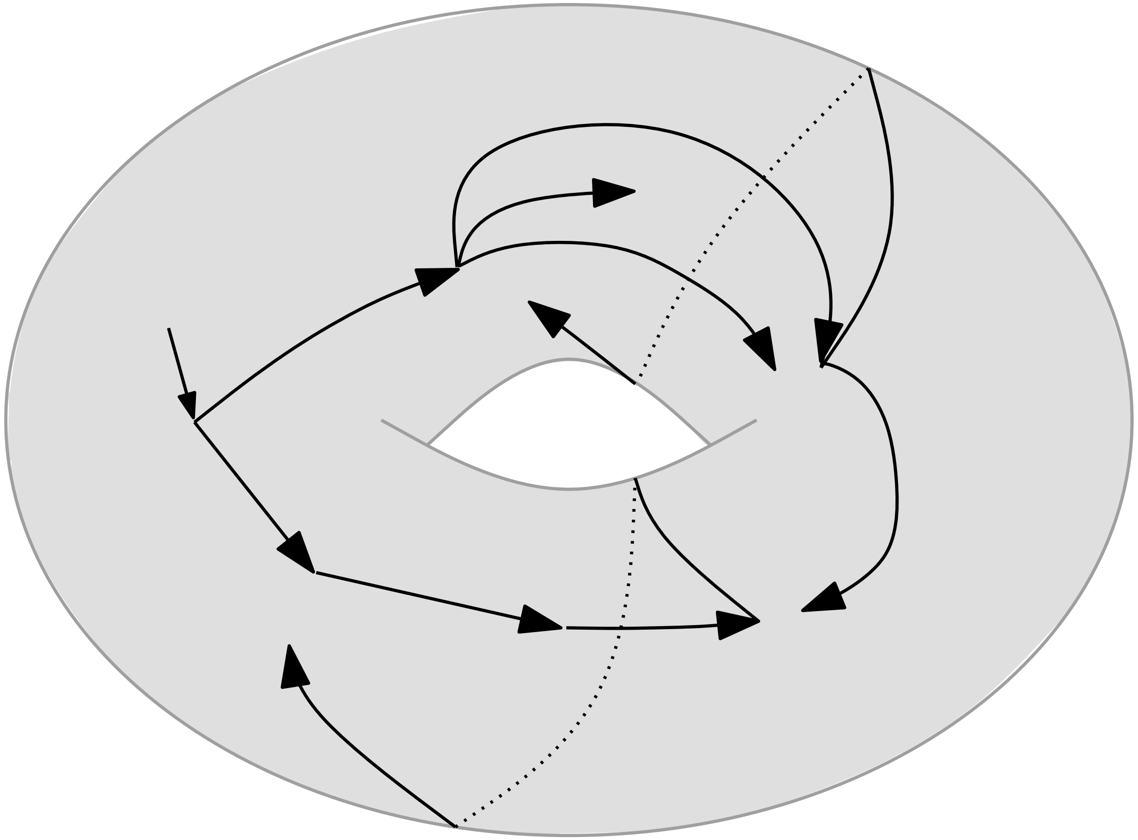
Unfolding a vertex:



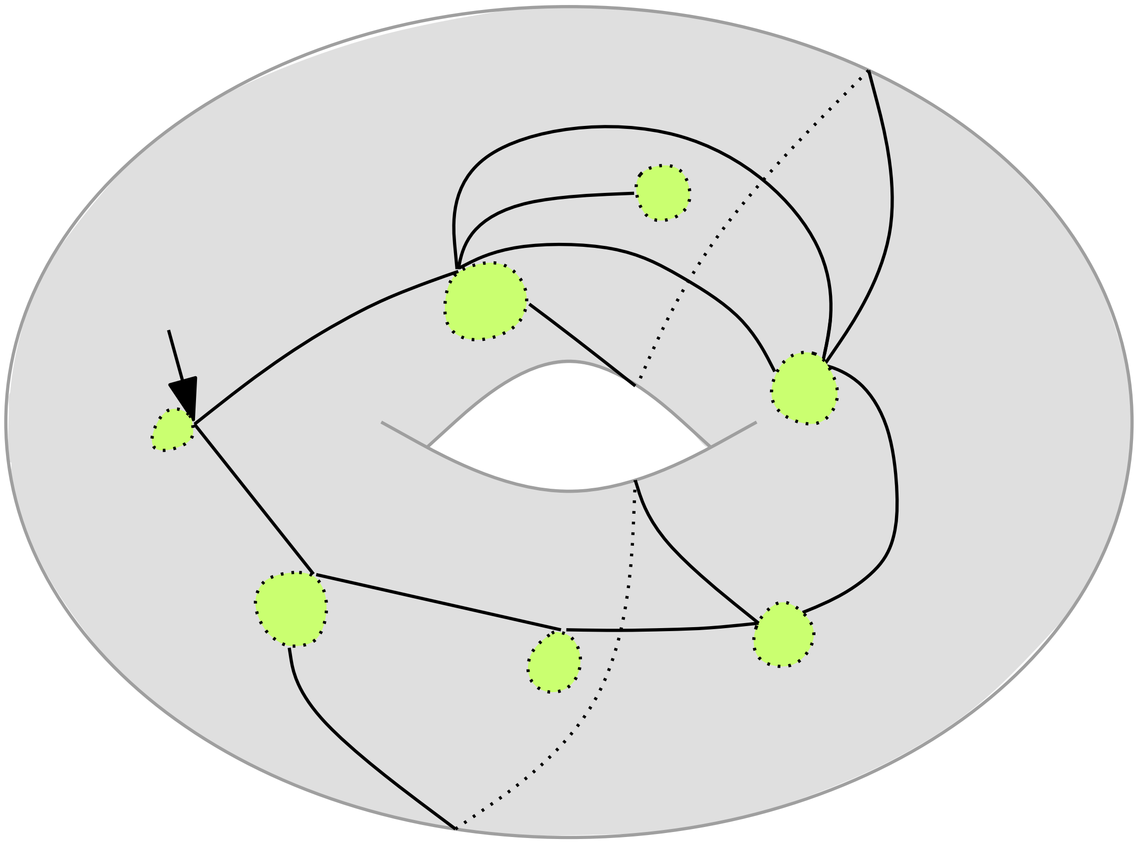
Fact 1: we obtain a tree.



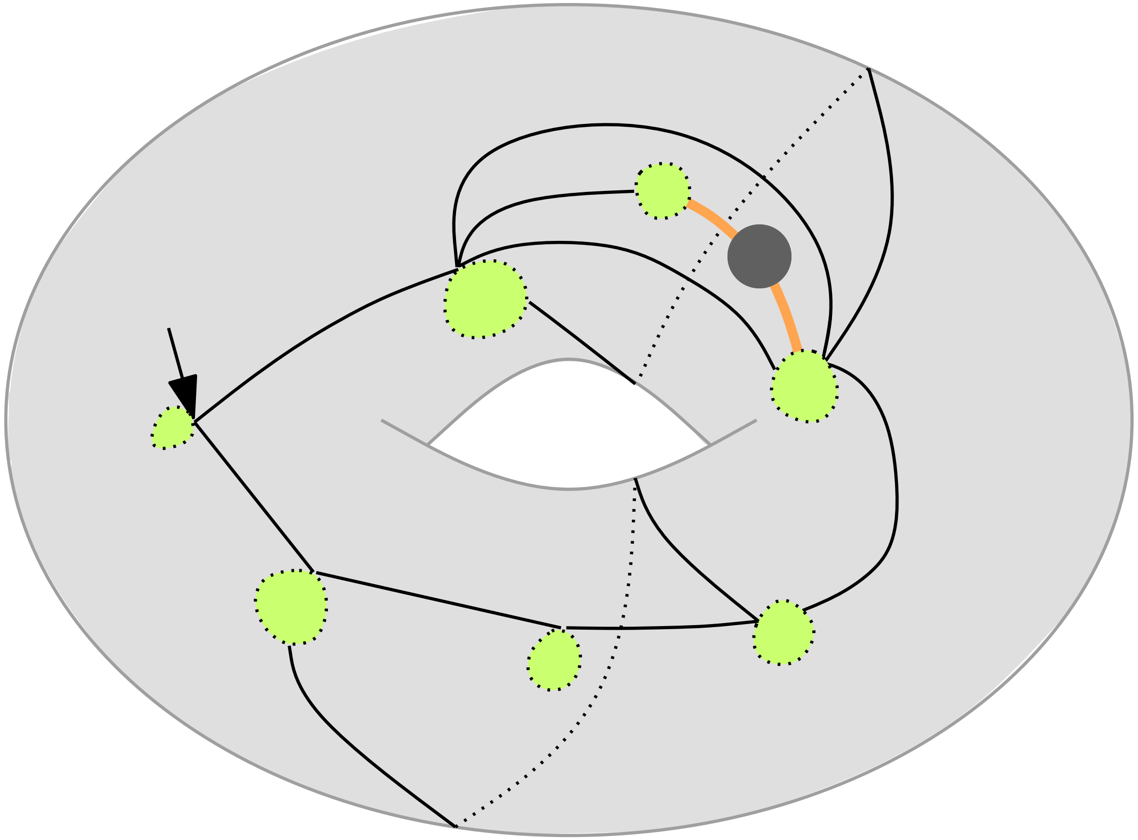
Fact 1: we obtain a tree.



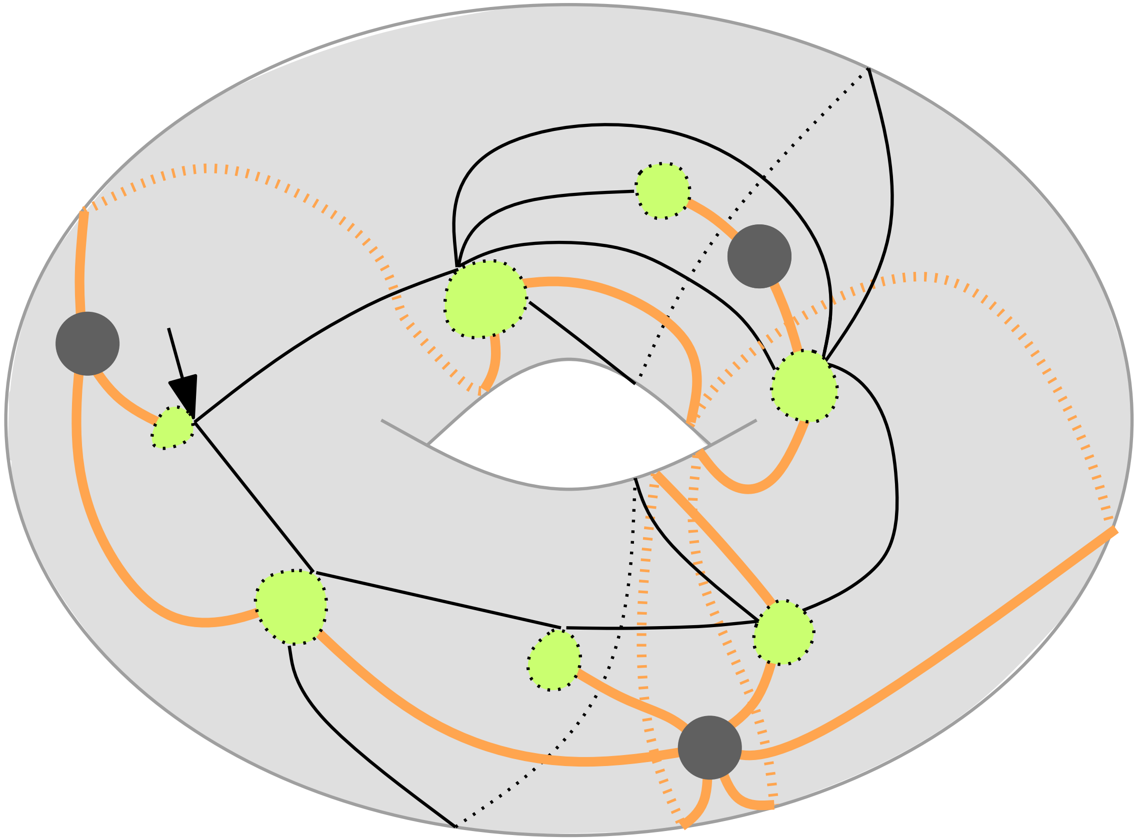
Fact 2: the gluing skeleton is a unicellular bipartite map.



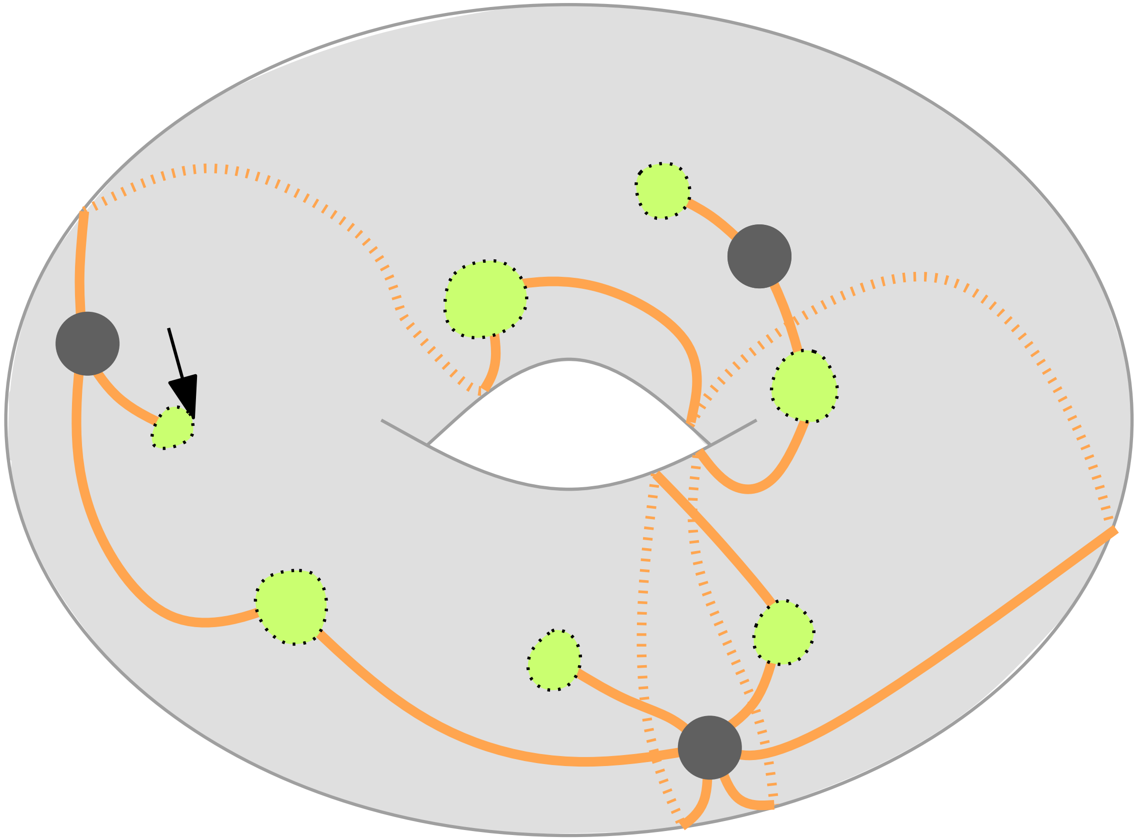
Fact 2: the gluing skeleton is a unicellular bipartite map.



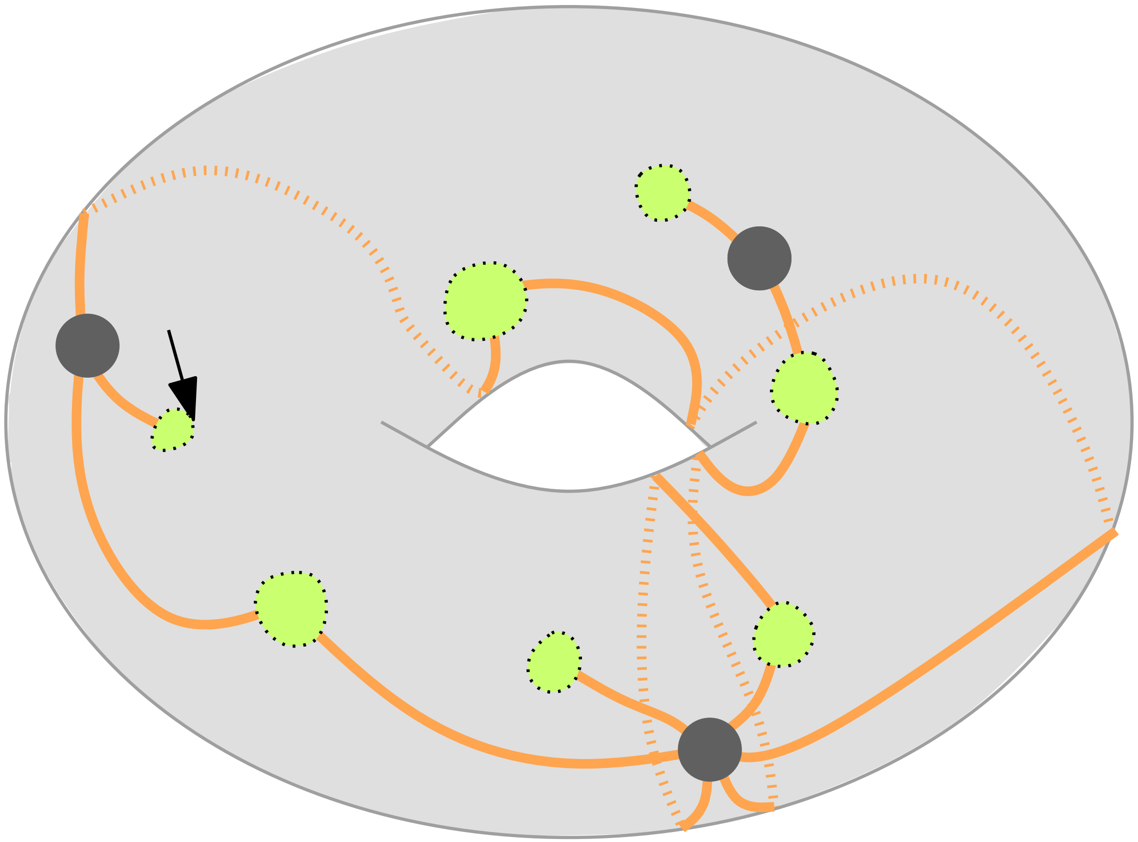
Fact 2: the gluing skeleton is a unicellular bipartite map.



Fact 2: the gluing skeleton is a unicellular bipartite map.



Fact 2: the gluing skeleton is a unicellular bipartite map.



To reconstruct the original map, just glue the tree along the border of the skeleton.
The construction is bijective.

Hence we have indeed:

$$\{\text{covered maps of genus } g \text{ with } n \text{ edges}\}$$
$$\updownarrow$$
$$\{\text{unicellular bipartite maps of genus } g,\ n+1 \text{ edges}\}$$
$$\times\ \{\text{plane trees, } n \text{ edges}\}$$

via left-orientations.

Hence we have indeed:

$$\{\text{covered maps of genus } g \text{ with } n \text{ edges}\}$$
$$\updownarrow$$
$$\{\text{unicellular bipartite maps of genus } g,\ n+1 \text{ edges}\}$$
$$\times\ \{\text{plane trees, } n \text{ edges}\}$$

via left-orientations.

Concluding remarks:

One has: $\dfrac{\#\{\text{tree-rooted maps}\}}{\#\{\text{covered maps}\}} \longrightarrow \dfrac{1}{2^g}$, but we do not see it on the bijection.

More generally, is it possible to enumerate tree-rooted maps in a bijective way ?