A bijection for covered maps on orientable surfaces.

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joint work with
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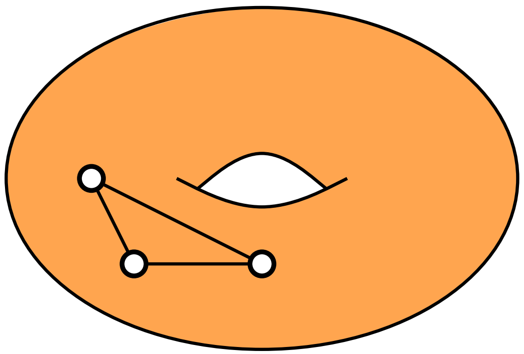
TGGT, May 2008.

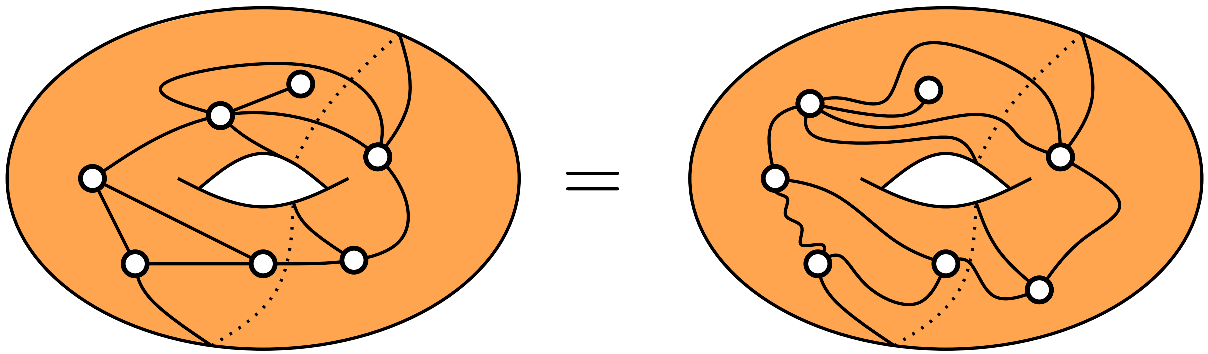
Map of genus g

$=$ drawing of a graph on the g-torus, such that the faces are simply connected.

Examples :



not a map



map on the torus

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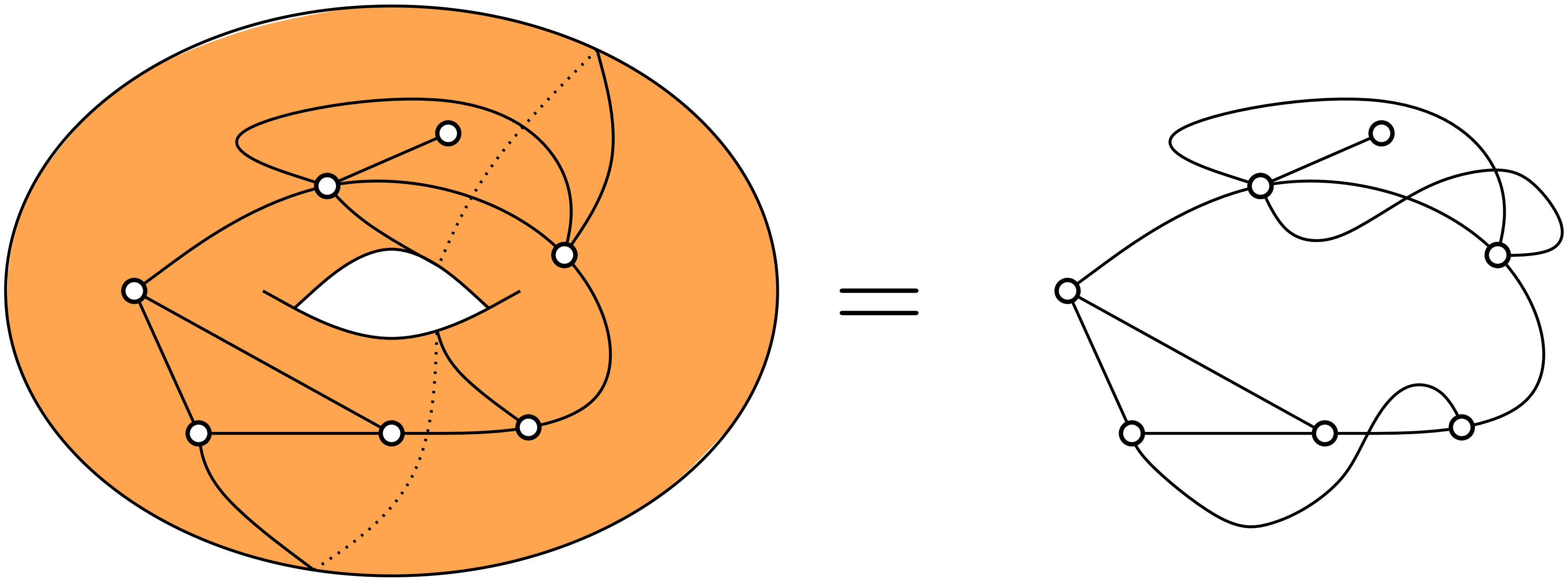

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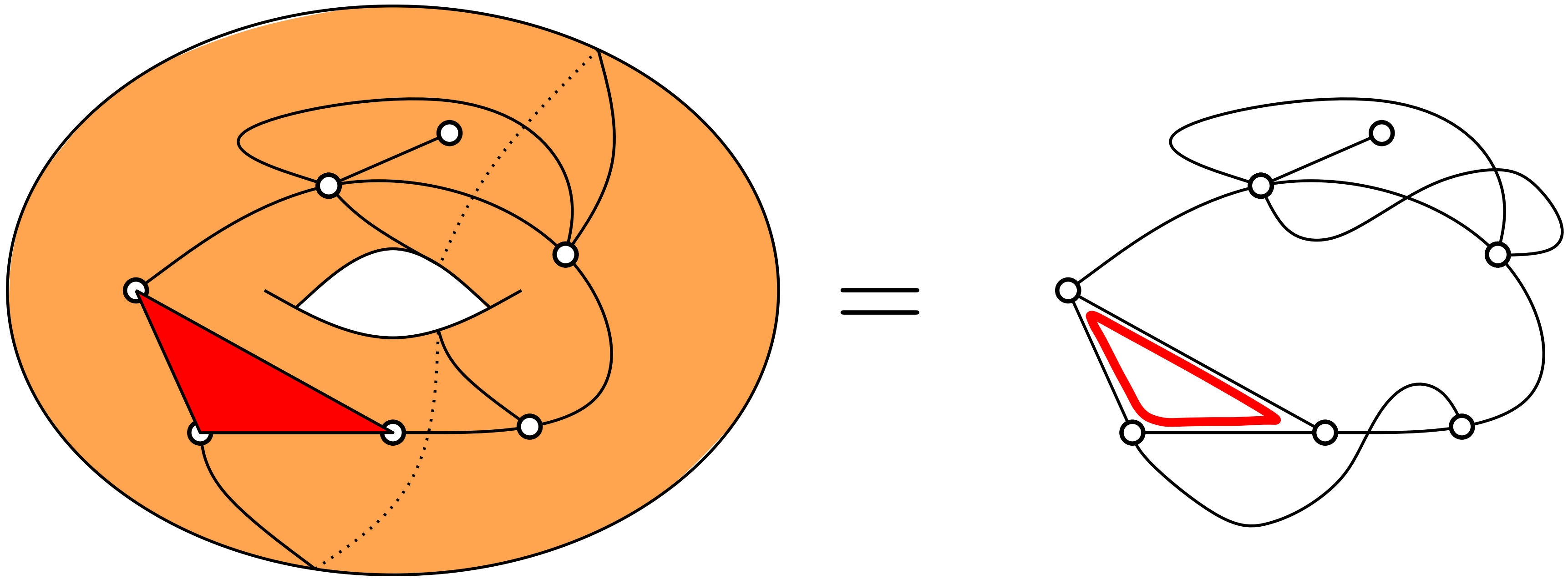

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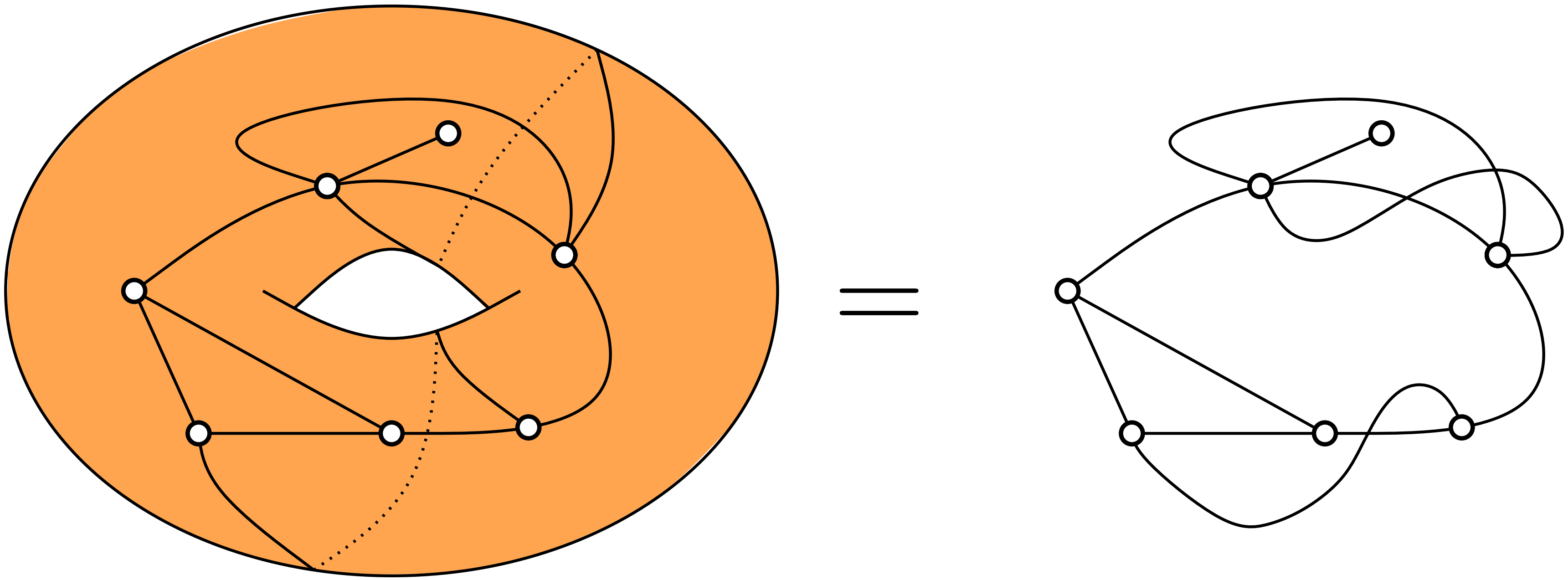

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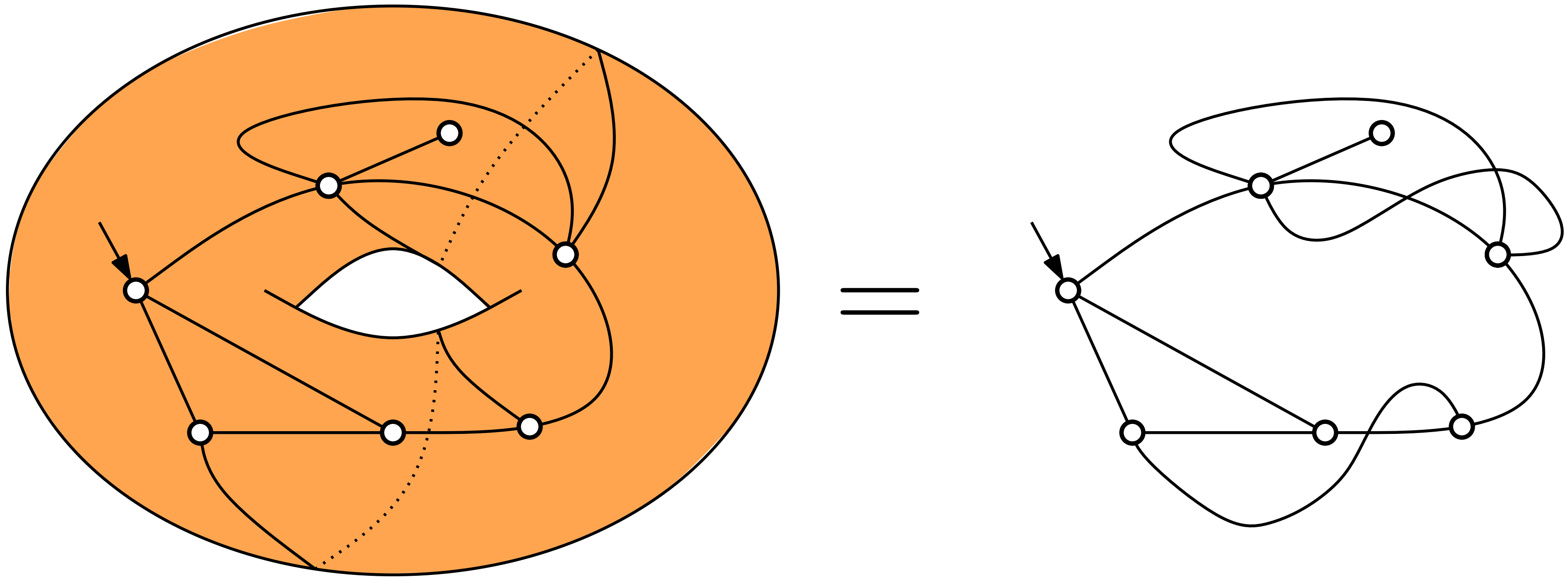

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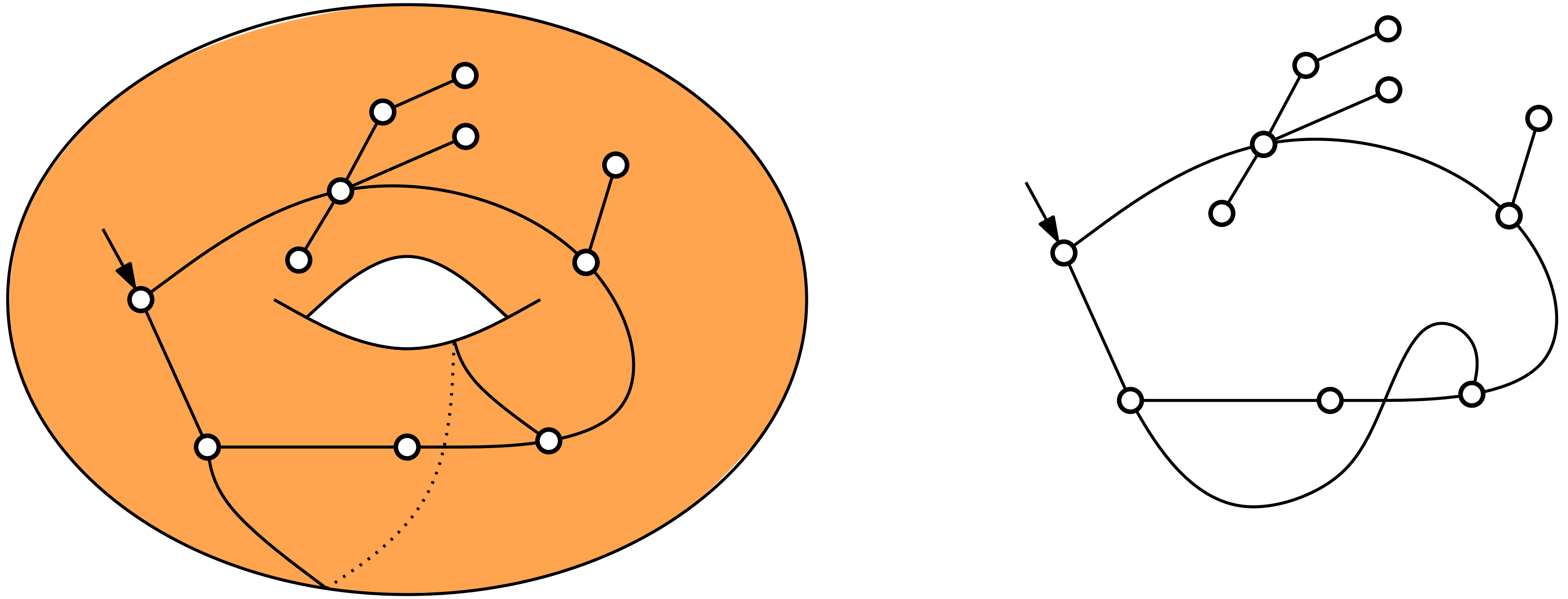

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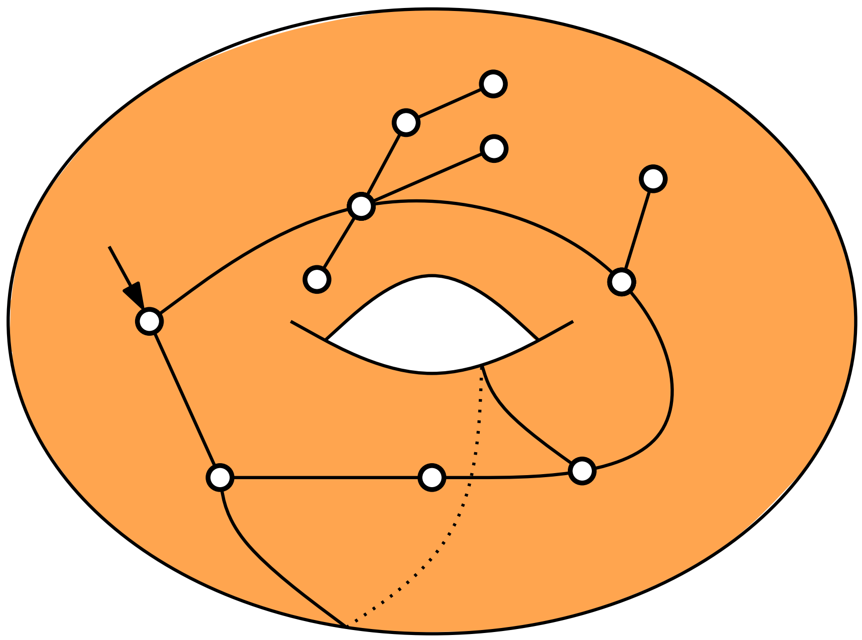

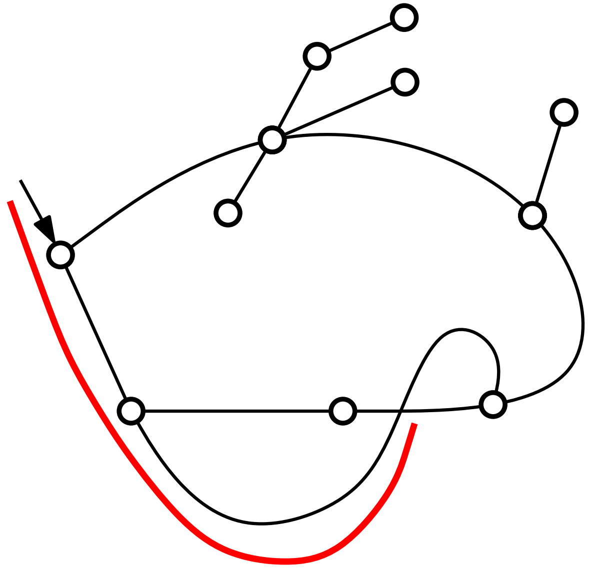

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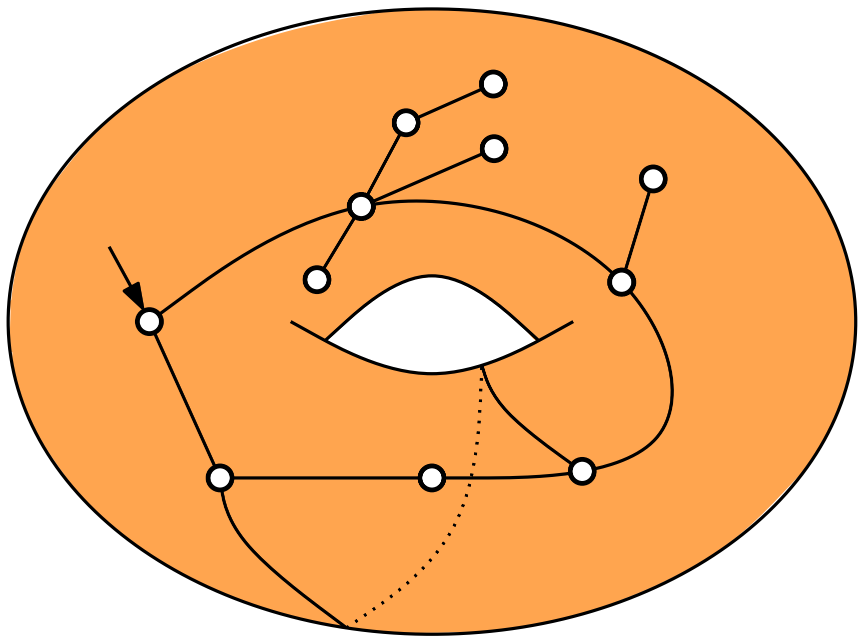

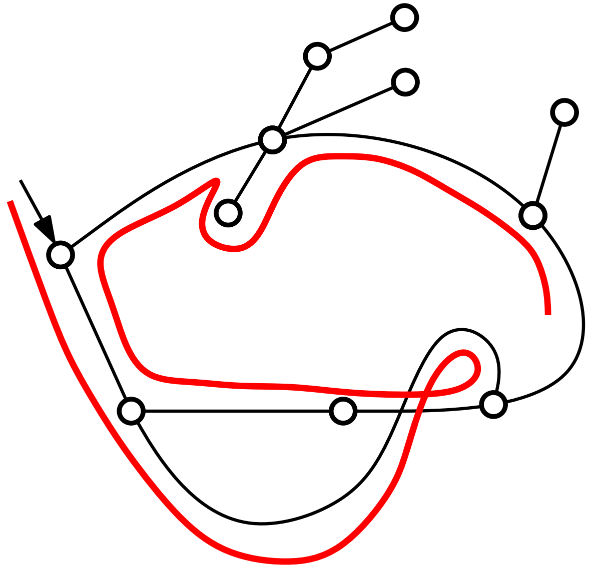

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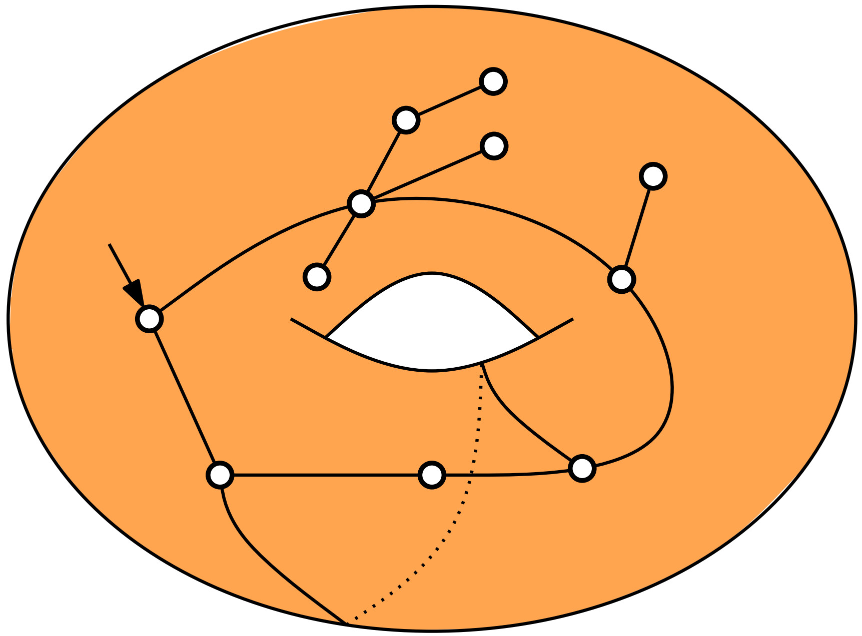

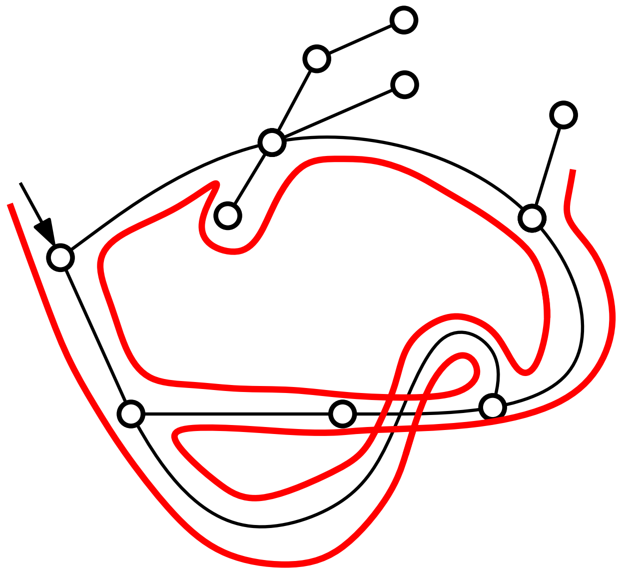

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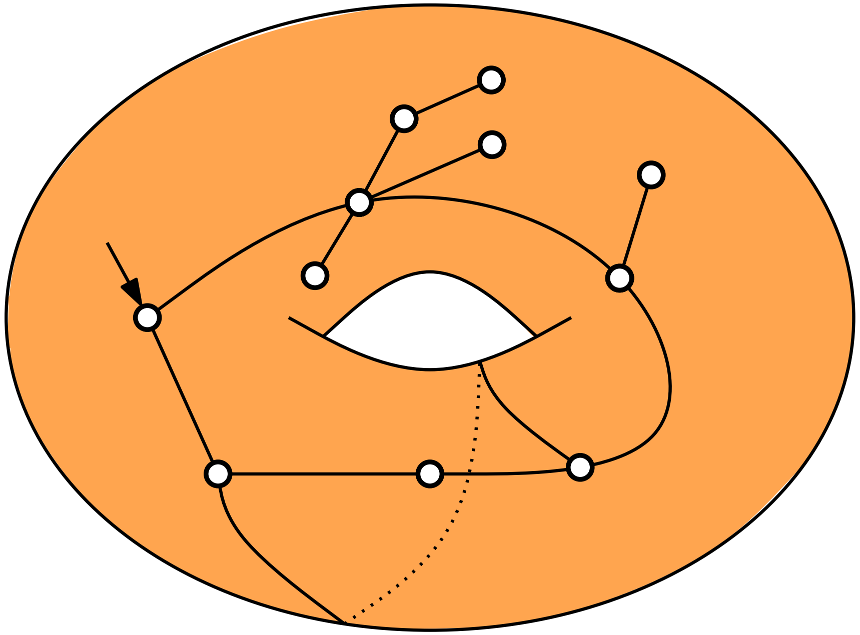

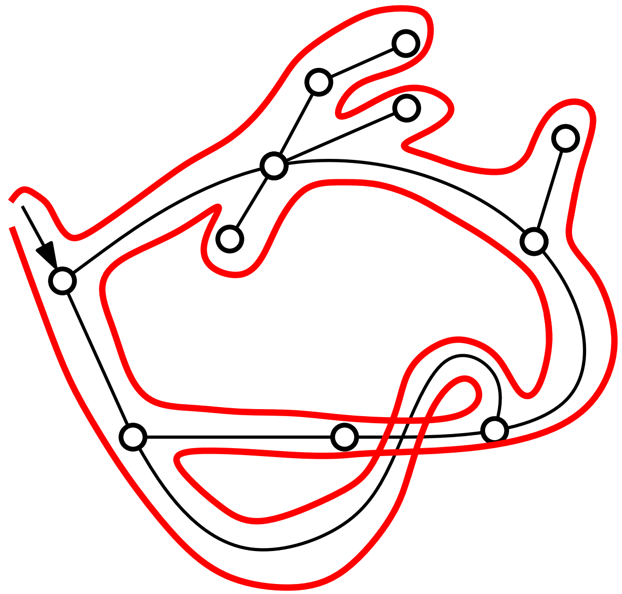

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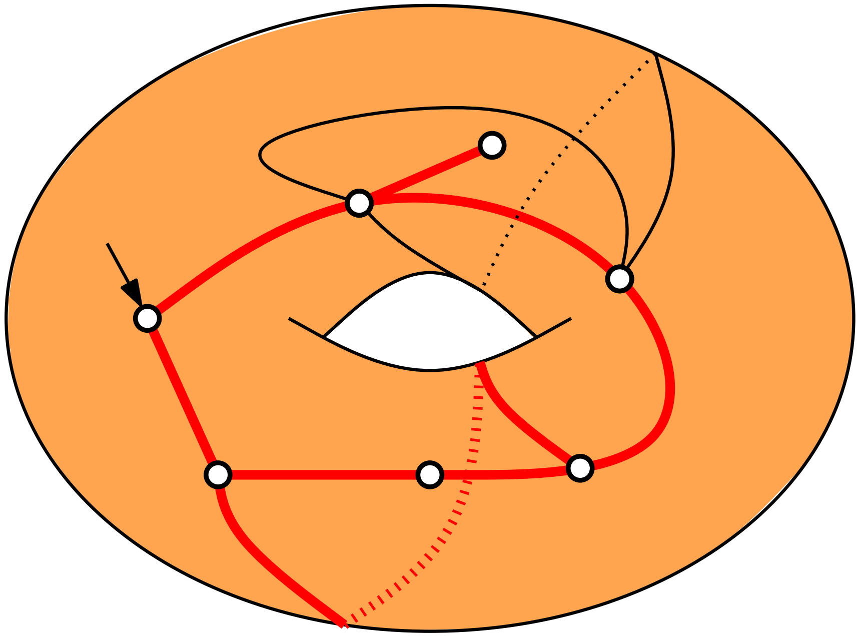
In genus 0, unicellular maps are exactly plane trees, but in positive genus, things are more complicated.

Covered maps.

A covered map is a map with a distinguished spanning unicellular submap.



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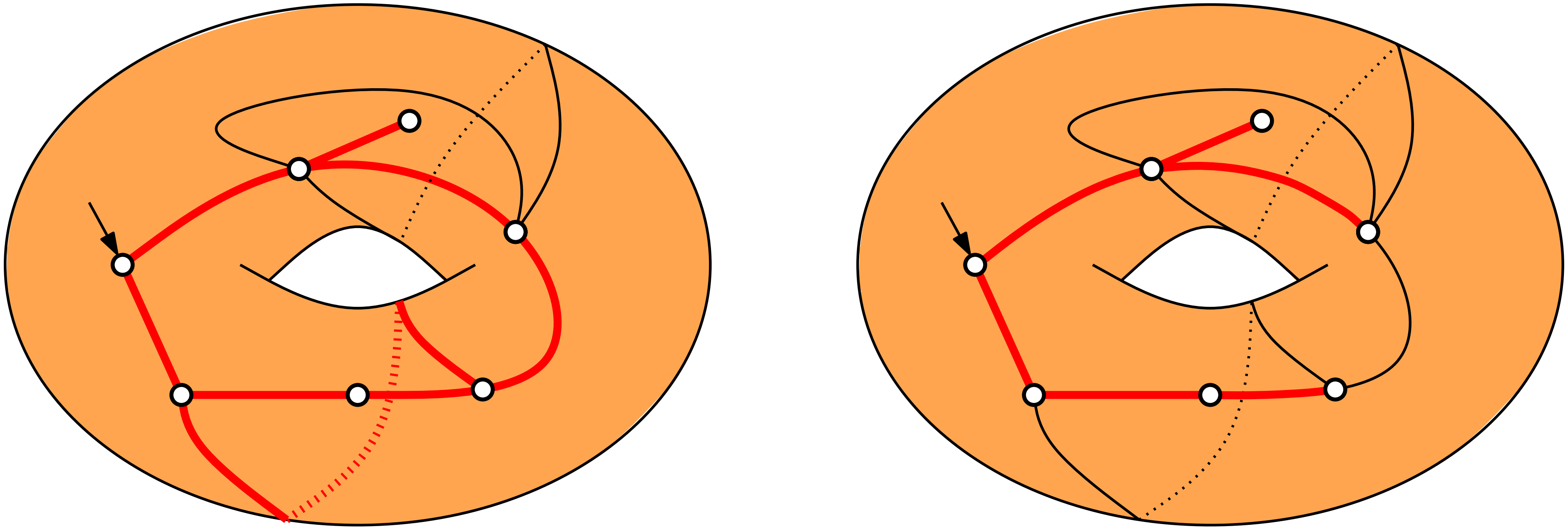

We do not impose that the spanning submap has genus g (it can have genus $0, 1, ..., g$).
Special case: map with a spanning tree = tree-rooted map.

Tree-rooted maps were previously studied.

In the planar case, a very nice formula from [Mullin 67]:

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Bijective proofs: [Cori, Dulucq, Viennot 82], [Bernardi 06].

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In higher genus:

[Lehman, Walsh 72]: nice formula for genus 1.
more complicated formulas for $g \geq 2$.

[Bender, Robert, Robinson 88]: asymptotics.

Theorem: There is a bijection:

$$\{\text{covered maps of genus } g \text{ with } n \text{ edges}\}$$
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Corollary:

For each g, there is a closed formula for the number of covered maps.

$$C_0(n) = Cat_n \times Cat_{n+1}$$
$$C_1(n) = Cat_n \times \frac{(2n-2)!}{12(n-1)!(n-3)!}$$
$$C_2(n) = Cat_n \times \frac{(5n^2-7n+6)(2n-5)!}{720(n-3)!(n-5)!}$$

Corollary: Nice formulas for tree rooted-maps.

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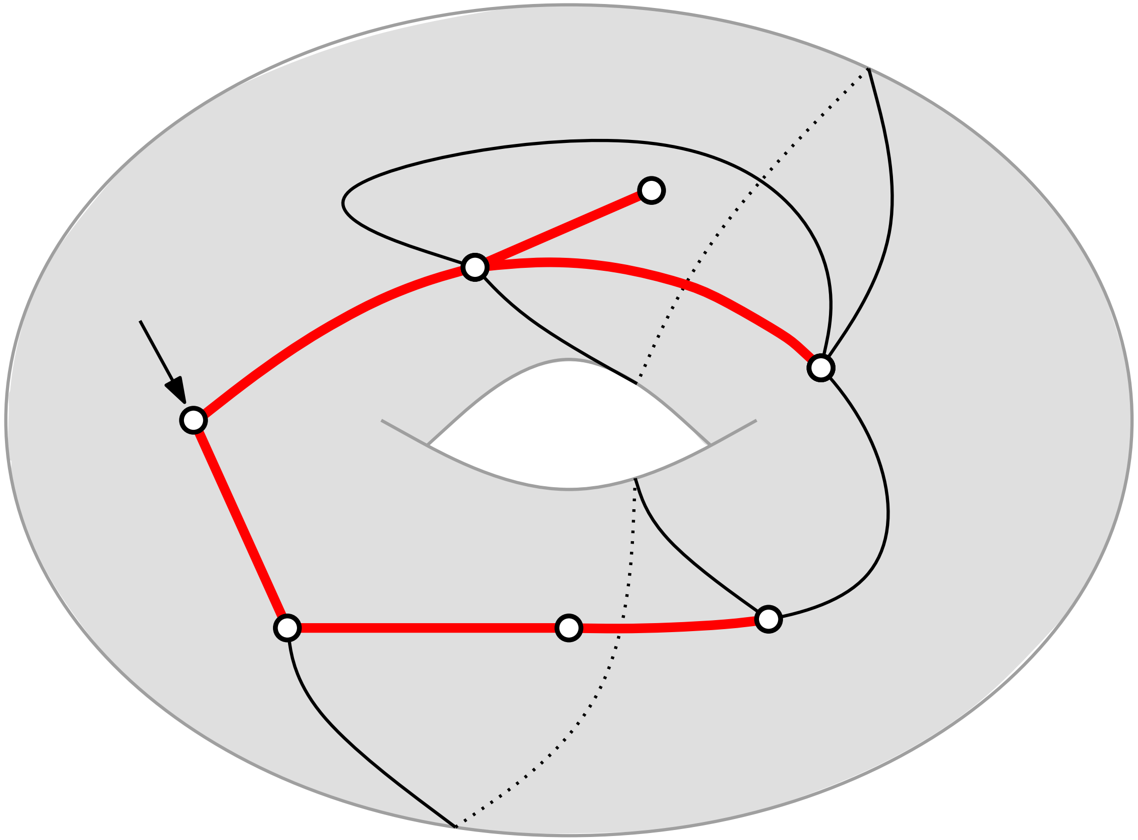
- No similar argument for $g \geq 2$: explains (?) why formulas for tree-rooted maps seem to be more complicated.

The bijection.

step 1: from covered maps to orientations.



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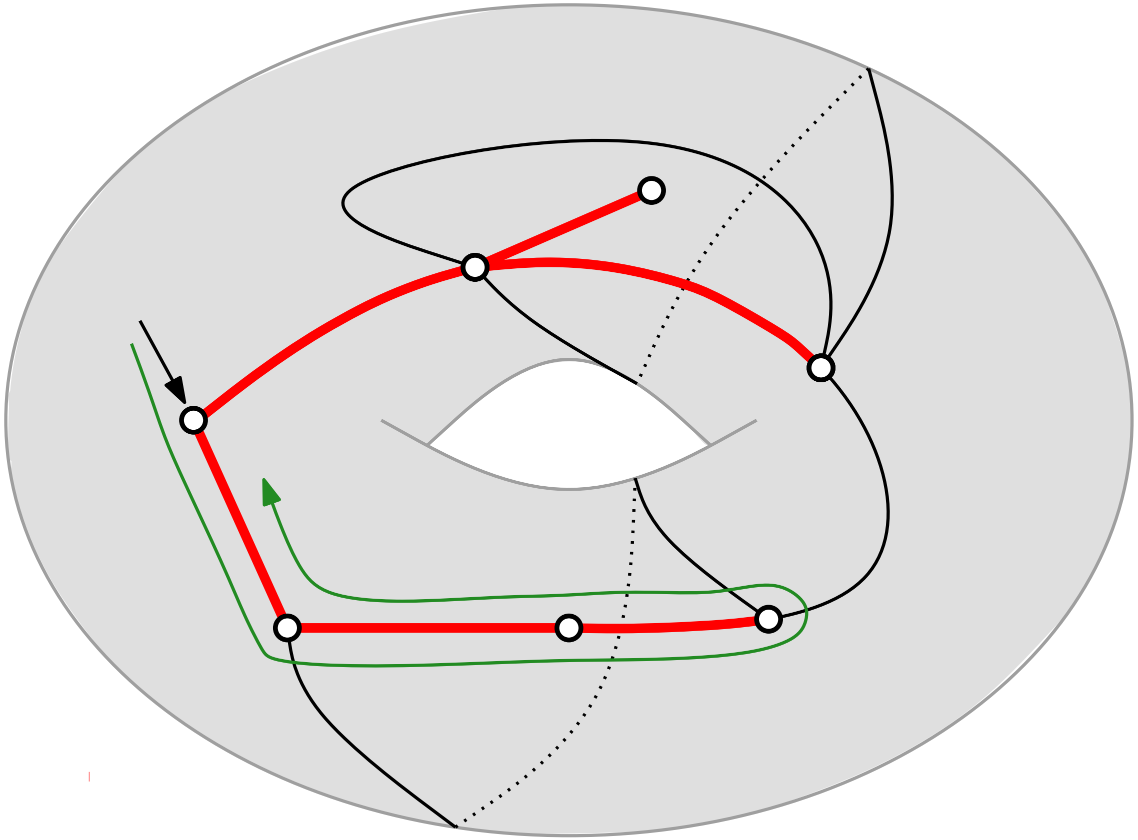

We make the tour of the submap, and orient:

- red edges as we followed them for the first time
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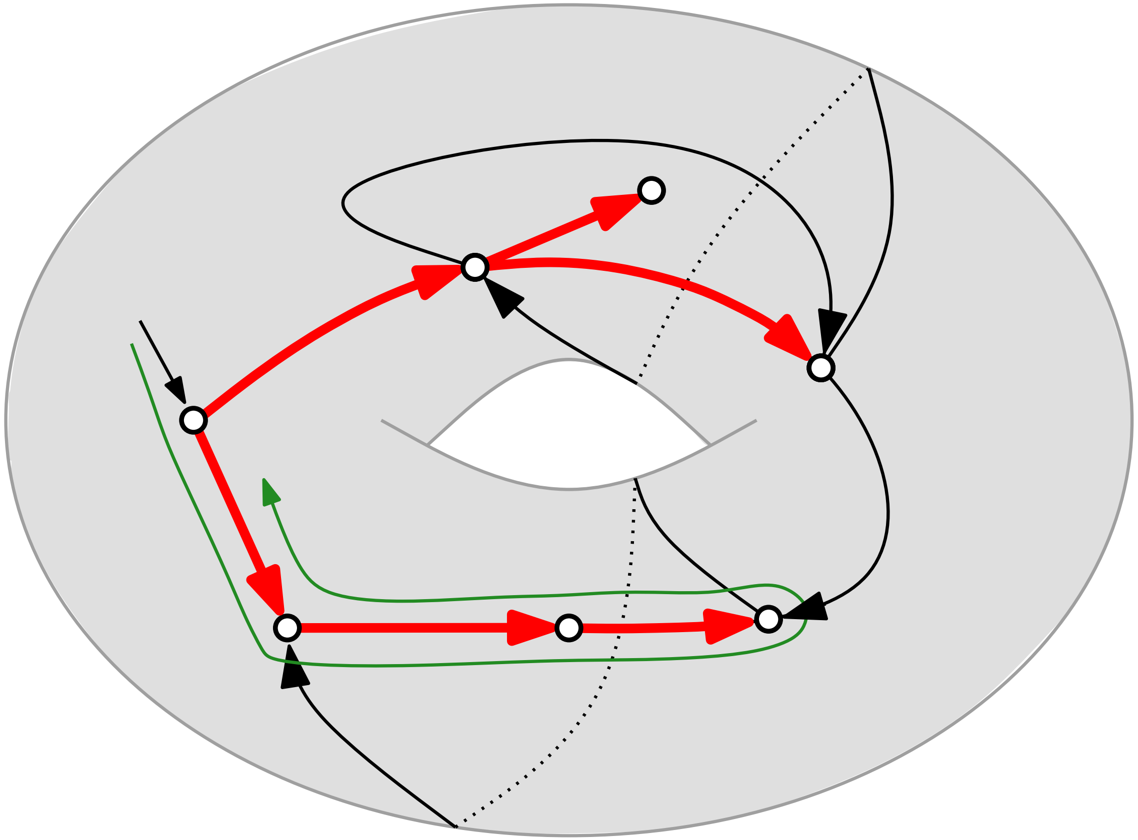

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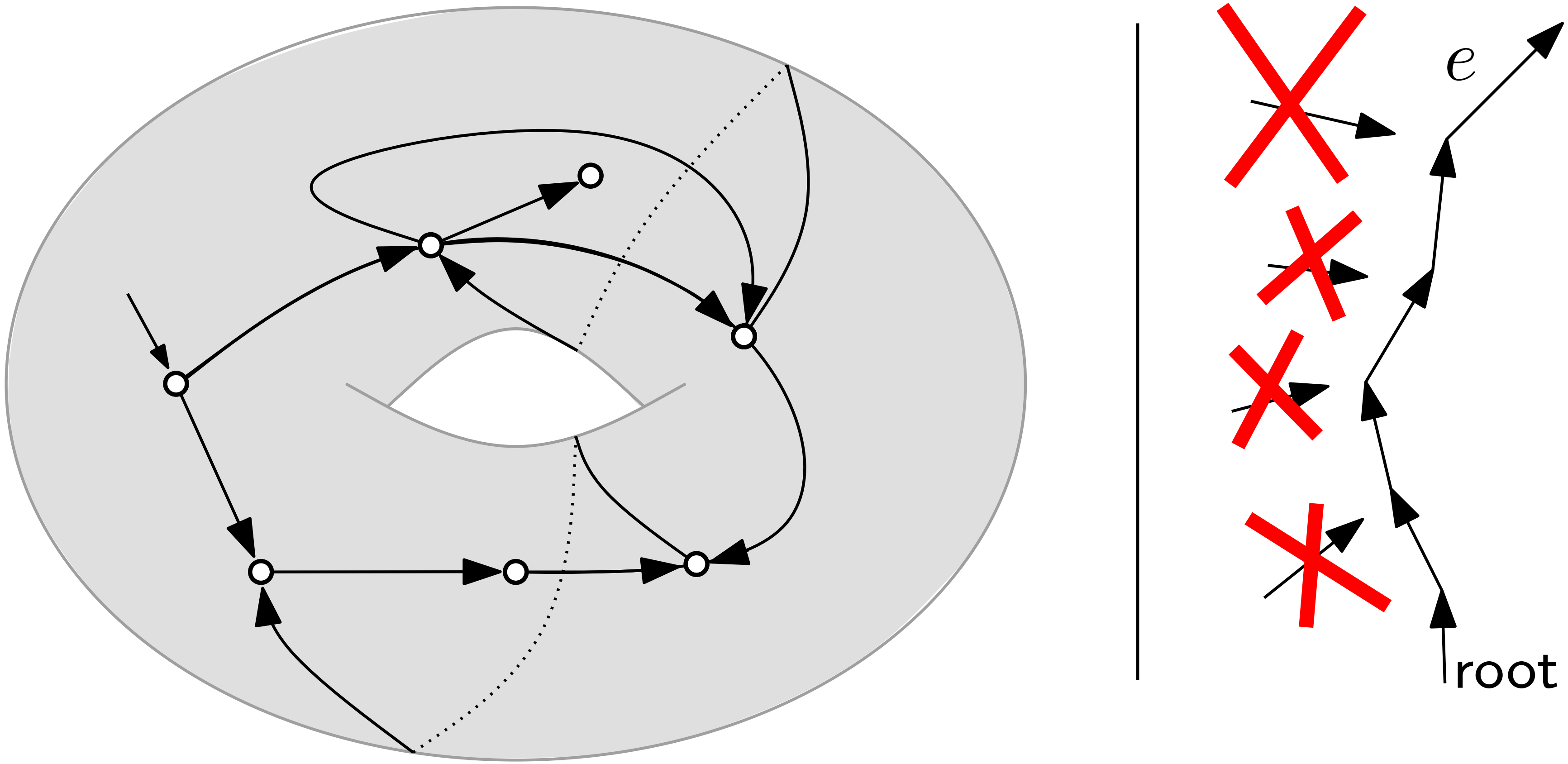

We obtain a left-orientation: each edge e can be reached from the root by a left-path. The construction is bijective.

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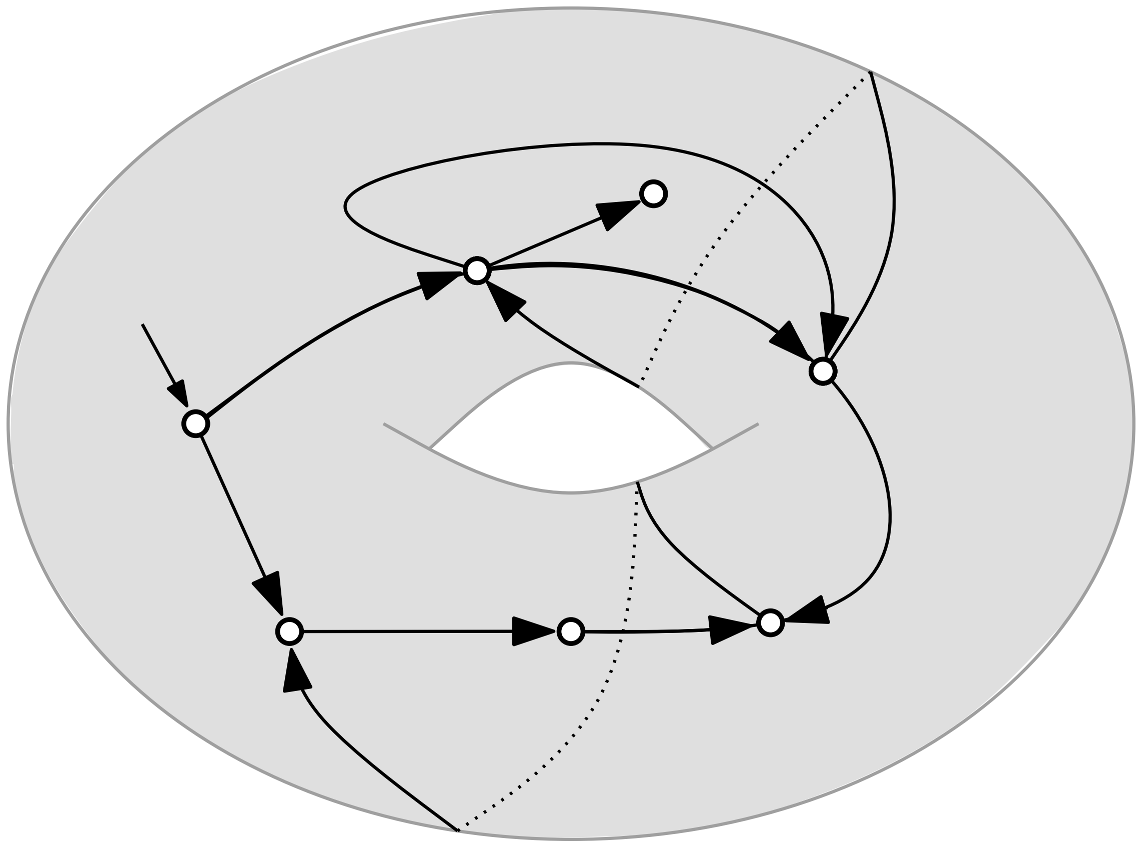
step 2: from left-orientations to pairs (tree, unicellular bipartite map).



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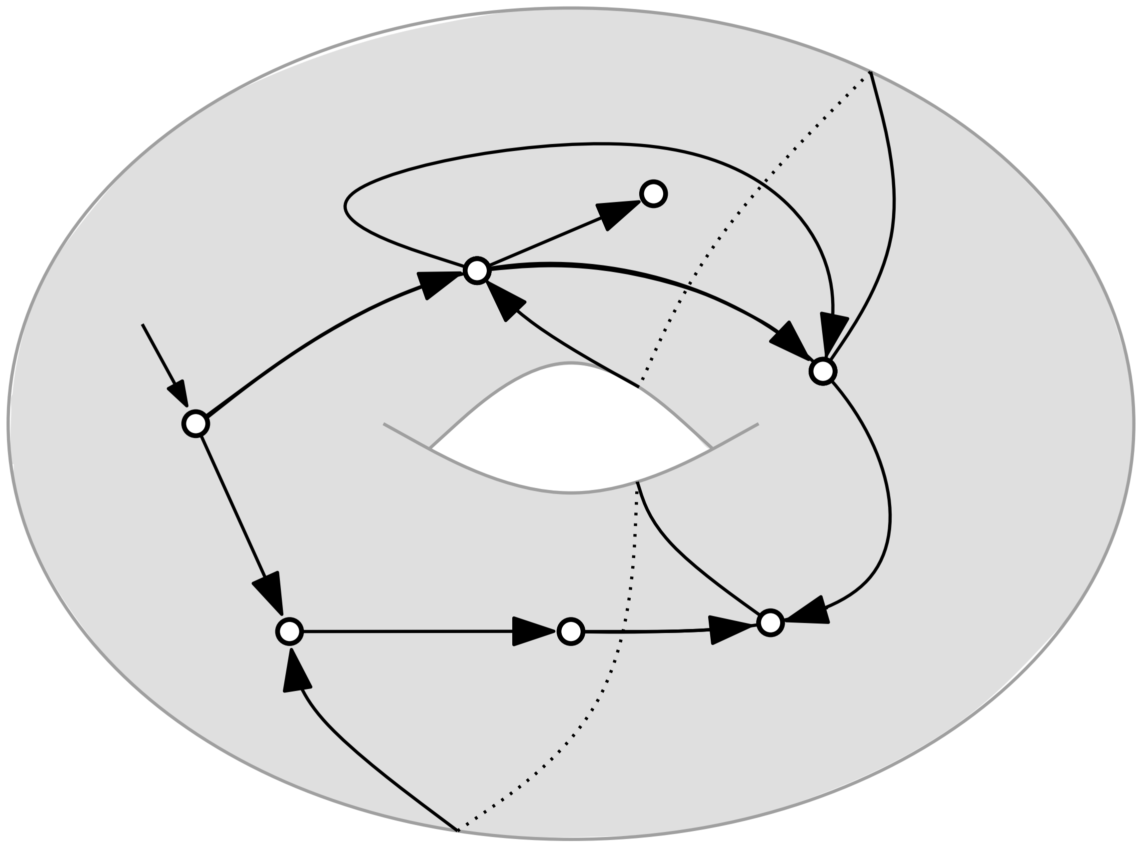
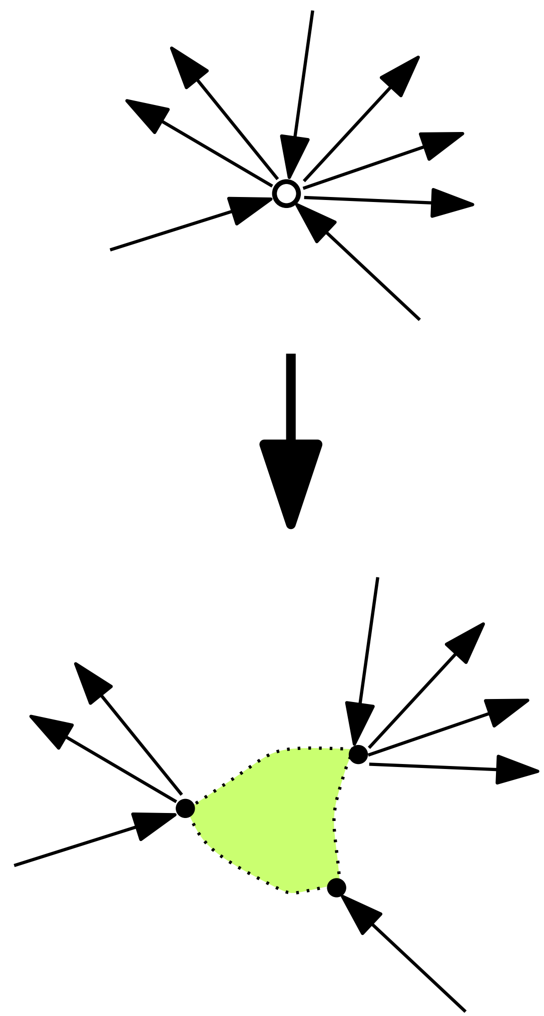


Unfolding a vertex:



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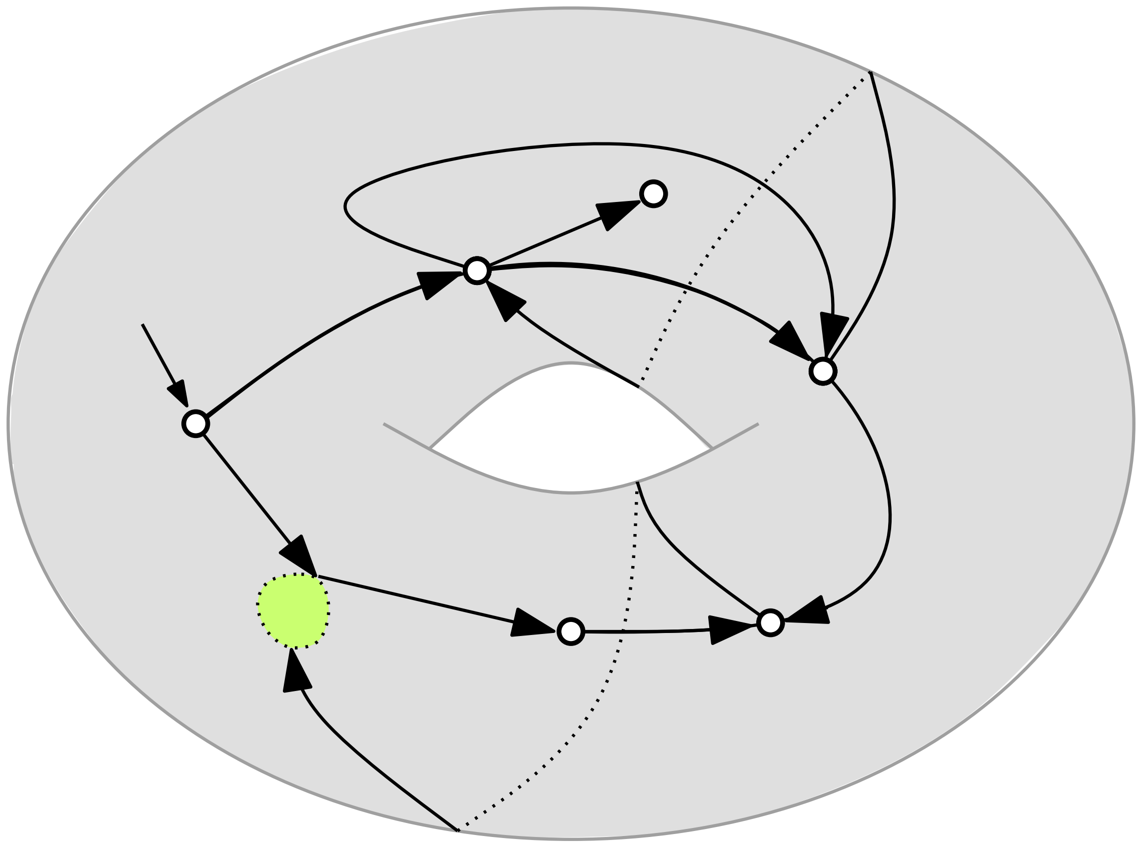

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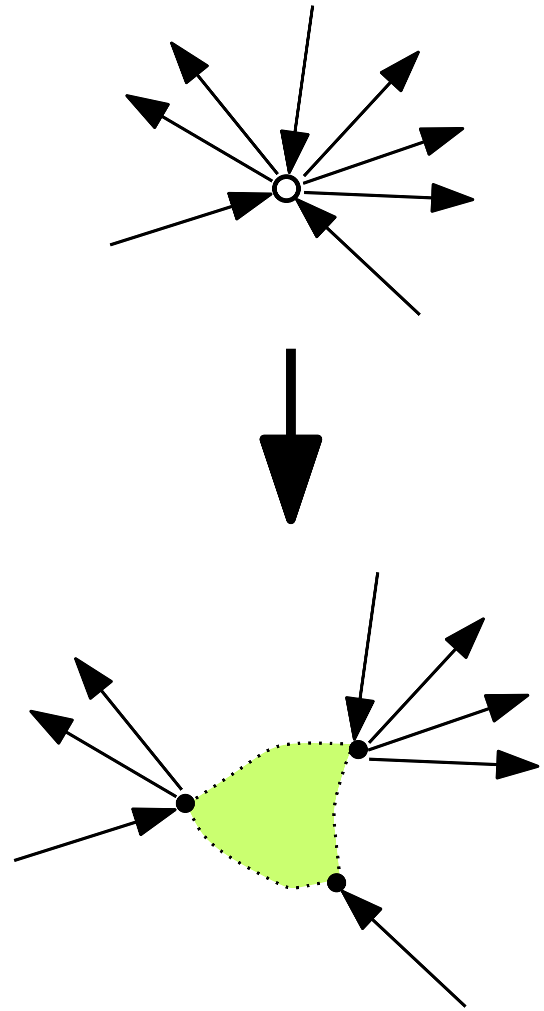

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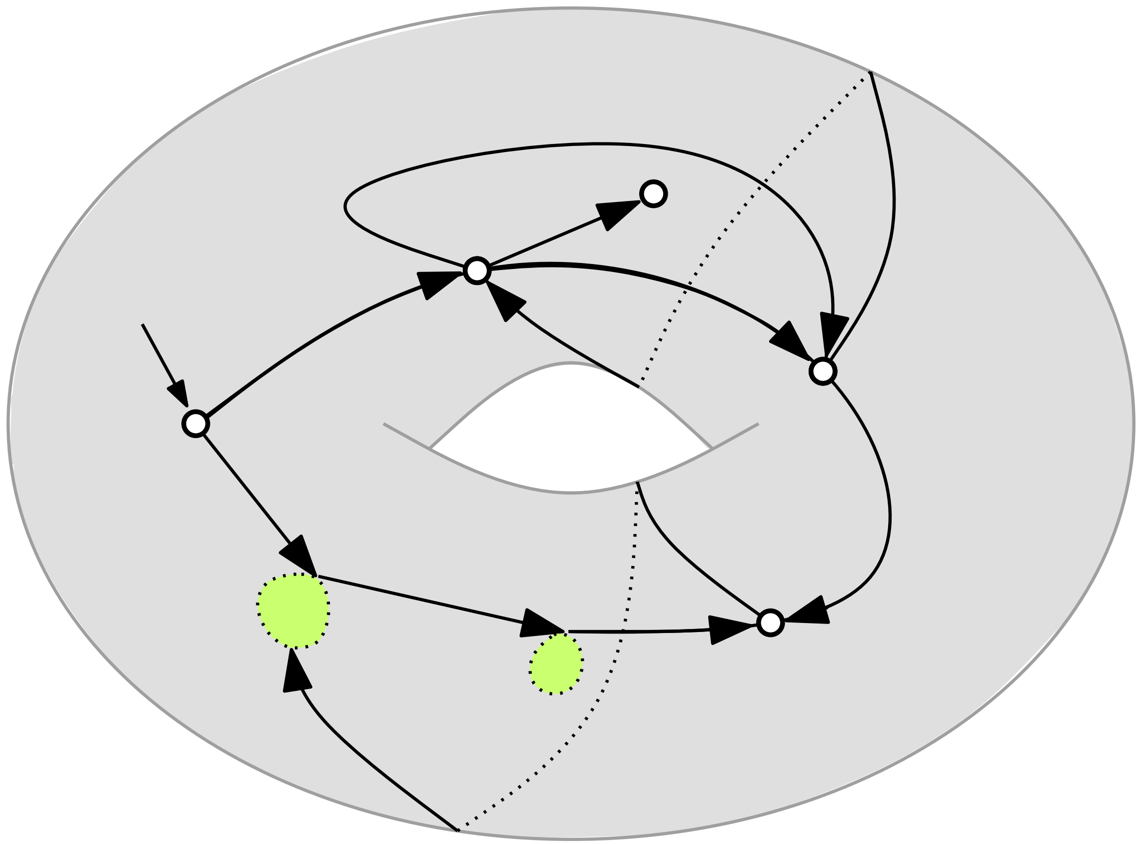

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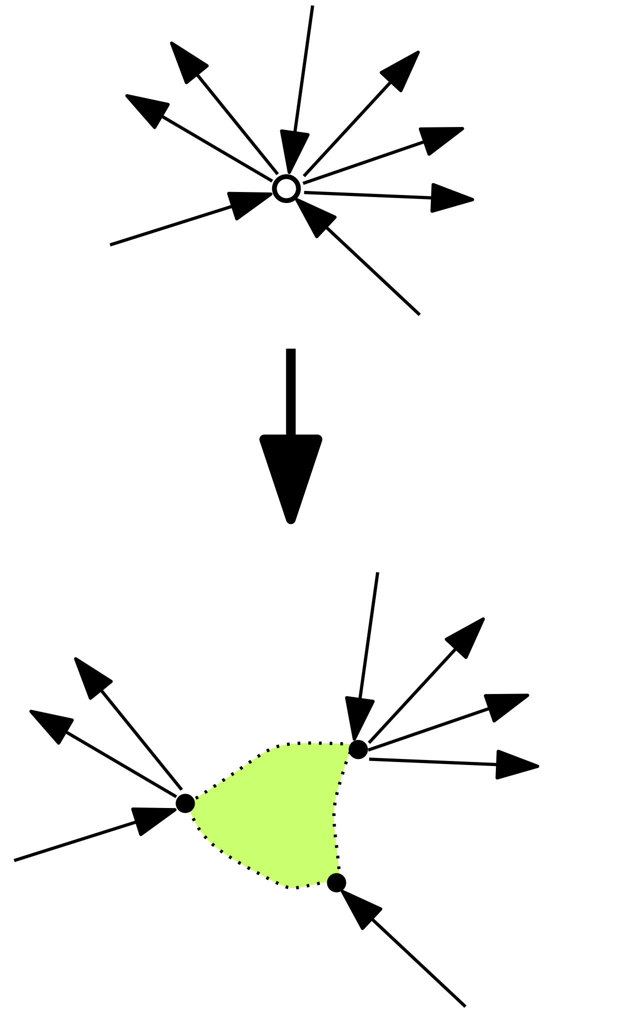

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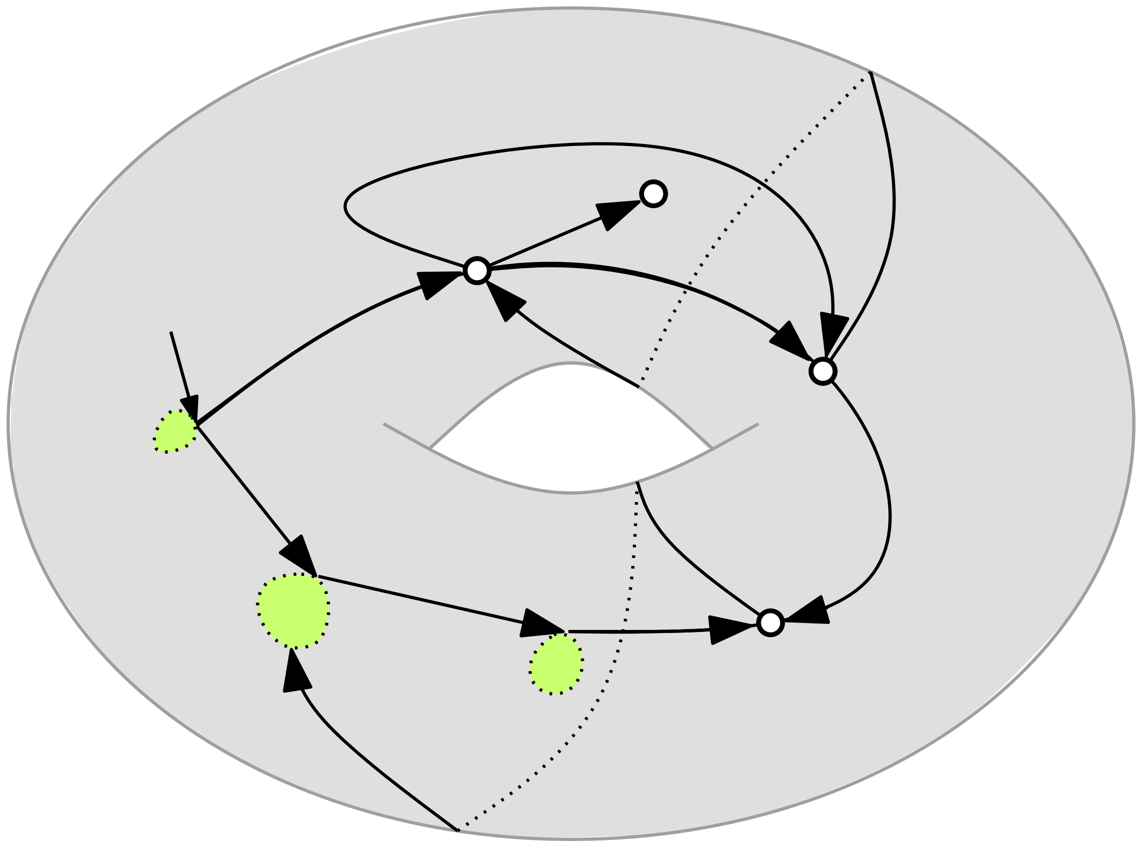

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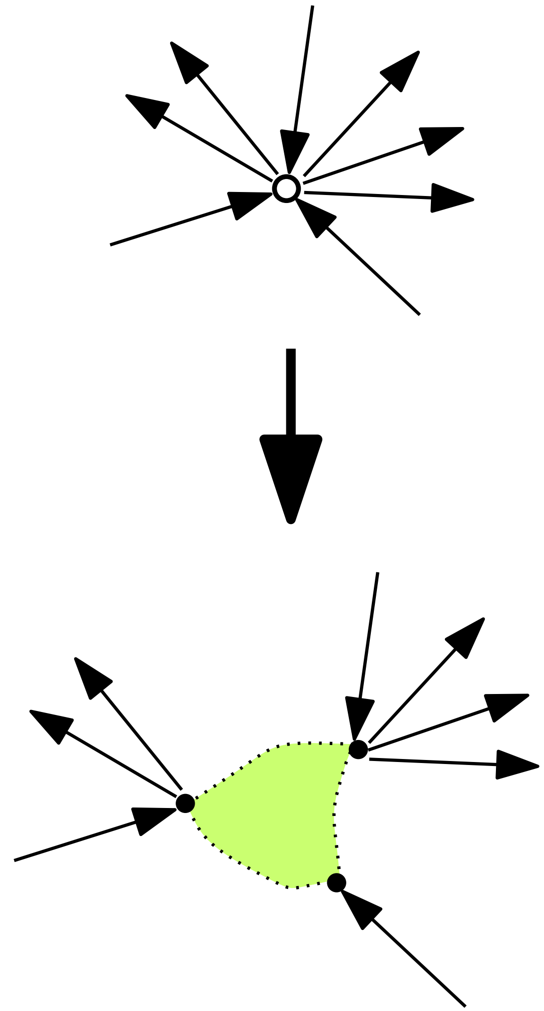

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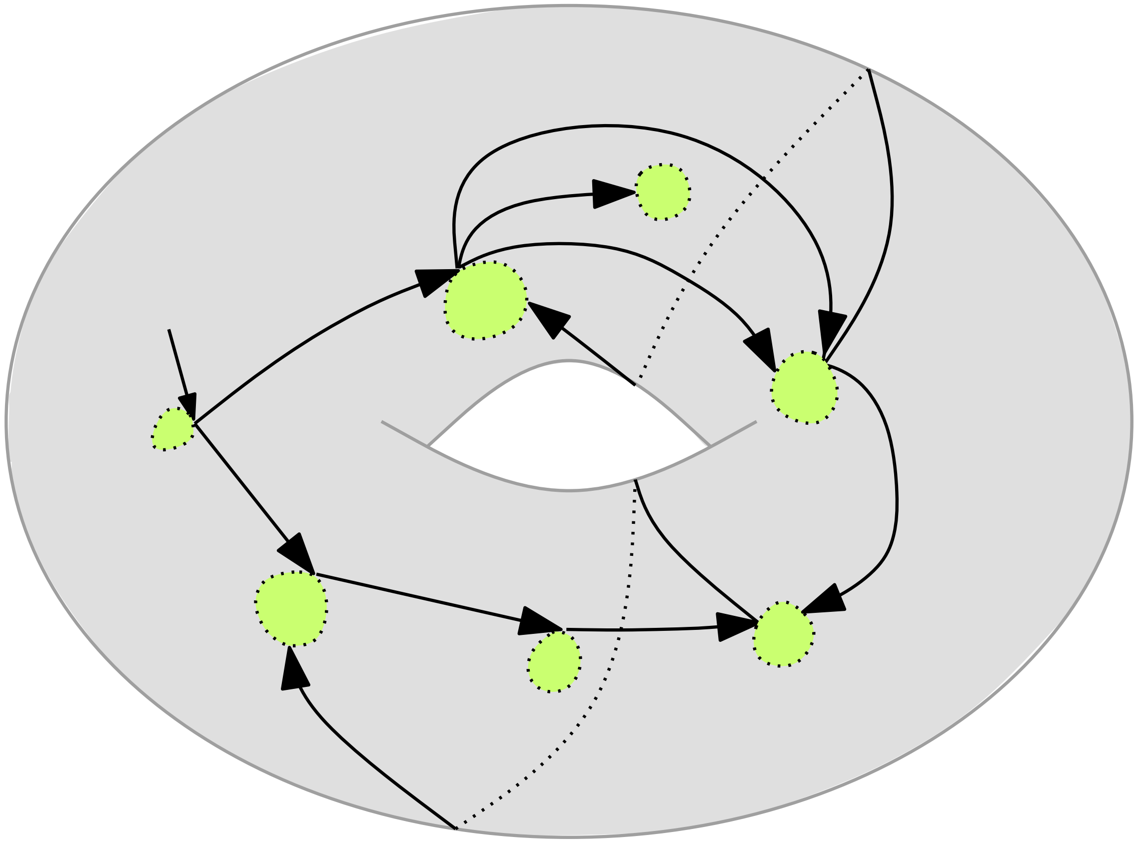

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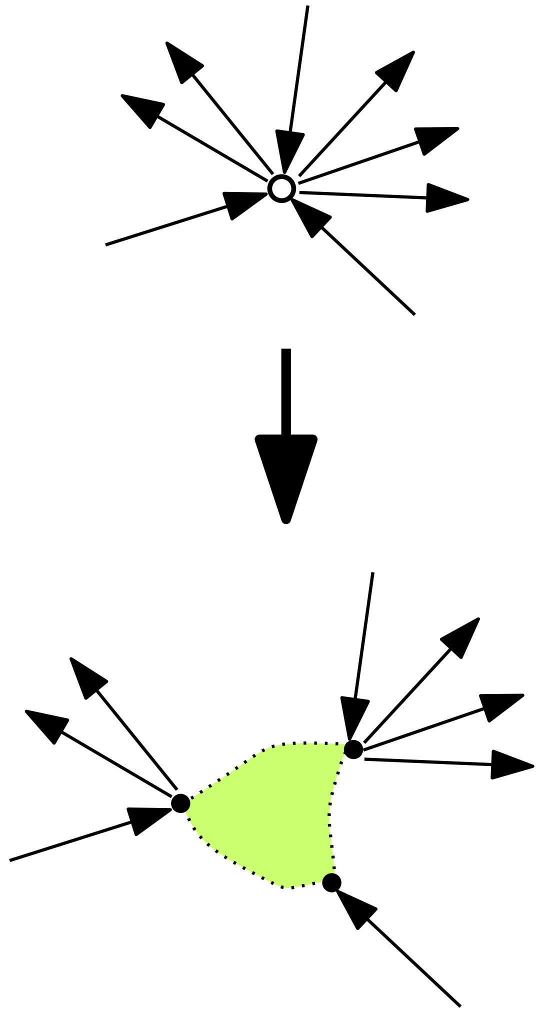

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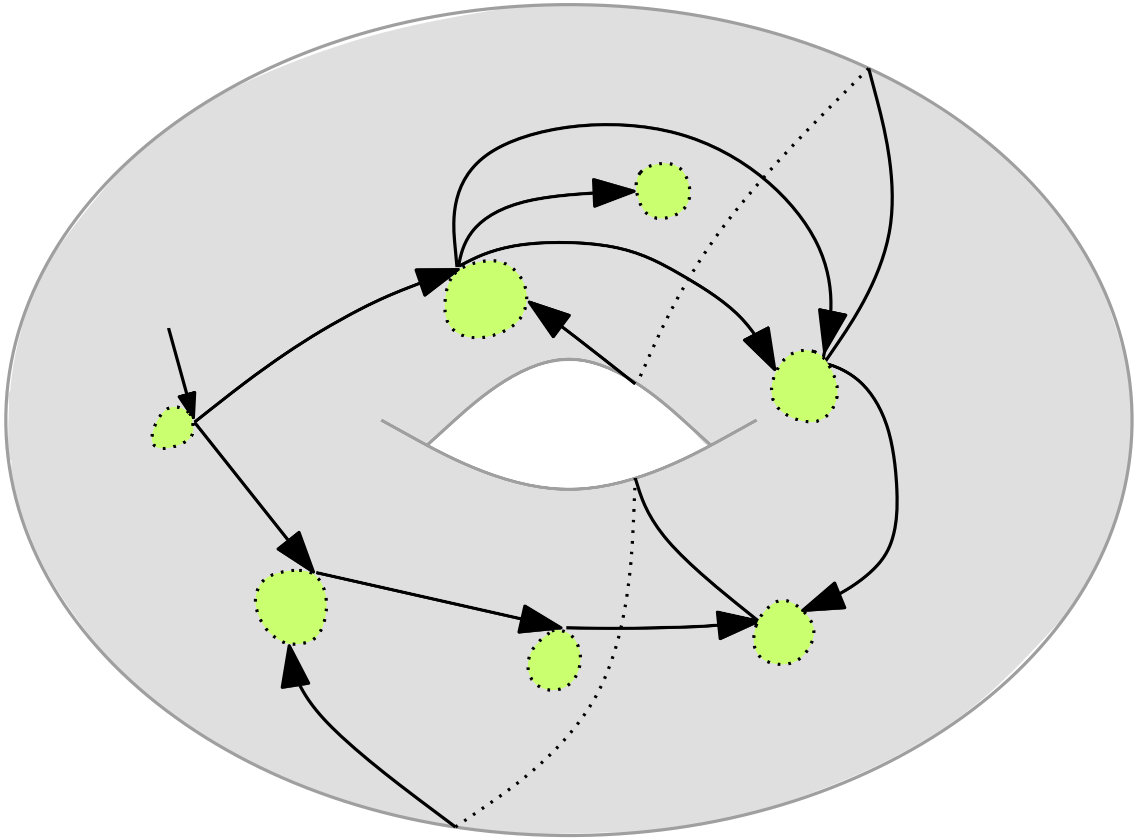

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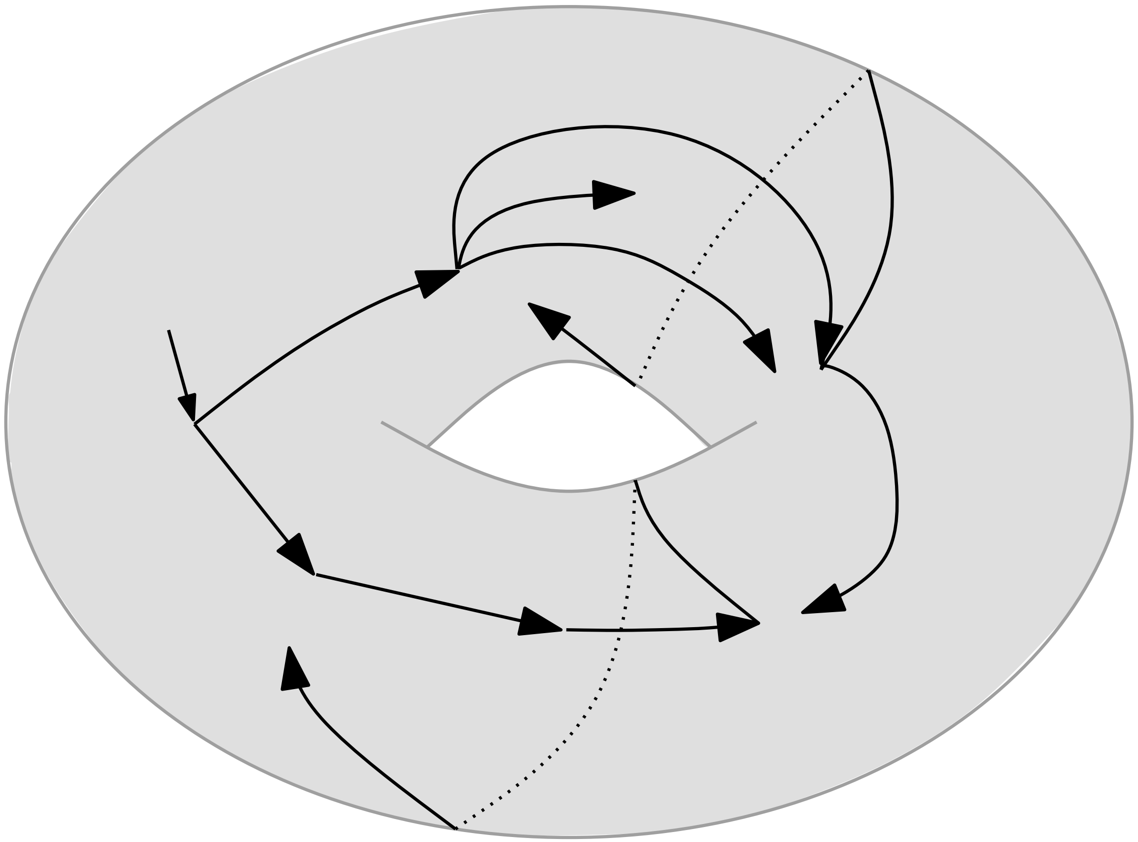
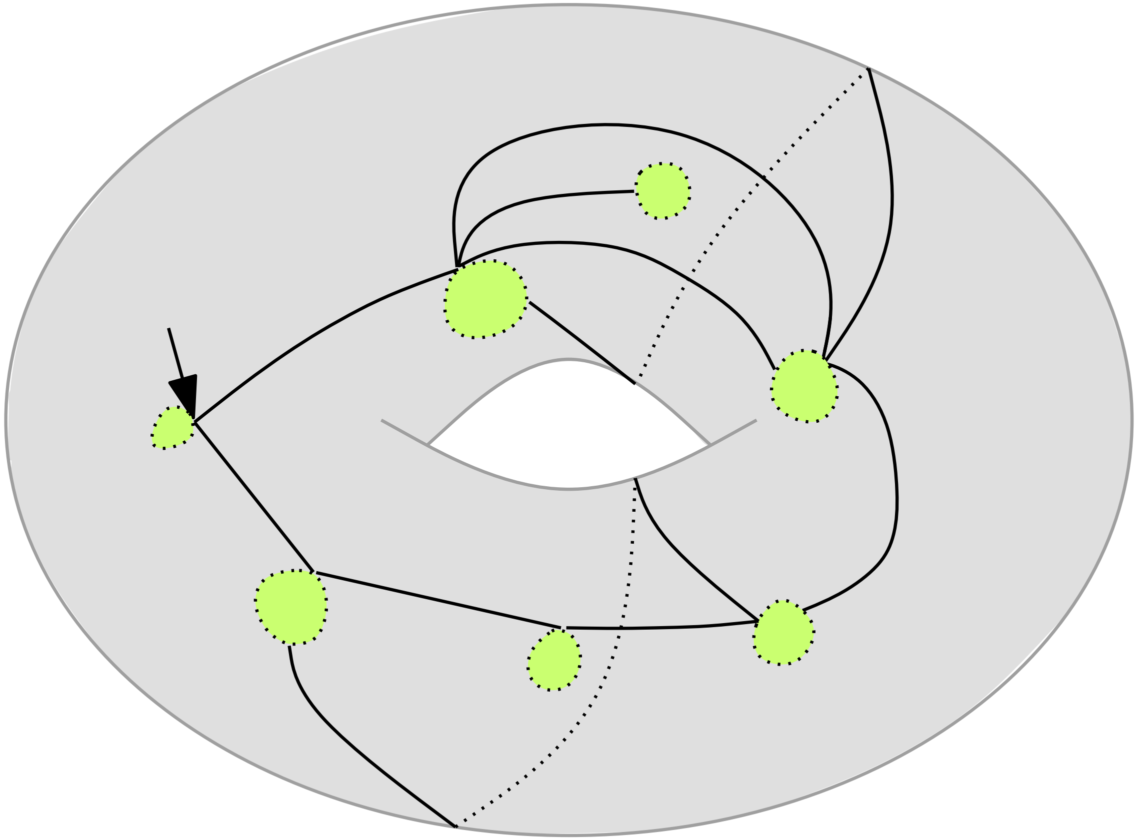


Fact 2: the gluing skeleton is a unicellular bipartite map.



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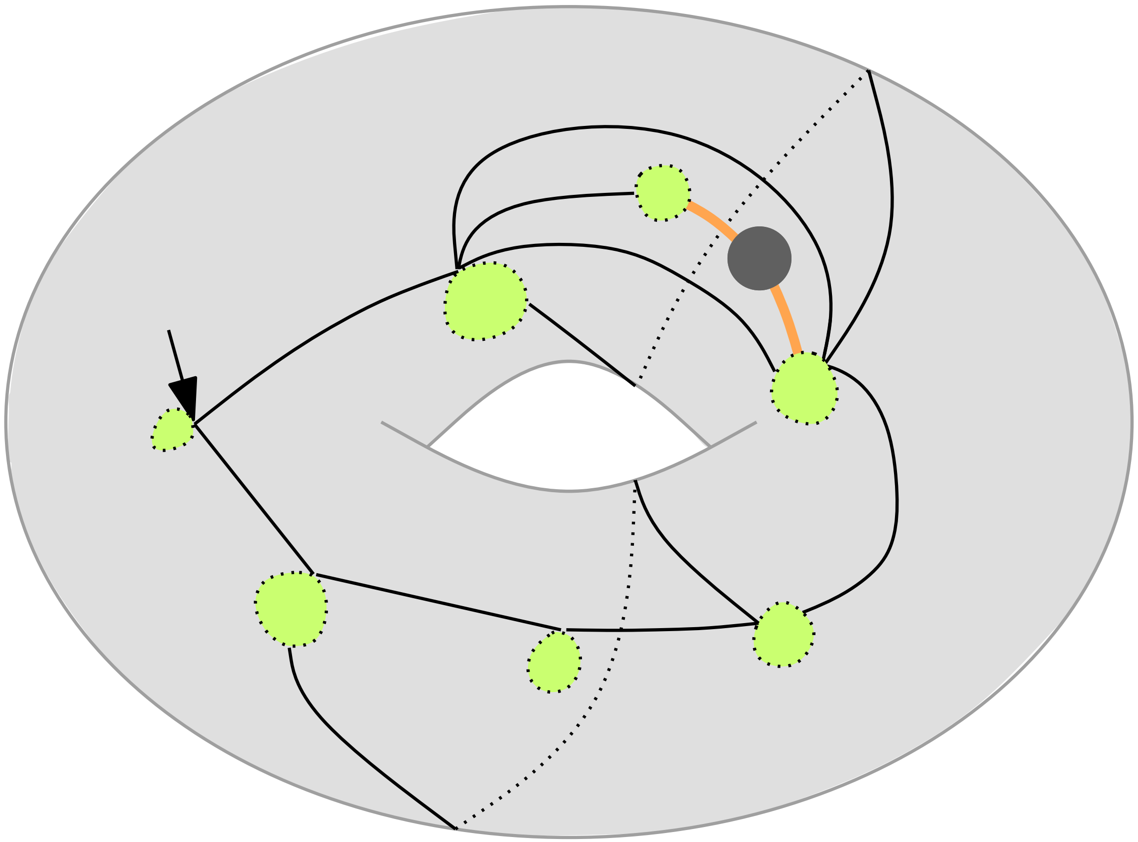

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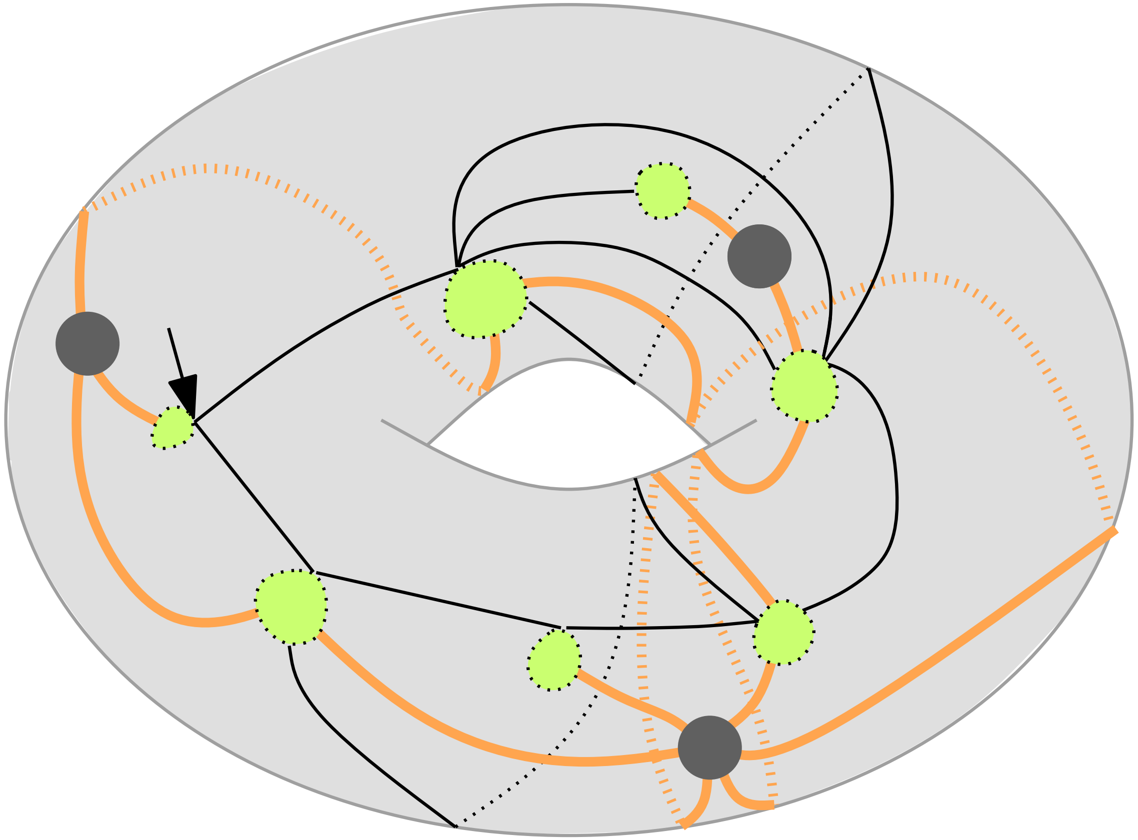

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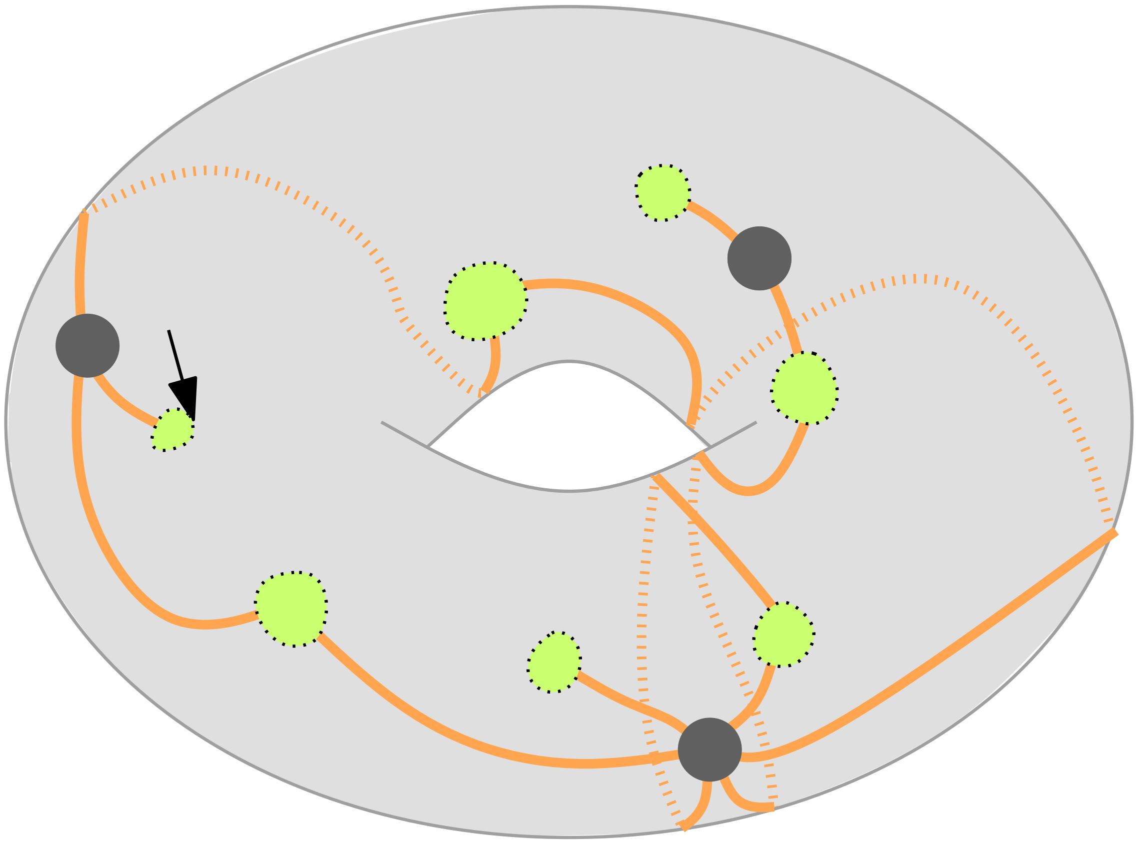

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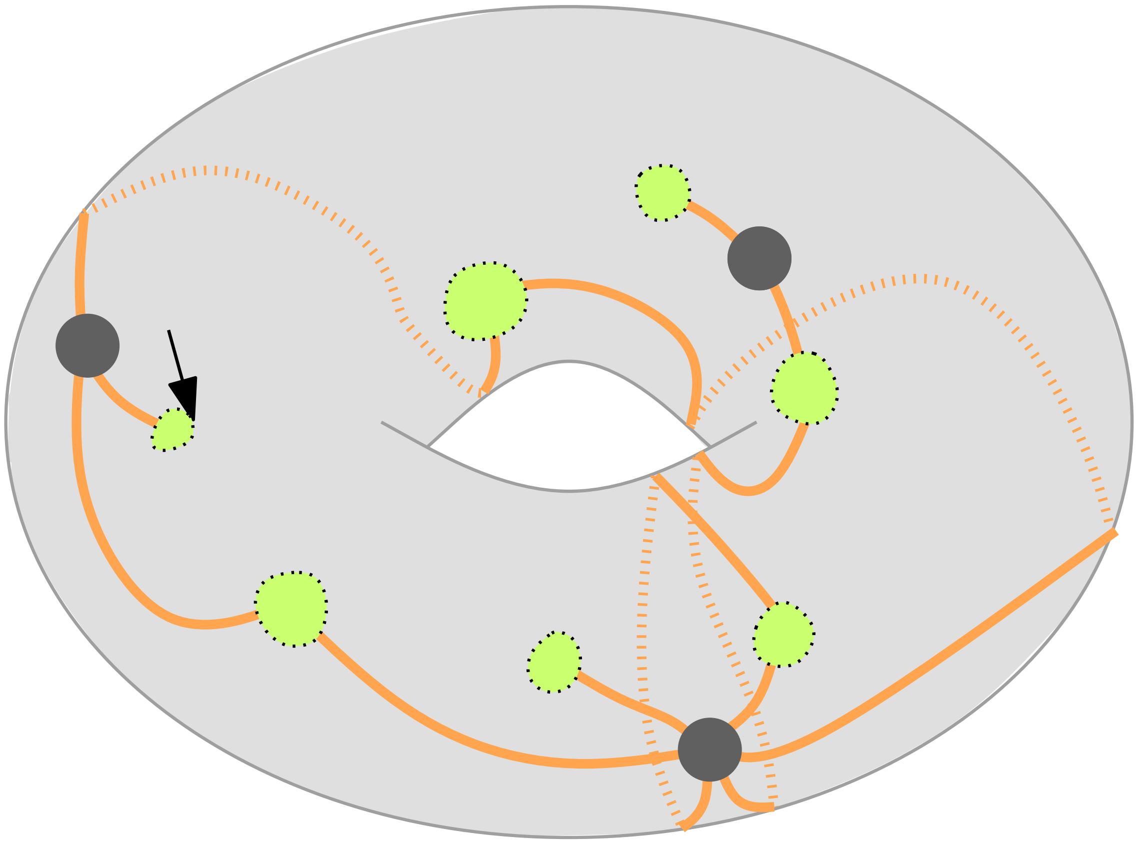

To reconstruct the original map, just glue the tree along the border of the skeleton.

The construction is bijective.

Hence we have indeed:

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Concluding remarks:

One has: $\dfrac{\#\{\text{tree-rooted maps}\}}{\#\{\text{covered maps}\}} \longrightarrow \dfrac{1}{2^g}$, but we do not see it on the bijection.

More generally, is it possible to enumerate tree-rooted maps in a bijective way ?