A bijection for covered maps on orientable surfaces.

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Map of genus $g$

$= \text{drawing of a graph on the } g\text{-torus, such that the faces are simply connected}.$

Examples:

- not a map
- map on the torus
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A unicellular map is a map which has only one face.

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In genus 0, unicellular maps are exactly plane trees, but in positive genus, things are more complicated.
Covered maps.

A covered map is a map with a distinguished spanning unicellular submap.
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A covered map is a map with a distinguished spanning unicellular submap.

We do not impose that the spanning submap has genus $g$ (it can have genus $0, 1, \ldots, g$).

Special case: map with a spanning tree = tree-rooted map.
Tree-rooted maps were previously studied.

In the planar case, a very nice formula from [Mullin 67]:

\[(\text{nb. of tree-rooted maps w. } n \text{ edges}) = \text{Cat}_n \times \text{Cat}_{n+1}\]
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Bijective proofs: [Cori, Dulucq, Viennot 82], [Bernardi 06].
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In higher genus:

[Lehman, Walsh 72]: nice formula for genus 1.
more complicated formulas for \( g \geq 2 \).

[Bender, Robert, Robinson 88]: asymptotics.
Theorem: There is a bijection:

\[ \{ \text{covered maps of genus } g \text{ with } n \text{ edges} \} \leftrightarrow \{ \text{unicellular bipartite maps of genus } g, n + 1 \text{ edges} \} \times \{ \text{plane trees, } n \text{ edges} \} \]
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\times \{\text{plane trees, } n \text{ edges}\}
\]

**Corollary:**
For each \( g \), there is a closed formula for the number of covered maps.

\[
C_0(n) = \text{Cat}_n \times \text{Cat}_{n+1}
\]
\[
C_1(n) = \text{Cat}_n \times \frac{(2n - 2)!}{12(n - 1)!(n - 3)!}
\]
\[
C_2(n) = \text{Cat}_n \times \frac{(5n^2 - 7n + 6)(2n - 5)!}{720(n - 3)!(n - 5)!}
\]
Corollary: Nice formulas for tree rooted-maps.

- $g = 0$: there is a closed formula for the number of tree-rooted planar maps:

$$T_0(n) = C_0(n)$$
**Corollary:** Nice formulas for tree rooted-maps.

- $g = 0$: there is a **closed formula** for the number of tree-rooted **planar** maps:
  \[ T_0(n) = C_0(n) \]

- $g = 1$: there is a **closed formula** for the number of tree-rooted **toroidal** maps:
  \[ T_1(n) = \frac{1}{2} C_1(n) \] by a duality argument.
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- No similar argument for $g \geq 2$: explains (?) why formulas for tree-rooted maps seem to be more complicated.
The bijection.

**step 1**: from *covered maps* to *orientations*. 
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We make the **tour of the submap**, and orient:
- red edges as we followed them for the first time
- black edges s.t. we see their head before their tail
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**step 1**: from **covered maps** to **orientations**.

We obtain a **left-orientation**: each edge $e$ can be reached from the root by a **left-path**. The construction is **bijective**.
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**step 2:** from left-orientations to pairs (tree, unicellular bipartite map).
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**Unfolding a vertex:**
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Unfolding a vertex:
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Unfolding a vertex:
Fact 1: we obtain a tree.
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Fact 2: the gluing skeleton is a **unicellular bipartite map**.
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To reconstruct the original map, just glue the tree along the border of the skeleton. The construction is **bijective**.
Hence we have indeed:

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via \textit{left-orientations}.
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**Concluding remarks:**

One has:

\[ \frac{\# \{ \text{tree-rooted maps} \}}{\# \{ \text{covered maps} \}} \rightarrow \frac{1}{2^g} \], but we do not see it on the bijection.

More generally, is it possible to enumerate tree-rooted maps in a bijective way?