Colouring graphs excluding fixed subgraphs

joint work

with S. Thomassé, M. Bonamy
Problem

Very General Question:
What does having large chromatic number say about a graph?

First case: maybe it contains a big clique as a subgraph.

Is it the only case?

Of course not. There even exists triangle-free families of arbitrarily large $\chi$ (Mycielski, Tutte, Zykov...).

Even more: For every $k$, there exists graphs with arbitrarily large girth (size of a min cycle) and arbitrarily large $\chi$. (Erdős).
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Now our question is: what families $\mathcal{F}$ are chi-bounding?
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$\mathcal{F}$ of size 1

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Conjecture (Gyarfás–Sumner)

If $F$ is a forest, the class of graphs excluding $F$ as an induced subgraph is chi-bounded.
\[ \mathcal{F} = T \text{ tree} \]

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Scott proved the following very nice "topological" version of the conjecture:

- For every tree $T$, the class of graphs excluding all subdivisions of $T$ is chi-bounded.
Larger families $\mathcal{F}$

Erdős says that if $\mathcal{F}$ is finite, then $\mathcal{F}$ must contain a forest to be $\chi$-bounding.

What about excluding infinite families that do not contain a forest? What about excluding families of cycles?

- excluding all cycles: trees
- excluding all cycles of length at least 4: chordal graphs are perfect
- excluding all cycles of length at least $k$: Open conjecture of Gyarfas.
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Families of cycles

Gyárfás made in fact three conjectures about cycles.

Conjecture (Gyárfás, ’87)

1. The set of all cycles of length at least $k$ is chi-bounding
2. The set of odd cycles is chi-bounding.
3. The set of all odd cycles of length at least $k$ is chi-bounding

The second conjecture was proven very recently by Seymour and Scott.

Graphs that do not contain any odd hole nor any complement of odd hole: Berge graphs.

Strong Perfect Graph Theorem: $\chi = \omega$.

No simple proof of any (even much worse) other chi-bounding function.
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Could the following conjecture be also true?

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NO
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**NO**

Using Erdős Theorem construct a sequence $F_i$ such that

- $\chi(F_i) \geq i$
- $\text{girth}(F_i) > |2^{F_i-1}|$

Let $\mathcal{F}$ be the set of cycles that do not occur in any $F_i$. Then $\mathcal{F}$ is NOT chi-bounding and is infinite (it contains at least all the $|F_i|$). Even more it has upper density 1 since it contains every interval $[|F_i|, 2|F_i|]$. 
Conjecture (Scott-Seymour, 2014)

If $l \subset \mathbb{N}$ has bounded gaps (\(\exists k \text{ s.t. every } k \text{ consecutive integers contains an element of } F\)), then \(\{C_i, i \in l\}\) is $k$-bounding.
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Theorem (Bonamy, C., Thomassé)

*Every graph with sufficiently large chromatic number must contain a cycle of length 0 mod 3.*
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  Chudnovsky et al recently proved that \( \chi > 4 \) implies the existence of a 3k cycle as a (not necessarily induced) subgraph.
▶ The question originally came as a sub case of a more general question of Kalai and Meschulam.
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- Trinity changing paths: try to find vertices $x$ and $y$ such that many independent paths exist between the two.
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- If this other is present prove it.
\[ \{ C_{3k}, k > 1 \} \text{ is chi-bounding} \]
\[ \{ C_{4k} \} \text{ is chi-bounding} \]