Semidefinite method and Caccetta-Häggvist conjecture

Jan Volec

ETH Zürich

joint work with Jean-Sébastien Sereni and Rémi De Joannis De Verclos

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Conjecture (Caccetta-Häggvist, 1978)

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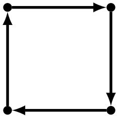

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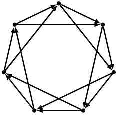

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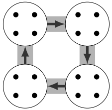

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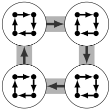

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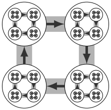

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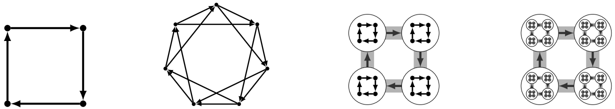

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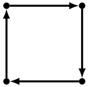
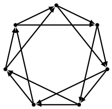
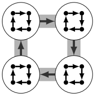
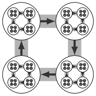

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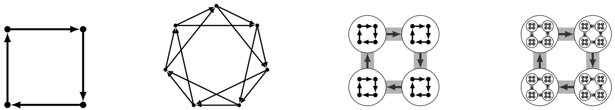

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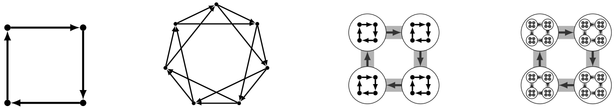

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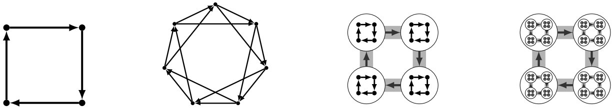

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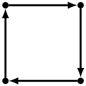
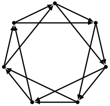
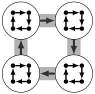
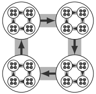

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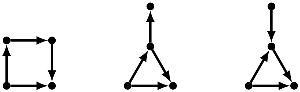

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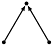
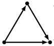
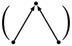
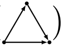
- we optimize on $\text{LIM}^{\text{EXT}} = \{q \in \text{LIM} : q \text{ is extremal for C-H}\}$

Flag Algebras – basic properties of q

- linear extension of q:

$q\big(\alpha_1 \times$  $+ \alpha_2 \times$  $\big) := \alpha_1 \cdot q\big($  $\big) + \alpha_2 \cdot q\big($  $\big)$

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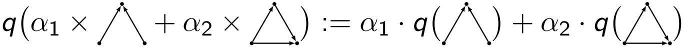

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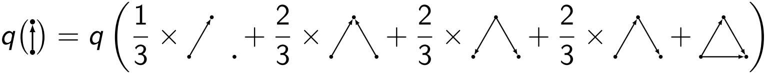

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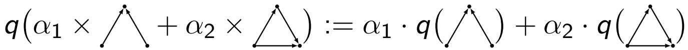

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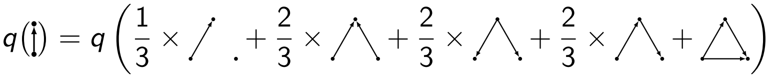

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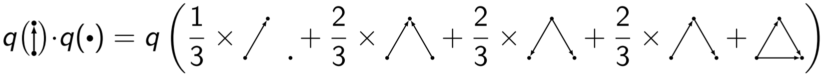

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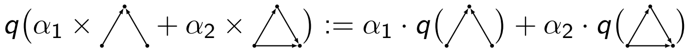

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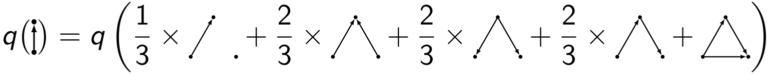

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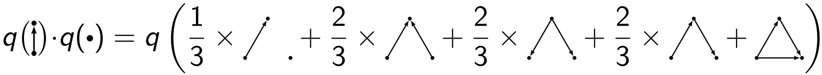

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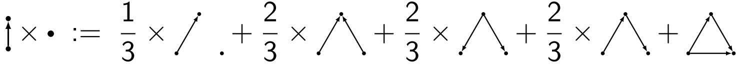

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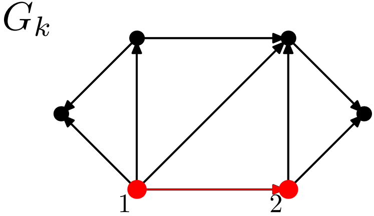
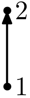
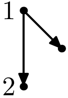


$\sigma =$ 

$F^\sigma =$ 

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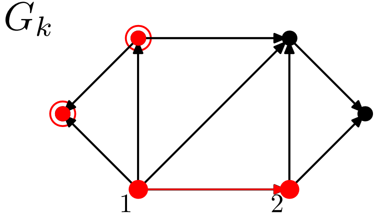

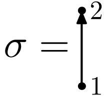

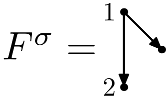

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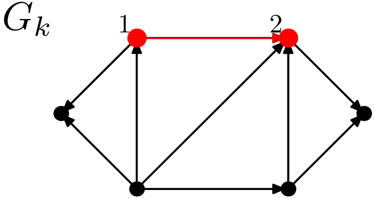
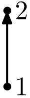
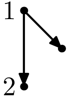


$\sigma =$ 

$F^\sigma =$ 

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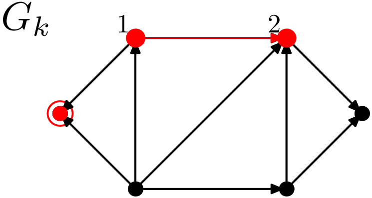
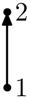
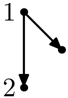


$\sigma =$ 

$F^\sigma =$ 

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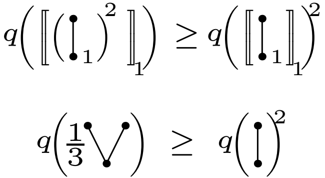

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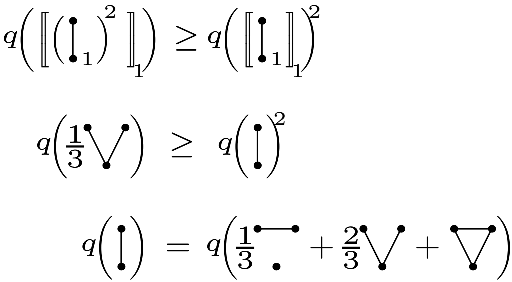

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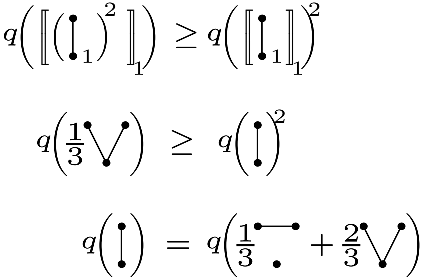

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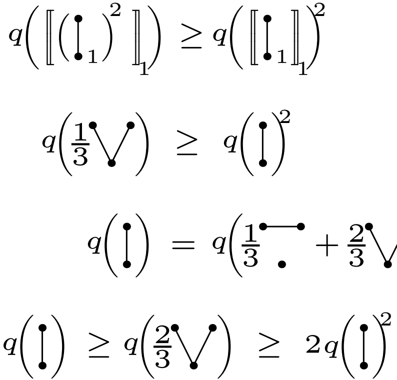

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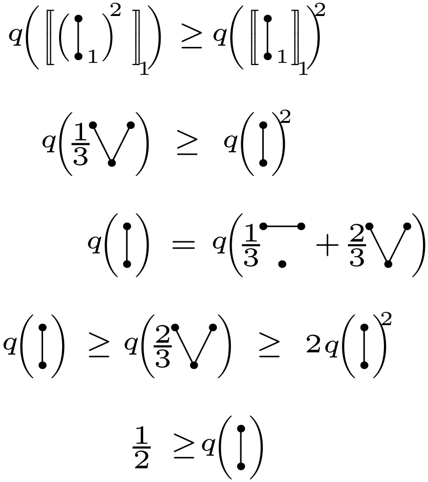

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Theorem

Every n-vertex oriented graph with minimum out-degree at least $0.3386n$ contains an oriented triangle.

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Thank you for your attention!