Infinite locally random graphs

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Abstract

Motivated by copying models of the web graph, Bonato and Janssen [3] introduced the following simple construction: given a graph G, for each vertex x and each subset X of its closed neighbourhood, add a new vertex y whose neighbours are exactly X. Iterating this construction yields a limit graph $\uparrow G$. Bonato and Janssen claimed that the limit graph is independent of G, and it is known as the *infinite locally random graph*. We show that this picture is incorrect: there are in fact infinitely many isomorphism classes of limit graph, and we give a classification. We also consider the inexhaustibility of these graphs.

1 Introduction

The *Rado graph* $\mathcal{R}$ is the unique graph with countably infinite vertex set such that, for any disjoint pair X, Y of finite subsets of vertices, there is a vertex z that is joined to every vertex in X and no vertex in Y. If $0 < p < 1$, and G is a random graph in $\mathcal{G}(\mathbb{N}, p)$, then with probability 1 we have $G \cong \mathcal{R}$. For this reason, the Rado graph is also known as *the infinite random graph* (see [5] for a survey).

The Rado graph can be obtained deterministically by beginning with any finite (or countably infinite) graph G and iterating the following construction:

[E1] For every finite subset X of $V(G)$ add a vertex y with neighbourhood $N(y) = X$.

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Here $N(x) = \{y \in V(G) : xy \in E(G)\}$ is the *neighbourhood* of x; we also write $N[x] = N(x) \cup \{x\}$ for the *closed neighbourhood* of x.

Motivated by copying models of the web graph, Bonato and Janssen [3] (see also [1] and [4]) introduced the following interesting construction. For a finite graph G, the *pure extension* $PE(G)$ of G is obtained from G by the following construction:

[E2] For every $x \in V(G)$ and every finite $X \subseteq N[x]$ add a vertex y with neighbourhood $N(y) = X$.

Iterating this construction, we obtain a limit graph, denoted by $\uparrow G$.

Bonato and Janssen ([3], Theorem 3.3) claimed that $\uparrow G \cong \uparrow H$ for every pair G, H of finite graphs. The (claimed) unique limit graph, which has become known [1] as the *infinite locally random graph* (see Proposition 1 below for the reason for this name). As we show below, Bonato and Janssen's claim is incorrect. There are in fact infinitely many limit graphs G (for instance, $\uparrow C_5, \uparrow C_6, \uparrow C_7, \ldots$ are all distinct), and we give a simple criterion that determines when $\uparrow G \cong \uparrow H$.

In the next section, we give a few simple properties of limit graphs $\uparrow G$; we prove our classification result in section 3. Finally, in section 4, we prove that for every finite G, $\uparrow G$ is inexhaustible, that is $(\uparrow G) \setminus x \cong \uparrow G$ for all $x \in V(\uparrow G)$. This corrects another result from [3].

2 Simple properties of $\uparrow G$

We begin with some notation. We shall refer to the vertices y that are introduced in [E2] above with neighbourhoods contained in $N[x]$ as *clones* of x. Thus a vertex of degree d in G has 2^{d+1} clones in $PE(G)$ (note that we take all subsets of the *closed* neighbourhood $N[x]$), and $PE(G)$ contains $|G|$ isolated vertices, each one a clone of a different vertex from G. As indicated above, iterating construction [E2] gives a sequence of graphs $G \subseteq PE(G) \subseteq PE^2(G) \subseteq \cdots$, where $PE^n(G) = PE(PE^{n-1}(G))$; we write $\uparrow G$ for the limit of this sequence. We define the *level* $L(x)$ of a vertex of $\uparrow G$ to be the least integer k such that it is contained in $PE^k(G)$ (where $L(x) = 0$ for all $x \in V(G)$), and for a finite subset $X \subseteq V(\uparrow G)$, we write $L(X) = \max_{x \in X} L(x)$. We also write $L^{(k)}(\uparrow G)$ for the vertices of level k in $\uparrow G$, and $L^{(\leq k)}(\uparrow G)$ for the vertices of level k or less. Note that, by the construction, $L^{(k)}(\uparrow G)$ is an independent set for every $k \geq 1$.

Given a graph H, a graph G is *locally* H if, for every vertex x of G, the graph induced by the neighbourhood $N(x)$ of x is isomorphic to H.

Bonato and Janssen note the following property of the construction defined above.

Proposition 1. *[3] For every finite graph G, $\uparrow G$ is locally $\mathcal{R}$*

Proof. For every $x \in V(\uparrow G)$, and every X and Y finite disjoint subsets of $N(x)$, we want to find a vertex z such that z is adjacent to every vertex in X and to none in Y. This is possible by the definition of $\uparrow G$ by taking a suitable vertex z of level $L(X \cup Y) + 1$. □

Since $\mathcal{R}$ is the (unique) infinite random graph, it therefore makes sense to refer to $\uparrow G$ as an *infinite locally random graph.*

Corollary 2. *Let G be a finite graph. Then $\uparrow G$ is $\aleph_0$-universal (that is, $\uparrow G$ contains every countable graph H as an induced subgraph).*

Another easy but important remark concerns the distance between vertices.

Proposition 3. *Let G be a finite graph and x and y two vertices of $PE^k(G)$, for some integer $k \geq 0$. Then the distance between x and y is the same in $PE^k(G)$ and in $\uparrow G$.*

Proof. It is sufficient to note that the pure extension construction [E2] does not change the distance between vertices. □

We also note the following simple property.

Lemma 4. *Let G be a finite graph and x a vertex of $\uparrow G$. Let X be a finite subset of $N(x)$. Then there exists a vertex y with $L(y) \leq L(X)$ such that $X \subseteq N[y]$.*

Proof. Let x_0 be a vertex of minimal level with $X \subseteq N[x_0]$. If $L(x_0) \leq L(X)$ then we can take $y = x_0$. Otherwise, $L(x_0) > L(X)$ and so $x_0 \notin X$. But x_0 was constructed on level $L(x_0)$ as the clone of some vertex x_1 with $L(x_1) < L(x_0)$. In particular, $N(x_0) \cap L^{(<L(x_0))}(\uparrow G) \subseteq N[x_1]$ and so $X \subseteq N[x_1]$, which contradicts the minimality of $L(x_0)$. □

For $x \in V(\uparrow G)$, we write

$$N^{-}(x) = N(x) \cap L^{(<L(x))}(\uparrow G).$$

Note that $N^{-}(x)$ is the set of neighbours assigned to x at time $L(x)$, when x is first introduced. We say that a subgraph G_1 of G is *good* if it is an induced subgraph of G and, for all x in $V(G_1)$, $N^{-}(x) \subseteq V(G_1)$. Equivalently, G_1 is an induced subgraph such that $N(y) \cap V(G_1) \subseteq N^{-}(y)$ for all $y \in V(G) \setminus V(G_1)$.

In this context, Lemma 4 gives the following result.

Lemma 5. *Let G be a finite graph and suppose that H is a good subgraph of $\uparrow G$. Then*

$$\forall x \in V(\uparrow G),\ \exists y \in V(H) \text{ such that } N(x) \cap V(H) \subseteq N[y] \cap V(H)$$

Proof. We can assume that $x \notin V(H)$. Let $X = N(x) \cap V(H)$. Then $X \subseteq N^{-}(x)$, and by Lemma 4 there exists y of level at most $L(X)$ with $X \subseteq N[y]$. If $L(y) = L(X)$ then, since the levels are independent sets and $X \subseteq N[y]$, y must belong to X, and thus to H. If $L(y) < L(X)$, then y belongs to H as H is a good subgraph of $\uparrow G$. □

3 Classification

We now investigate when $\uparrow G$ and $\uparrow H$ are isomorphic. In [3], the authors claim that $\uparrow G \cong \uparrow H$ for any pair of finite graphs G and H (this is their Theorem 3.3). Here we disprove this. Their proof seems to fail on page 209 at the end of the first paragraph: the equality $H_{n+1} - S \cong G_1 \uplus \overline{K_m}$ does not hold because these vertices can be linked by edges. Moreover, it is not clear why this equality would imply $H - S \cong \uparrow (G_1 \uplus \overline{K_m})$ on the following line, as some vertices in H can be constructed by cloning elements in S.

We begin with the following useful consequence of Lemma 5.

Theorem 6. *Let G and H be finite graphs. Suppose that $G_1 \supseteq G$ is a good subgraph of $\uparrow G$ and $H_1 \supseteq H$ is a good subgraph of $\uparrow H$. If $G_1 \cong H_1$ then $\uparrow G \cong \uparrow H$*

Proof. Let $\phi : V(G_1) \to V(H_1)$ be an isomorphism (note that, as G_1 and H_1 are good, they are induced subgraphs of $\uparrow G$ and $\uparrow H$, respectively, so this is an isomorphism between induced subgraphs). Using a classical 'back and forth' argument, we extend ϕ one vertex at a time until, in the limit, we obtain an isomorphism between $\uparrow G$ and $\uparrow H$. Let $x \in V(\uparrow G)$ be a vertex of minimal level with $x \notin V(G_1)$. By Lemma 5, there exists $y \in V(G_1)$ such that $N(x) \cap V(G_1) \subseteq N[y] \cap V(G_1)$. Let $z \notin V(H_1)$ be a clone of $\phi(y)$ with

$$N^-(z) = N(z) \cap V(H_1) = \phi(N(x) \cap V(G_1)).$$

Such a clone is easily found: let $k = L(V(H_1))$, and take the clone of $\phi(y)$ on level $k+1$ with exactly this neighbourhood in $L^{(\leq k)}(\uparrow H)$. Then $V(H_1) \cup \{z\}$ induces a good subgraph of $\uparrow H$ and, by minimality of x, $V(G_1) \cup \{x\}$ induces a good subgraph of $\uparrow G$. We can therefore extend ϕ by setting $\phi(x) = z$. Repeating the construction in alternate directions we clearly obtain an isomorphism between $\uparrow G$ and $\uparrow H$. □

We shall say that a vertex x of a graph G is *inessential* if there exists $y \in V(G)$, $y \neq x$ such that $N(x) \subseteq N[y]$. A graph is *essential* if it contains no inessential vertices. Given a graph G, a sequence of vertices $x_1, \ldots, x_k$ is a *maximal sequence of removals* if x_i is inessential in $G \setminus \{x_1, \ldots, x_{i-1}\}$ for each i, and $G \setminus \{x_1, \ldots, x_k\}$ is an essential graph.

We shall show below that every maximal sequence of removals yields the same essential graph (up to isomorphism). However, we first prove a simple lemma. We say that two vertices x and y in a graph G are *equivalent* if $N(x) = N(y)$ or $N[x] = N[y]$. Equivalently, $N(x) \subseteq N[y]$ and $N(y) \subseteq N[x]$. Clearly, if x and y are equivalent in G then $G \setminus x \cong G \setminus y$, with the obvious isomorphism given by exchanging x for y and leaving the other vertices fixed.

Equivalent vertices play an important role in the removal of inessential vertices.

Lemma 7. *Suppose that x and y are inessential in a graph G, but x is not inessential in $G \setminus y$. Then x and y are equivalent.*

Proof. Note first that since x and y are inessential in G, there are x' and y' such that $N(x) \subseteq N[x']$ and $N(y) \subseteq N[y']$. If $x' \neq y$ then considering the vertex x' in $G \setminus y$ shows that x is inessential in $G \setminus y$, a contradiction. So $x' = y$, and $N(x) \subseteq N[y]$.

Now consider y'. If $y' \neq x$ then $N(x) \subseteq N[y] = \{y\} \cup N(y) \subseteq \{y\} \cup N[y']$ implies that $N(x) \setminus y \subseteq N[y']$, and so y' shows that x is inessential in $G \setminus y$, a contradiction. Thus we have $y' = x$, and so $N(y) \subseteq N[x]$. It follows that x and y are equivalent. □

We now show that maximal sequences of removals define a unique graph up to isomorphism.

Theorem 8. *Suppose that* $x_1, \ldots, x_k$ *and* $y_1, \ldots, y_l$ *are two maximal sequences of removals in a finite graph* G. *Then* $G \setminus \{x_1, \ldots, x_k\} \cong G \setminus \{y_1, \ldots, y_l\}$.

Proof. We claim that we can modify the sequence $\{y_1, \ldots, y_l\}$ to obtain the sequence $\{x_1, \ldots, x_k\}$ without changing the isomorphism type of the resulting essential graph $G \setminus \{y_1, \ldots, y_l\}$.

Suppose first that $x_1 \notin \{y_1, \ldots, y_l\}$. Then (by maximality) x_1 is inessential in G but not in $G \setminus \{y_1, \ldots, y_l\}$. Let i be maximal such that x_1 is inessential in $G \setminus \{y_1, \ldots, y_i\}$. Then, by Lemma 7, x_1 and y_{i+1} are equivalent in $G \setminus \{y_1, \ldots, y_i\}$, and so we can replace y_{i+1} by x_1 in the sequence $y_1, \ldots, y_l$, without effecting the isomorphism type of $G \setminus \{y_1, \ldots, y_l\}$ (the isomorphism is given by exchanging x_1 and y_{i+1}). We may therefore assume that $x_1 \in \{y_1, \ldots, y_l\}$.

We now show that we can modify $y_1, \ldots, y_l$ so that $y_1 = x_1$. Suppose that $x_1 = y_{i+1}$ for some $i \geq 1$. If there exists some $0 \leq j < i-1$ such that x_1 is inessential in $G \setminus \{y_1, ..., y_j\}$ and not in $G \setminus \{y_1, ..., y_{j+1}\}$, Lemma 7 implies that x_1 and y_{j+1} are equivalent in $G \setminus \{y_1, ..., y_j\}$. Therefore we can exchange them in the sequence. We can repeat this operation as long as such an integer j exists, and thus we can assume that $x_1 = y_{i+1}$ is inessential in $G \setminus \{y_1, \ldots, y_j\}$ for all $j \leq i$. Now, if y_i is not inessential in $G \setminus \{y_1, \ldots, y_{i-1}, x_1\}$ then (as it is inessential in $G \setminus \{y_1, \ldots, y_{i-1}\}$) Lemma 7 shows that x_1 and y_i are equivalent in $G \setminus \{y_1, \ldots, y_{i-1}\}$. It is clear that we may therefore exchange y_i and $y_{i+1} = x_1$ in the sequence $y_1, \ldots, y_l$. Repeating this argument, we move x_1 forward in the sequence $y_1, \ldots, y_l$ until $x_1 = y_1$.

Finally, if $x_1 = y_1$, we can work instead with the graph $G \setminus x_1$ and the sequences $x_2, \ldots, x_k$ and $y_2, \ldots, y_l$, continuing until one (and hence both) of the sequences is exhausted. □

We shall denote the (isomorphism type of the) subgraph of G obtained by deleting a maximal sequence of removals $\downarrow G$. For instance, $\downarrow K_n = \downarrow C_4 = K_1$, but $\downarrow C_k = C_k$ for all $k \geq 5$.

We next show that inessential vertices have no effect on limit graphs.

Corollary 9. *Let* G *be a finite graph and* x *an inessential vertex of* G. *Then* $\uparrow G \cong \uparrow (G \setminus x)$

Proof. Let $H = G \setminus x$. Since x is inessential, there exists y in G such that $N(x) \subseteq N[y]$ in G. In $\uparrow H$, y has a clone x' such that $N^-(x') = N(x) \cap V(G)$. Clearly $G_1 = G$ is a good subgraph of $\uparrow G$ and $V(H) \cup \{x'\}$ induces a good subgraph H_1 of $\uparrow H$. Thus it suffices to apply Theorem 6 to G_1 and H_1. $\square$

Corollary 9 implies the following theorem.

Theorem 10. *Let G be a finite graph. Then $\uparrow G \cong \uparrow(\downarrow G)$*

If H is an induced subgraph of $\uparrow G$, then we define two kinds of transformations on this subgraph, called *reductions*.

(i) Delete an inessential vertex of H.

(ii) For a pair of vertices $x \in V(H)$ and $y \notin V(H)$ with $N(x) \cap V(H) \subseteq N(y) \cap V(H)$, replace H by the subgraph of $\uparrow G$ induced by $(V(H) \setminus x) \cup \{y\}$.

Lemma 11. *If H is a finite induced subgraph of $\uparrow G$, it is possible to apply a sequence of reductions to transform H into a subgraph of G.*

Proof. Define the *weight* $w(H')$ of an induced subgraph of $\uparrow G$ by

$$w(H') = \sum_{v \in V(H')} L(v).$$

If $w(H) = 0$ then H is a subgraph of G. If $w(H) > 0$, then we look for a reduction that decreases the weight or the number of vertices. If H contains an inessential vertex, then delete it (this can occur at most $|H| - 1$ times). Otherwise, let $x \in V(H)$ be a vertex of highest level. Then $N(x) \cap V(H) = N(x) \cap V(H) \cap L^{(<L(x))}(\uparrow G)$, as $L^{(L(x))}(\uparrow G)$ is an independent set. Since x was built at level $L(x)$ as the clone of some vertex y that satisfies $N(x) \cap V(H) \cap L^{(<L(x))}(\uparrow G) \subseteq N[y] \cap V(H)$ and $L(y) < L(x)$, we can replace x by y, to obtain H' with $w(H') < w(H)$. Repeating this process, we eventually obtain an induced subgraph of $\uparrow G$ with weight 0 which, as already noted, is a subgraph of G. $\square$

We are now ready to prove our main result.

Theorem 12. *Let G and H be finite graphs. Then $\uparrow G \cong \uparrow H \iff \downarrow G \cong \downarrow H$*

Proof. By Theorem 10, we may assume that G and H do not contain any inessential vertices, that is $\downarrow G = G$ and $\downarrow H = H$. Suppose that $\uparrow G \cong \uparrow H$, and fix an isomorphism.

Let $\{1, 2, \ldots, n\}$ be the vertices of G. We partition the vertices of $\uparrow G$ into n classes in the following way. For $i = 1, \ldots, n$, let $A_{i,0} = \{i\}$, and for $j \geq 1$, let $A_{i,j}$ be the vertices of $\uparrow G$ which are clones of vertices in $A_{i,j-1}$. We then define $A_i = \bigcup_{j=0}^{\infty} A_{i,j}$. Thus A_i is the smallest set of vertices containing i and closed under taking clones. It is easy to see that, for $i \neq k$, there is an edge between class A_i and A_k if and only if there is an edge between i and k (as creating a clone cannot create adjacencies between a new pair of classes). We shall say that *edges between classes respect G*.

Now consider an isomorphic embedding ϕ of G into $\uparrow G$. We say that ϕ is *good* if $\phi(i) \in A_i$ for every $i \in V(G)$. Suppose that ϕ is good and let G' be the image of G under ϕ. If we apply a type (ii) reduction to some vertex of G', say $v_i := \phi(i)$, then it is replaced by a vertex x such that $N(x) \cap V(G') \supseteq N(v_i) \cap V(G')$. Let A_j be the class containing x. Since ϕ is good, there is an edge between A_j and A_k whenever $k \in N(i)$. Since edges between classes respect G, this implies $N[j] \supseteq N(i)$. But since we assumed that G contains no inessential vertices, this is possible only if $i = j$. Indeed, $N(x) \cap V(G') = N(v_i) \cap V(G')$, or else we would introduce edges between new pairs of classes. It follows that we obtain a good embedding ϕ' of G by setting $\phi'(i) = x$ and $\phi'(j) = \phi(j)$ otherwise. This remains true for any sequence of reductions starting from a good embedding. In particular, any sequence of reductions starting from G produces an induced copy of G (note that reductions of type (i) are not possible at any stage).

By Lemma 11, any induced subgraph of $\uparrow H$ isomorphic to G can be reduced to a subgraph of H. It follows that G must be isomorphic to a subgraph of H. Arguing similarly the other way round, we see that H is isomorphic to a subgraph of G, and so $G \cong H$. $\square$

Now it is clear that $\uparrow G$ is not independent of G: it suffices to consider two circuits of different length (larger than 4). In fact, Theorem 12 immediately gives the following classification of possible limit graphs.

Corollary 13. *The isomorphism classes of limit graphs $\uparrow G$ of finite graphs G are in bijective correspondence with the class of essential finite graphs.*

4 Inexhaustibility

A graph G is *inexhaustible* if $G \setminus x \cong G$ for every vertex $x \in V(G)$. For instance, the infinite complete graph K_ω and its complement are trivially inexhaustible; the Rado graph $\mathcal{R}$ is also inexhaustible. On the other hand, the infinite two-way path is not inexhaustible, as deleting any vertex increases the number of components. For results on inexhaustible graphs, see Pouzet [7], El-Zahar and Sauer [6] and Bonato and Delić [2].

Bonato and Janssen [3] consider the inexhaustibility of infinite graphs satisfying various properties, and claim a rather general result. Let us define define two properties of (infinite) graphs as follows. We say that a graph G has *Property A* if it satisfies the following condition.

(A) For every vertex x of G, every finite $X \subseteq N[x]$, and every finite $Y \subseteq V(G) \setminus X$, there is a vertex $z \notin X \cup Y$ such that $X \subseteq N(z)$ and $Y \cap N(z) = \emptyset$,

and G has *Property B* if it satisfies the following.

(B) For every vertex x of G, every finite $X \subseteq N(x)$, and every finite $Y \subseteq V(G) \setminus X$, there is a vertex $z \notin X \cup Y$ such that $X \subseteq N(z)$ and $Y \cap N(z) = \emptyset$.

Note that the only difference between (A) and (B) is that (A) is concerned with closed neighbourhoods, while (B) is only concerned with neighbourhoods. Clearly Property A implies Property B; furthermore, for any finite G, it is clear from the constructive step [E2] that $\uparrow G$ has Property A (and therefore Property B).

Bonato and Janssen ([3], Theorem 4.1) claim that every graph with Property B is inexhaustible. However, there is a simple counterexample to this assertion: let G be the Rado graph $\mathcal{R}$ with an additional isolated vertex x. Since the Rado graph is connected, and G is not, it is clear that $G \setminus x \not\cong G$. (The proof of Bonato and Janssen in [3] appears to fail with the definition of their sets S_i.)

In fact, even the stronger Property A does not imply that a graph is inexhaustible. Consider the graph G defined by starting from the path $x_1x_2x_3x_4$ of length 3, and alternating the pure extension construction [E2] with the following step.

[E3] For every pair of vertices $\{x, y\} \neq \{x_1, x_4\}$, add a vertex z with $N(z) = \{x, y\}$.

Note that x_1 and x_4 are at distance 3 in the initial graph. The pure extension step [E2] does not change the distance between vertices, while [E3] does not create a path of length 2 from x_1 to x_4. Thus x_1 and x_4 are at distance 3 in the limit graph. On the other hand, there are infinitely many paths of length 2 between any other pair of vertices. Thus $G \setminus \{x_1, x_4\} \not\cong G$, and so G cannot be inexhaustible (if G is inexhaustible, then clearly $G \setminus X \cong G$ for every finite $X \subseteq V(G)$).

On the positive side, we can show that for any finite G, the limit graph $\uparrow G$ is actually inexhaustible.

Theorem 14. *For every finite graph G, $\uparrow G$ is inexhaustible.*

Proof. Let v be any vertex of $\uparrow G$. We shall show that $\uparrow G \cong (\uparrow G) \setminus v$. Note that since $\uparrow G \cong \uparrow PE^{L(v)}(G)$, we can replace G by $PE^{L(v)}(G)$, and so we may assume that $v \in V(G)$.

On the first level above G, v has a clone v' with $N(v) \cap G = N(v') \cap G$. Thus we have an isomorphism between $G_1 = G$ and $G_2 = G \setminus v \cup \{v'\}$. It is clear that G_1 and $G_2 \cup \{v\}$ are good subgraphs. We will extend this isomorphism by a 'back and forth' argument.

Suppose we are given a partial isomorphism ϕ between two subgraphs G_1 and G_2 of $\uparrow G$, with the following properties:

1. G_1 and $G_2 \cup \{v\}$ are good subgraphs of $\uparrow G$
2. $V(G) \subseteq V(G_1)$, $V(G) \setminus v \subseteq V(G_2)$ and $v \notin V(G_2)$
3. There is a vertex $\tilde{v} \in V(G_2)$ such that $N(v) \cap V(G_2) \subseteq N(\tilde{v}) \cap V(G_2)$

The vertex $\tilde{v}$ (in the third property) will change at each step of our construction. We begin by setting $\tilde{v} = v'$, and note that our initial G_1 and G_2 satisfy the conditions above.

Let $x \in V(\uparrow G)$ be a vertex of minimal level with $x \notin V(G_1)$. This property implies that $N^-(x) \subseteq V(G_1)$ and so $G_1 \cup \{x\}$ is still a good graph. By Lemma 5, there exists $y \in V(G_1)$ such that $N(x) \cap V(G_1) \subseteq N[y] \cap V(G_1)$ and we can define $\phi(x)$ by taking a clone of $\phi(y)$ of level greater than $L(V(G_1) \cup V(G_2))$ such that

$$N^-(\phi(x)) = N(\phi(x)) \cap V(G_2) = \phi(N(x) \cap V(G_1)).$$

This extends the isomorphism, implies that $G_2 \cup \{\phi(x), v\}$ is still a good graph and that the vertex $\tilde{v}$ still satisfies the desired property.

We now go in the opposite direction. Let z be a vertex of minimal level with $z \notin V(G_2) \cup \{v\}$: we attempt to define $\phi^{-1}(z)$.

We distinguish two cases:

- $zv \notin E(\uparrow G)$, or $zv \in E(\uparrow G)$ and $z\tilde{v} \in E(\uparrow G)$.

  As before, we can apply Lemma 5 to get $y \in V(G_2) \cup \{v\}$ such that $N(z) \cap V(G_2) \subseteq N[y] \cap V(G_2)$. If $y = v$, we can instead choose $y = \tilde{v}$. We can then define $\phi^{-1}(z)$ as previously to be a suitable clone of $\phi^{-1}(y)$.

- $zv \in E(\uparrow G)$ and $z\tilde{v} \notin E(\uparrow G)$.

  In this case we will have to change $\tilde{v}$, because we want the condition $N(v) \cap V(G_2) \subseteq N(\tilde{v}) \cap V(G_2)$ to hold after adding z to G_2. Let w be a clone of v such that $L(w) > L(V(G_1) \cup V(G_2))$ and $N^-(w) = (N(v) \cap V(G_2)) \cup \{z\}$. Such a vertex exists, since z is a neighbour of v. The only reason why the subgraph induced by $V(G_2) \cup \{v, w\}$ might not be a good graph is the edge zw. We therefore extend the isomorphism to $G_2 \cup \{z, w\}$. Since G_2 is a good graph, we can use Lemma 5 as before to first extend the isomorphism to z. Since, by minimality of z, the subgraph induced by $V(G_2) \cup \{z, v\}$ is also a good graph, we can use Lemma 5 again to extend the isomorphism to w. Finally, the definition of w implies that $G_2 \cup \{z, w, v\}$ is a good graph, and we can choose the new $\tilde{v}$ to be w, as it satisfies the desired property.

Repeating the argument gives, in the limit, an isomorphism between $\uparrow G$ and $(\uparrow G) \setminus v$. $\square$

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