Graphs with large girth not embeddable in the sphere.

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Abstract

In 1972, M. Rosenfeld asked if every triangle-free graph could be embedded in the unit sphere S^d in such a way that two vertices joined by an edge have distance more than $\sqrt{3}$ (i.e. distance more than $2\pi/3$ on the sphere). In 1978, D. Larman [4] disproved this conjecture, constructing a triangle-free graph for which the minimum length of an edge could not exceed $\sqrt{8/3}$. In addition, he conjectured that the right answer would be $\sqrt{2}$, which is no better than the class of all graphs. Larman's conjecture was independently proved by M. Rosenfeld [7] and V. Rődl [6]. In this last paper it was shown that no bound better than $\sqrt{2}$ can be found for graphs with arbitrarily large odd girth. We prove in this paper that this is still true for arbitrarily large girth. We discuss then the case of triangle-free graphs with linear minimum degree.

Fix a real $\alpha > 0$ and an integer $d \geq 1$. The *Borsuk graph* $Bor(d, (1+\alpha)\pi/2)$ is the (infinite) graph defined on the d-dimensional unit sphere where two points are joined by an edge if and only if the distance on the sphere is more than $(1+\alpha)\pi/2$. A graph is *α-spherical* if it is a subgraph of $Bor(d, (1+\alpha)\pi/2)$ for some d. D. Larman in [4] asked if for every $\alpha > 0$, there exists a triangle-free graph which is not *α-spherical*. The problem was popularized by P. Erdős, and was independently proved by M. Rosenfeld [7] and V. Rődl [6]. See [5] for a survey. In [6] was also proved that for every $\alpha > 0$, there exists a graph with arbitrarily large odd-girth which is not α-spherical. We generalize this result to graphs with arbitrarily large girth. Revisiting the problem twenty years later is much easier: one reason is that the probabilistic method is now widely spread, and the other reason is that the work of Goemans and Williamson on max-cut [3] highlighted the close relationship between sphere-embedding of graphs and cuts.

Here is the key-observation:

Lemma 1 *If G is α-spherical there exists a cut of G which has at least $(1+\alpha)m/2$ edges, where m is the total number of edges.*

Proof. Embed G in some S^d in such a way that every edge has spherical length at least $(1+\alpha)\pi/2$. Observe that a random hyperplane cut an edge of G with probability $(1+\alpha)/2$. By double counting, there is some hyperplane which cuts at least $(1+\alpha)m/2$ edges. This is our cut. ■

Now the proof is almost finished, since a graph G satisfying Lemma 1 is certainly far from being random. And indeed, Erdős' random graphs with large girth and large χ, when they have enough vertices, are not α-spherical. This is the next result:

Lemma 2 *For every $\alpha > 0$ and every integer k, there exists a graph G, with girth at least k, in which every cut has less than $(1+\alpha)m/2$ edges, where m is the number of edges of G.*

Proof. We consider for this random graphs on n vertices with independently chosen edges with probability p. We first want to bound the size of a maximum cut.
Let A be a subset of vertices. Let X denotes the number of edges between A and its complement. The

worst case being when $|A| = n/2$, X is at most a binomial law $Bin(n^2/4, p)$. Its expectation is then at most $\frac{pn^2}{4}$

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) \leq \Pr\left(Bin(n^2/4, p) - \frac{pn^2}{4} \geq \alpha\frac{pn^2}{4}\right)$$

We use the following form of the Chernoff Bound,
for any $0 \leq t \leq Np$:

$$\Pr(|Bin(N,p) - Np| > t) < 2e^{-t^2/3Np}$$

Thus, with $t = \alpha\frac{pn^2}{4}$, we get

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) < 2e^{-4\alpha^2p^2n^4/48pn^2} = 2e^{-\alpha^2n^2p/12}$$

Now, the probability that there exists a cut of size more than $(1+\alpha)\frac{pn^2}{4}$ is less than $2^n 2e^{-\alpha^2n^2p/12}$, the value 2^n being a bound to the number of cuts.
Choosing $p = n^{-k/k+1}$, this probability goes to 0 as n tends to infinity.

The other part of the proof is extracted from [1]. Let Y denotes the number of cycles of length at most k.

$$E(Y) = \sum_{i=3}^{k}\binom{n}{i}\frac{(i-1)}{2}p^i \leq \frac{1}{2}\sum_{i=3}^{k}n^ip^i \leq \frac{1}{2}(k-2)n^kp^k.$$

where the last inequality holds because $np = n^{\frac{1}{k+1}} \geq 1$.
Applying Markov's inequality,

$$\Pr(Y \geq \frac{n}{2}) \leq \frac{E(Y)}{n/2} \leq (k-2)n^{-\frac{1}{k+1}}.$$

Thus for n sufficiently large, we can find a graph on n vertices with $m = n^{\frac{k+2}{k+1}}$ edges such that the number of cycles of length at most k is less than $n/2$ and the size of a maximum cut is at most $(1+\alpha)m/2$. We can delete less than $n/2$ edges to make a graph of girth $\gamma > k$ and since $n/2$ is negligeable in front of m, we still have that the size of a maximum cut is at most $(1+\alpha)m/2$. ■

An interesting case arises when considering triangle-free graphs G with minimum degree δ such that $\delta \geq c.n$ for some fixed constant $c > 0$. In this case, there exists a constant α_c, depending on c, such that every triangle-free graph G with minimum degree $\geq cn$ is α_c-spherical. To prove this, enumerate the vertices $v_1, \ldots, v_n$ of G. Fix $c := \delta/n$. For every v_i, we fix a set of neighbours N_i of v_i such that $|N_i| = \delta$. The unit vector associated with v_i is then $V_i = \frac{1}{\sqrt{c(1-c)n}}(x_1^i, \ldots, x_n^i)$ where $x_j^i = -c$ if $j \notin N_i$ and $x_j^i = 1-c$ if $j \in N_i$.

Since $N_i \cap N_j$ is empty whenever v_iv_j is an edge of G, we have $V_i.V_j = \frac{-c}{1-c}$. And thus the spherical length of an edge is exactly $Arcos(\frac{-c}{1-c})$.

One particular case of the previous construction is when $c = 1/3$, in which case we find $V_i.V_j = -1/2$. This means that if a triangle-free graph has minimum degree $\geq n/3$, its vertices can be positioned on some unit-sphere in such a way that edges have length on the sphere at least $2\pi/3$. Let us mention to conclude that the class of triangle-free graphs with minimum degree $> n/3$ has been recently entirely characterized [2] - they have chromatic number at most four. It would be interesting to try to obtain this bound from a geometric point of view. Moreover, a construction of Hajnál shows that minimum degree $> cn$, when $c < 1/3$ does not yield any bound on the chromatic number. So the last question would be: is the chromatic number of a triangle-free graph with minimum degree $n/3$ bounded?

References

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