$N°$d'ordre : Année 2005

THÈSE

présentée devant

L'UNIVERSITÉ CLAUDE BERNARD - LYON I

Ecole Doctorale de Mathématiques et Informatique Fondamentale

pour l'obtention du

DIPLÔME DE DOCTORAT (arrêté du)

présentée et soutenue publiquement le

12 Décembre 2005

Pierre CHARBIT

Circuits in Graphs and Digraphs via Embeddings

Directeur de thèse :

J.A.Bondy

<u>RAPPORTEURS :</u>

Noga Alon
Vašek Chvátal
Sylvain Gravier

<u>JURY :</u>

John Adrian Bondy
Jean Fonlupt
Jean-Luc Fouquet
Sylvain Gravier
Stéphan Thomassé

Contents

Chapter 1

Introduction

The main topic of this work is the study of combinatorial properties of graphs through different embeddings. The idea is to know how it is possible to deduce some results on classical invariants and functions of graphs by looking at geometrical properties of these embeddings. These considerations have always been central in graph theory, the relation between colouring and planarity has been for example one of the first topic studied by combinatorists. Then, people got interested into embeddings into more complicated surfaces, and a lot of open conjectures on this topic, for instance the Tutte flow conjecture or The Double Cycle Cover Conjecture, have motivated the work of numerous combinatorists during the second part of the century.

The first representation that we will be studied here is the embedding of the vertices of a digraph into the unit circle. More precisely, we will in the first chapters of this thesis give a detailed study of a notion promoted by S. Bessy and S. Thomassé : cyclic orders. They have introduced and used very efficiently this notion in their proof of a forty years old conjecture of Tibor Gallai. The conjecture was to prove that the vertices of a strongly connected digraph can be covered by at most α circuits, where α denotes the stability of the digraph. The central idea, given a cyclic ordering, is to study circuits through their index (or winding number), that is the number of times the circuit goes around the circle. Then, instead of directly proving that the vertices can be covered by less than α circuits, they prove that given a well-chosen cyclic order, the vertices can be covered by a set of circuits, whose sum of indices is less than α. In Chapter 2, we will describe how with Adrian Bondy we made a very simple proof of this result by using duality in linear optimization. We will also use such considerations in order to prove other min-max theorems concerning cyclic orders. In fact we will see that one of the interesting aspects of this notion is that a lot of classical invariants of graph, such as stability, minimum number of circuits in a vertex or arc-cover, chromatic number, minimum feedback arc set ... that are hard to calculate, fall into polynomial when we study their cyclic counterpart, because they are related by min-max linear programming results. We will also give a theorem obtained with András Sebő that characterize the equivalence between cyclic orders, and see that these notions extends to non-oriented graphs ([20]).

Chapter 3 will be devoted to the notion of cyclic length of a circuit, that is, given a cyclic order, its usual length divided by its index. The aim is to try to get results about the usual length by studying the cyclic one. For example, a result about the maximum cyclic length of a circuit in a coherent cyclic order had been proved by Bessy and Thomassé. This Theorem implies in particular a result of Bondy that asserts that any strongly connected digraph contains a circuit of length greater than it's chromatic number. Again we will give simple proofs based on linear programming of this result and also some generalizations and extensions. This chapter will also contain some results in the opposite direction, that is the minimum cyclic length of circuits in a given cyclic order. The Theorems proved there allow us to give a characterization in terms of cyclic length of the possibility for a given graph partition its vertex set into k acyclic induced digraphs. This will permit us conclude this chapter by proving a theorem that states that for any digraph, it is possible to reverse the orientations of circuits in order to get a digraph that can be partitioned in to 2 acyclic digraphs.

In Chapter 4, we want to present some results and ideas about a now classical conjecture in Graph Theory, stated first in 1978 : The Caccetta-Haggkvist Conjecture. This conjecture states that if a digraph on n vertices has minimum out-degree n/k, where k is an integer, then it contains a circuit of length at most k. It has been studied by numerous great combinatorists, apart from L. Caccetta and R. Häggkvist original article ([17]), we can mention the works of J.A. Bondy ([13], this method will be discussed shortly in one section of this chapter), of V. Chvátal and E. Szemeredi ([18]), of Y.O. Hamidoune, who proved the conjecture for vertex-transitive digraphs ([31, 32]), of C.T. Hoàng and B. Reed ([33]), or J. Shen ([45, 46, 47]). Working on this topic, P. Seymour also formulated the Second Neighbourhood Con-

jecture, which is still open and implies a weaker case of Caccetta-Häggkvist conjecture.
In fact, this problem motivated a great deal of the work shown in this thesis, and was for example a reason why we started studying the minimum cyclic length of a circuit in Chapter 4. First, we define in this chapter a large natural class of graphs with no circuits shorter than a given size, that contains for instance all the classical extremal examples of Caccetta-Häggkvist conjecture. This family is constructed by considering the digraphs that can be embedded in toruses with constraints on the length or the arcs. The study of this class has permitted us to infirm a Conjecture of Joseph Myers related to the Caccetta-Häggkvist Conjecture, related in the sense, that if it had been true, then it would have implied a particular case of the Conjecture. The conjecture was the following : suppose you have a digraph in which every pair of vertices has a common out-neighbour, is it true that this digraph contains a circuit of length 3. After that, we study what we call majority digraphs, that are digraphs that can be constructed from a family of total orders on n elements by putting arcs ij for all pairs such that i is before j in more than a certain ratio, say 2/3, of our family of orders. We will see how these digraphs are also related to the Caccetta-Häggkvist Conjecture. Then, in the other sections of this Chapter, we will describe some equivalent (sometimes stronger) statements of the Caccetta-Häggkvist Conjecture. These statements are expressed in terms of choice functions, homogeneity structures, weightings of digraph, or linear algebra. In the last sections, we will also give insights of interesting trails other people have studied in order to prove this conjecture.

In Chapter 5, we will describe another embedding problem, this one about non-oriented graphs. The context is the study of triangle free graphs. It is interesting to note here that the knowledge on these graphs has improved recently, since S. Brandt and S. Thomassé have proved in 2004 that the chromatic number of these graphs was bounded by 4 as long as one assumes that their minimum degree is at least $n/3$. Though they are non-oriented graphs, studying triangle free graphs with degree at least $n/3$, is of course not so far from the conjecture of the previous chapter... In fact, the embeddings studied here are analogues of the toric ones we consider in chapter 4, since we are still trying to represent a large class of triangle free graphs (here non-oriented) with embeddings for which we put constraints on the length of the edges. More precisely, we are interested here in a problem raised in 1972 by

M. Rosenfeld, and popularized by P. Erdős in the following years. The initial question was to know if every triangle-free graph could be embedded in the unit sphere in such a way that two vertices joined by an edge have a distance strictly greater than $\sqrt{3}$ (i.e. the angle they form with the origin is more than $2\pi/3$). In 1978 D. Larman disproved this conjecture, constructing a triangle free-graph for which the minimal length could not exceed $\sqrt{8/3}$. In addition, he conjectured that the right answer would be $\sqrt{2}$, which is not better than the class of all graphs. Larman's conjecture was independently proved by M. Rosenfeld and V. Rődl. In his paper Rődl even proved that no better bound than $\sqrt{2}$ can be found for graphs with arbitrarily large odd girth. With S. Thomassé, we have proved that it is still true for arbitrarily large girth ([21]), this is the subject of this chapter. The proof will use two main ingredients : first the now well-known probabilistic methods initiated by Erdős, and also the work of Goemans and Williamson ([28]) on the max-cut problem that highlighted the close relationship between sphere-embeddings and cuts.

In the last chapter, we are still working on circuits since we will focus on feedback arc sets. These are the set of arcs whose removal produces an acyclic digraph, and thus it is a central notion when one studies the structure of circuits in a digraph. We will have in fact already discussed these sets in chapter 2, as they are directly related to circular embeddings and cyclic orders. This sixth chapter, contains the answer to a conjecture of Bang-Jensen and Thomassen concerning minimum feedback arc set in digraphs. The question was to prove that the determination of the minimum size of such a set is a NP-Hard problem for the class of tournament, a thing that was already known for the class of all digraphs. The proved shown here, that was obtained in collaboration with S. Thomassé and A. Yeo ([22]), is in fact a reduction to the problem for graphs. It has to bas mentioned that this result was proved this year without knowing that another proof has been made by Noga Alon about six month earlier.

The proof given here is in fact contained in the second part of this chapter, the first one being devoted to a problem apparently disjoint but whose proof is one of the tools of the proof of the Bang-Jensen and Thomassen Conjecture. The question, related to an article of D. Grigoriev, is the following. Consider a finite set X and a fixed collection $\mathcal{F}$ of subsets of X. Now, given a subset A, we can ask about he number of elements in $\mathcal{F}$ that have an odd intersection with A. Grigoriev showed, as a tool in is work, that he can guarantee the existence of a subset A such that this number is between 1/3 and 2/3 of the elements of $\mathcal{F}$ ([30]). Of course, the intuition is that there exist a subset that intersects roughly half of $\mathcal{F}$ in an odd way. What we prove here, is that there exists a set such that this number is between $|\mathcal{F}|/2 - \sqrt{|\mathcal{F}|}/2$ and $|\mathcal{F}|/2| + \sqrt{|\mathcal{F}|}/2$ and that we can find it in polynomial time. This work is a joint work with P. Koiran, S. Perifel and S. Thomassé ([19]).

Chapter 2

Preliminary Definitions

In this first chapter, we give the classical definitions of graph theory as well as the conventions we will use through this thesis. All these definitions can be found with more details in books as [5], [8], [29], [25] and of course [14].

2.1 Graphs et digraphs.

A *non-oriented graph* G (or more simply a *graph*) is a pair of finite sets $(V(G), E(G))$ where $E(G)$ consists of (non-ordered) pairs of elements of $V(G)$. The elements of $V(G)$ are called *vertices* of G and those of $E(G)$ *edges* of G. If no confusion is possible, we will note V and E instead of $V(G)$ and $E(G)$. We will also use the simplified notation xy or yx for the edge $\{x, y\}$. An edge of the type xx is called a *loop* of G.
If $e = xy$ is an edge of G, we say that x and y are *neighbours* in G and that they are the *endpoints* of e. The *neighbourhood* of a vertex x in a graph G is the set of his neighbours, and we denote it by $N_G(x)$ or $N(x)$ if if no confusion is possible. The *degree* of x in G is the cardinality of $N_G(x)$, et is denoted by $d_G(x)$, or $d(x)$. For a subset X of V we define $N_G(X)$ as the union of the neighbourhoods of the vertices in X.

graph
vertices
edges
xy
$N_G(x)$, $d_G(x)$
$N_G(X)$

Similarly, a *digraph* D is a pair of finite sets $(V(D), A(D))$ where the elements of $A(D)$ are ordered pairs of $V(D)$, called *arcs* of D. The elements of $V(D)$ are also called *vertices* of D. Again if there is no confusion possible, we will note V and A instead of $V(D)$ and $A(D)$. Similarly we will denote by xy the arc (x, y), and an arc of type xx is still called a loop of D. If for

digraph
arcs
xy

each pair x an y, D does not contain both arcs xy and yx, then D will be called an *oriented graph* .

oriented graph

If $a = xy$ is an arc of D, we say that x *dominates* y, x is then the *tail* of a and y its *head*.

$N_D^+(x)$, x^+

$d_D^+(x)$

$N_D^+(X)$

X^+

The *outneighbourhood* (respectively *inneighbourhood*) of a vertex x is the set of vertices dominated by x (resp. which dominate x), and is denoted by $N_D^+(x)$ or x^+ if (resp. $N_D^-(x)$ or x^-). The cardinality of $N_D^+(x)$ (resp. $N_D^-(x)$) is called *outdegree* (resp. *indegree*) of x in D and is denoted by $d_D^+(x)$ (resp. $d_D^-(x)$). Similarly, for a subset X of V, we use the notation $N_D^+(X)$ (resp. $N_D^-(X)$) for the union of the out-neighbourhoods (resp. in-neighbourhoods) of the vertices in X. If there is no confusion possible, this set will be denoted by X^+ (resp. X^-).

subgraph

spanning subgraph

induced subgraph

A *subgraph* G' of a graph G is a graph that satisfies $V(G') \subset V(G)$ and $E(G') \subset E(G)$. In that case we say that G contains G'. If $V(G') = V(G)$, G' is a *spanning subgraph* of G. By noting $V(G') * V(G')$ the set of non ordered pairs of $V(G')$, if $E(G') = E(G) \cap (V(G') * V(G'))$, we say that G' is an *induced subgraph* of G.

$G[X]$

$G \setminus X$

$G - F$

If $X \subset V$, we denote by $G[X]$ the induced subgraph of G which vertex set is X, by $G \setminus X$ the induced subgraph which vertex set is $V \setminus X$. In the same way, if F is a subset of edges of G, $G - F$ denotes the graph with same vertex set as V and with $E \setminus F$ as the set of edges..

G/X

contraction

For a subset X of V, the graph G/X obtained from G after *contraction* of X is the graph with vertex set $(V \setminus X) \cup v_X$ and edge set $\{xy \in E : x \notin X, y \notin X\} \cup \{v_X y : \text{ if there exists } x \in X \text{ such that } xy \in E\}$. For digraphs, we define in the same way *subgraph*, *induced subgraph*, *spanning subgraph*, *digraph induced by a subset* and *digraph obtained by contraction*.

2.2 Usual structures.

The definitions for paths and circuits are given for an arbitrary digraph $D = (V, A)$. Nevertheless, if we forget the orientation of the arcs , we obtain the same definitions for non-oriented graphs.

path

An *oriented path* P of D (or simply *path* of D) is a subgraph of D, with vertices $\{x_0, x_1, \dots, x_p\}$, $p \geq 0$ and arcs $\{x_i x_{i+1} : i = 0, \dots, p-1\}$. We say that P is a path from x_0 to x_p and write $P = x_0x_1 \dots x_p$, the integer p being the *length* of P. The *endpoints* of P are x_0 et x_p, x_0 being the *beginning* of P and x_p its *end*.

$P = x_0x_1 \dots x_p$

An *oriented cycle* C of D (or simply *cycle* of D) is a subgraph of D, with vertices $\{x_1, x_2, \cdots, x_l\}$, $l \geq 1$ and arcs $\{x_ix_{i+1} : \ i = 1, \dots, l-1\} \cup \{x_lx_1\}$. We note $C = x_1 \dots x_lx_1$, where the integer l is the *length* of C.
If all the vertices of C are distinct, then C is called a *circuit* of D. A digraph that does not contain any circuit is called *acyclic* . The *girth* of a digraph is the minimum length of a circuit and is denoted by $g(D)$.

cycle

$C = x_1 \dots x_lx_1$

circuit

acyclic

girth

$g(D)$

We also define the notion of *feedback vertex set* of a digraph $D = (V, A)$. This is a set of vertices $X \subset V$ such that $D \setminus X$ is an acyclic digraph. Similarly, a *feedback arc set* is a set of arcs $F \subset A$ such that $D - A$ is an acyclic digraph. We will denote by *FVS(D)* (resp. *FAS(D)*) the set of all feedback vertex (resp. arc) sets of D.

feedback vertex set

feedback arc set

FVS(D)

FAS(D)

2.3 Connexity

A graph is said to be *connected* if for any two vertices, there is a path in the graph from one to another. A *connected component* of a graph is a maximal connected subgraph. It is straightforward to check that the connected components of a graph form a partition of its vertices.
A connected and acyclic graph is called a *tree* .

connected

tree

A digraph is *strongly connected* (or simply strong) if for every pair of vertices x and y, there exists a path from x to y. For a digraph D, we consider the following transitive and reflexive relation on the vertices of D : there exists a path from x to y in D. The inclusionwise maximal subsets of vertices for which this relation is symmetric are called *strongly connected components* of D.

strongly connected

strongly connected components

2.4 Classical Functions of Graphs.

For a graph, or a digraph D with set of vertices V, a *stable set* is a subset of vertices which induce a subgraph with no arcs (or edges). The *stability* of D, denoted by $\alpha(D)$, is the cardinality of a maximal stable set of D. An

stable set

$\alpha(D)$

tournament

oriented graph with stability 1 is a *tournament* .

separator

In a digraph $D = (V, A)$, for every two subsets of vertices X and Y, we say that $S \subset V$ is a *separator from X to Y* if every path from X to Y intersects S. More generally, if for a subset S, we can find two non empty subsets of V that are separated by S, we say that S is a *separator*

k-connected

Then, we say that a digraph D is *k-connected* , for $k \geq 1$, if $|D| > k$ and if D does not contain any separator of size strictly less than k. The *connectivity* of D is the smallest integer $k \geq 1$ such that D is k-connected, we denote it by $\kappa(D)$.

$\kappa(D)$

coloration

For a graph or a digraph $D = (V, E)$, a *k-coloration* of D is a partition of its vertices into k stable sets of D. The smallest integer k for which D admits a k-coloration is called the *chromatic number* of D, and is denoted by $\chi(D)$.

$\chi(D)$

2.5 Tools of Linear Programming.

In the next chapters, we will often use results of combinatorial optimization, especially linear programming. We recall here some classic definitions and results, for which more details can be found in reference books such as [23], [43], or chapter 30 of the Handbook of Combinatorics [29].

polyhedron

We call *polyhedron* a subset of $\mathbb{R}^n$ defined by a finite intersection of half-spaces. In other words, P is a polyhedron means that $P = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, where $\mathbf{A}$ is a matrix of size $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$ (the notation $\mathbf{u} \leq \mathbf{v}$ for two vectors of $\mathbb{R}^m$ means that $u_i \leq v_i$ for every $1 \leq i \leq m$). A bounded polyhedron is called a *polytope* . We are interested in problems of combinatorial optimization of the form :

polytope

$$\max\{\mathbf{c}.\mathbf{x} \,:\, \mathbf{A}\mathbf{x} \leq \mathbf{b}\,, \mathbf{x} \geq \mathbf{0}\}. \tag{2.1}$$

The dual of such a problem is :

$$\min\{\mathbf{y}.\mathbf{b} \,:\, \mathbf{y}\mathbf{A} \geq \mathbf{c}\,, \mathbf{y} \geq \mathbf{0}\}. \tag{2.2}$$

By convention, we fix to $+\infty$ (respectively $-\infty$) the value of the max (resp. min) if the corresponding definition set is empty.
We have the following duality theorem :

Theorem 2.1
If at least one of the two sets defined in 2.1 and 2.2 is non empty, then

$$\max\{\mathbf{c}.\mathbf{x} \,:\, \mathbf{A}\mathbf{x} \leq \mathbf{b}\,, \mathbf{x} \geq \mathbf{0}\} = \min\{\mathbf{y}.\mathbf{b} \,:\, \mathbf{y}\mathbf{A} \geq \mathbf{c}\,, \mathbf{y} \geq \mathbf{0}\}.$$

In particular, this theorem contains the following result :

Corollary 2.1 (Farkas Lemma) *The two following assertions are equivalent :*

- *There exists* $\mathbf{x} \geq \mathbf{0}$ *such that* $\mathbf{A}\mathbf{x} \leq \mathbf{b}$
- *For every* $\mathbf{y} \geq \mathbf{0}$*,* $\mathbf{y}\mathbf{A} \geq 0 \Rightarrow \mathbf{y}.\mathbf{b} \geq 0$

We also often use the property of *complementary slackness* . This means the following : if $x \in \mathbb{R}^n$ is an optimal solution of the primal problem, and if $x_i \neq 0$, then in the dual the i-th inequality constraint is in fact an equality for every optimal solution

complementary slackness

In combinatorics, we are often interested in integer linear programming, that is we want to find a vector with integer entries that is an optimum solution of a problem as the one defined in 2.1.

We give here a sufficient condition (but of course not necessary) to ensure the existence of integer valued solutions. If $\mathbf{A}$ is an integer valued matrix, we say that A is *totally unimodular* if all the square matrices that can be extracted from A have a determinant belonging to $\{-1, 0, 1\}$.

totally unimodular

Proposition 2.1 *Let* $\mathbf{A}$ *be a totally unimodular matrix. Then for every* $b \in \mathbb{Z}^n$ *et* $c \in \mathbb{Z}^m$*, the linear program 2.1 and its dual have both integer-valued optimal solutions.*

Proof
The proof is easy and can be founded in the reference books cited as the beginning of the section. It is based on the classic lemma from linear algebra that asserts that a matrix with integer coefficients and determinant 1 or -1 has an inverse with integer coefficients. ■.

2.6 Digraphs and Total unimodularity : Some Examples.

2.6.1 The Circulation Matrix.

A classic result is the fact that the incidence matrix of a digraph $D = (V, A)$ is totally unimodular. We recall that this matrix $\mathbf{M} = (m_{ij})$ is defined by

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } a_j \\ -1 & \text{if } v_i \text{ is the tail of } a_j \\ 0 & \text{otherwise} \end{cases}$$

There are many classical corollaries of this fact, for example the max-cut min-flow theorem.

In the proof of Gallai's conjecture exposed in chapter 3, we will use as a main ingredient the total unimodularity of a certain matrix. This matrix $\mathbf{U}(D)$ has $2n = 2|V|$ rows and $m = |A|$ columns and is defined as :

$$\mathbf{U} = \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} \tag{2.3}$$

where $\mathbf{M} = (m_{ij})$ is the incidence matrix of D and $\mathbf{N} = (n_{ij})$ is the $n \times m$ matrix defined by :

$$n_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } a_j \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.1 *For any digraph D, the matrix $U(D)$ defined in 2.3 is totally unimodular.*

Proof
If $\tilde{\mathbf{U}}$ is a square matrix extracted from $\mathbf{U}$, we have to prove that its determinant is either 0, 1 or -1. Let $\mathbf{B}$ be the matrix obtained from $\tilde{\mathbf{U}}$ the following way. For every index i such that the two corresponding rows of $\mathbf{M}$ and $\mathbf{N}$ appear in $\tilde{\mathbf{U}}$, we subtract the row of $\mathbf{N}$ to the one of $\mathbf{M}$. In each column of $\mathbf{B}$, there are at most two non-zero entries, and if there are two then there is one entry equal to 1 and the other equal to -1. It is straightforward to check by induction on the size of such a matrix that its determinant is either

1, -1 or 0. Since the determinant of $\mathbf{B}$ is equal to the one of $\tilde{\mathbf{U}}$, we have proved that $\mathbf{U}$ is totally unimodular.

■

In order to illustrate this property, we will in the two next paragraphs show how one can deduce from this the proof of two famous min-max theorems. We have chosen to write here these two proofs for two reasons : first these results are close two the topic of the next chapter and also because in the two cases, it is not the usual proof given for these results.

2.6.2 Menger's Theorem.

We start with a well-known connectivity result which is the following

Theorem 2.2 (K. Menger [39], 1927) *Let $D = (V, A)$ be a digraph and X, Y two subsets of V. The cardinality of minimal separator from X to Y is equal to the maximal number of disjoints (X, Y)-paths. In particular, D is k-connected if and only if $|D| > k$ and if for every subset X and Y of V with cardinality at least k, there exists k disjoints paths from X to Y.*

Proof :
We consider the digraph D' constructed from D by adding a vertex v_0 such that $N^+(v_0) = X$ and $N^-(v_0) = Y$ (see Figure 2.1). We also consider the matrix $\mathbf{U}(D')$ defined in the previous section. We extract from $\mathbf{U}(D')$ the matrix $\tilde{\mathbf{U}}$ obtained by removing the two rows corresponding to the vertex v_0. This matrix is also of course totally unimodular.

Then we define the vectors $\mathbf{b} = (b_1, \ldots, b_{2n})$ and $\mathbf{c} = (c_1, \ldots, c_m)$ as follows :

$$b_i = \begin{cases} 0 & \text{if } 1 \leq i \leq n \\ 1 & \text{otherwise} \end{cases}$$

and

$$c_j = \begin{cases} 1 & \text{if } v_0 \text{ is the tail of } a \\ 0 & \text{otherwise} \end{cases}$$

We consider the linear program (P):

$$\begin{array}{rrcl} \textit{maximize} & \mathbf{cx} & & \\ \textit{subject to the constraints} & \tilde{\mathbf{U}}\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

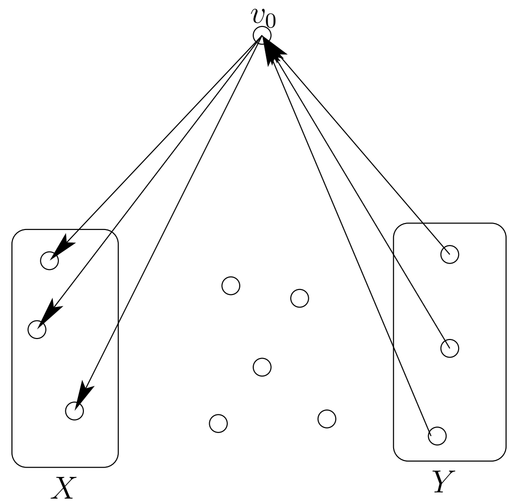


Figure 2.1: digraph D'

Thanks to total unimodularity, there are integer valued optimal solutions. Such a solution corresponds to a union of circuits of D' which can only intersect in v_0. In other words, these solutions represent disjoint unions of circuits and (X, Y)-paths and the objective function counts exactly the number of paths. Since it is clear that every union of disjoints (X, Y)-paths can be represented by some solution of the constraints, we get that this problem is exactly finding the maximum number of disjoints (X, Y)-paths.

Now we have to look at the dual problem. We get the following formulation for (P^*) :

$$\begin{array}{rrcl} \textit{minimize} & \mathbf{yb} & & \\ \textit{subject to the constraints} & \mathbf{yU} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

Let us write $\mathbf{y} := (z_1, \ldots, z_n, w_1, \ldots, w_n)$. Then (P^*) is the problem of minimizing $\sum_{i=1}^{n} w_i$ subject to the constraints

$$\begin{array}{rl} z_i - z_j + w_i \geq 0 & \text{if } v_i v_j \text{ is an arc of } D \\ z_i \geq 1 & \text{if } v_i \in X \\ w_i \geq z_i & \text{if } v_i \in Y \end{array}$$

Let us consider an optimal solution with integer entries.
Let $S = \{v_i \in V : w_i \geq 1\}$. We prove that S is a separator from X to Y. Indeed, if we sum the previous constraints along the arcs of a (X,Y)-path P in D extended in a circuit of D' going through v_0 , we get the inequality

$$\sum_{v_i \in V(P)} w_i \geq 1.$$

In other words, $P \cap S \neq \emptyset$. Moreover, the value of the objective function for this solution is greater than the size of S.
To conclude it remains to prove that every separator can be represented as a solution for which the objective function is equal to its cardinality. So, let S be a separator from X to Y. Define S_1 as the set of vertices v of D for which there is an oriented path from X to v in $D \setminus S$, and $S_2 = V \setminus (S \cup S_1)$.
We now define

$$\begin{aligned} (w_i, z_i) = &\quad (0,1) \quad \text{if } v_i \in S_1 \\ (w_i, z_i) = &\quad (1,1) \quad \text{if } v_i \in S \\ (w_i, z_i) = &\quad (0,0) \quad \text{if } v_i \in S_2 \end{aligned}$$

By definition, there are no arcs from S_1 to S_2 and as S is a separator, $Y \cap S_1 = \emptyset$. This implies that the constraints of the problem (P*) are satisfied and that we have $\sum w_i = |S|$.

■

2.6.3 Path Partitions.

Another classical result that follows from the total unimodularity of this matrix is the acyclic case of Berge conjecture on path partitions. In order to state the problem, we introduce the k-norm of a partition into paths of the vertices of a digraph.

$$||\mathcal{P}||_k = \sum_{P \in \mathcal{P}} \min(k, |P|).$$

We note $\pi_k = \min\{||\mathcal{P}||_k : \mathcal{P} \text{ path partition of } D\}$ and we call k-optimal a partition for which the k-norm is equal to π_k.
On the other hand, a k-coloration of the vertices of a digraph is defined as the union of k disjoints stable sets. We denote by α_k the maximal size of such a union.
Linial ([38]) formulated the following conjecture.

Conjecture 2.1 (Linial) *For every digraph D and every integer k, $\pi_k(D) \leq \alpha_k(D)$*

The result is true for acyclic transitive digraphs (digraph of a poset), Green-Kleitman Theorem even proves that equality holds in this particular case.

Berge strengthened this conjecture by introducing the notion of orthogonality. We say that a k-coloration is *orthogonal* to a path partition $\mathcal{P}$ if every path P in $\mathcal{P}$ meets exactly $min(|P|, k)$ different colour classes. Berge conjecture ([9]) is the following :

Conjecture 2.2 (Berge) *Let D be a digraph and k an integer. Every k-optimal path partition admits an orthogonal k-coloration.*

The result we are proving here concerns acyclic digraphs and is in fact stronger that Berge conjecture in this particular case.

Theorem 2.3 *Let $D = (V, A)$ be an acyclic digraph and k an integer. There is a k-coloration of D which is orthogonal to every k-optimal path partition.*

Proof :
As we did for Menger Theorem, we begin by modifying the digraph. We consider the digraph D' obtained by adding a vertex v_0 and all arcs (in both directions) between v_0 and the vertices of D. Finally we add a loop for each vertex of D (see Figure 2.2). Now, since D is acyclic, a path partition of D is a circuit cover of D' where all circuits of length greater than one go through v_0 and where two such circuits have no common vertex apart from v_0.

Consider now the matrix of circulations $\mathbf{U}(D')$ defined in 2.3. We extract from $\mathbf{U}(D')$ the matrix $\tilde{\mathbf{U}}$ obtained by removing the two rows corresponding to the vertex v_0. This matrix is also totally unimodular.

Define the vectors $\mathbf{b} = (b_1, \ldots, b_{2n})$ and $\mathbf{c} = (c_1, \ldots, c_{E(D')})$ by:

$$b_i = \begin{cases} 0 & \text{if } 1 \leq i \leq n \\ 1 & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

and

$$c_j = \begin{cases} k & \text{if } v_0 \text{ is the tail of } a_j \\ 1 & \text{if } a_j \text{ is a loop} \\ 0 & \text{otherwise} \end{cases}$$

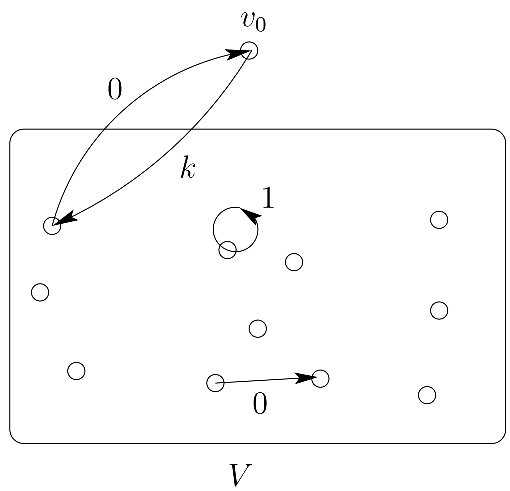


Figure 2.2: The digraph D' with the weights given by c

We study the linear program (P):

$$\begin{array}{rrcl} \textit{minimize} & \mathbf{cx} & & \\ \textit{subject to the constraints} & \widetilde{\mathbf{U}}\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

The total unimodularity of $\widetilde{\mathbf{U}}$ implies the existence of integer valued optimal solutions. The system of constraints given by the first n rows expresses the fact that these solutions represent integral flows on the vertices of G (and thus of D'), in other words, family of cycles and the n last rows imply that these cycles go exactly once through each vertex of D. The remark made at the beginning of the proof shows that these solutions are exactly path partitions of D. Furthermore, the objective function counts k times the number of circuits that go through x_0 and one for each loop. So for the path partition, it counts k times the number of paths of length more than one plus one for each path of length 0 (an isolated vertex). It is then clear that an integer solution that minimizes the objective function is a k-optimal path partition and the value of the objective function is its k-norm.

Consider now the dual problem. The variables are two vectors $\mathbf{y}$ and $\mathbf{z}$ corresponding to the vertices of D and we want to maximize

$$\sum_{i=1}^{n} y_i$$

subject to the constraints $[\mathbf{z},\mathbf{y}]\tilde{\mathbf{U}} \leq \mathbf{c}$, that is :

$$\left\{\begin{array}{rl} y_i + z_j - z_i \leq 0 & \text{if } v_i v_j \text{ is an arc of } D \\ y_i - z_i \leq 0 & \text{for every } i \text{ (constraints generated by the arcs } (v_i, v_0)) \\ z_i \leq k & \text{for every } i \text{ (constraints generated by the arcs} (v_0, v_i)) \\ y_i \leq 1 & \text{for every } i \text{ (constraints generated by the loops)} \end{array}\right. \tag{2.4}$$

We associate a k-colouring $\mathcal{C}^k = \{C_1, C_2, \ldots, C_k\}$ to such a solution in the following way :

$$C_r = \{v_i \, : \, y_i = 1 \text{ and } z_i = r\} \; , \, r = 1, 2, \ldots, k.$$

This defines a proper k-colouring. Indeed, note that if vertices v_i and v_j are in C_r, $y_i + z_j - z_i = 1$, and by (2.4), this implies that these vertices are not joined by an arc.

We shall now prove that $\mathcal{C}^k$ is orthogonal to every k-optimal path partition $\mathcal{P}$. Being an k optimal path partition, $\mathcal{P}$ is represented by a vector $\mathbf{x} \in \mathbb{N}^{E(D')}$ which is an optimal solution of the primal. Complementary slackness conditions means that if an entry of $\mathbf{x}$ is non zero, then the corresponding inequality constraint in the dual is an equality. We distinguish two cases : short paths (of length less than k) and long ones.

Let P be a short path in $\mathcal{P}$. We have seen that such paths are represented in $\mathbf{x}$ by loops on vertices of P. So if $v_i \in P$, by complementary slackness, we get $y_i = 1$, and since by (2.4), we have $y_i \leq z_i \leq k$, we know that v_i belongs to some C_r. Furthermore, if $(v_i, v_j) \in A$ and $v_i \in C_r$, then by (2.4) $z_j - z_i \leq -y_i = -1$, implying that the colours in P are strictly decreasing, and thus all distinct.

Assume now that P is a long path. To simplify notations , assume that $P = (v_1, \ldots, v_l)$. We have seen when studying the primal problem that P

is represented by a circuit going through v_0. Thus again, by complementary slackness, we know that equality is satisfied in all the constraints 2.4 corresponding to arcs of the circuit. Then :

$$\begin{aligned} z_1 &= k \\ z_{i+1} - z_i &= -y_i \quad \forall 1 \leq i \leq l-1 \\ z_l &= y_l. \end{aligned}$$

But y_i is at most 1, so the variable z_i can decrease only at most from 1 at a time along the arcs of the paths. Since $z_l = y_l \leq 1$ this means that if $z_i = t$, then for every t' with $1 \leq t' \leq t$ there exists $j \geq i$ such that $z_j = t'$, that is a kind of intermediate value property. Since $z_1 = k$, z takes all values between 1 and k and thus we can define $i_p = \max\{1 \leq i \leq l \,:\, z_i = p\}$. But this implies $z_{i_p+1} = p-1$ (by maximality of i_p), so $y_{i_p} = 1$, implying that for each p, $v_{i_p} \in C_p$. In other words, P meets all k colours in $\mathcal{C}^k$, which completes the proof. ∎

Chapter 3

Cyclic Orders : Equivalence and Duality

3.1 Introduction

The notion of cyclic order has been introduced by S. Bessy et S.Thomassé ([15]) in order to prove a forty year old of Gallai, which has become now the following Theorem :

Theorem 3.1 (Bessy - Thomassé, 2004)
Let D be a strongly connected digraph. The vertices of D can be covered by less than α circuits, where α denotes the stability of D.

The first remark we can make about this result is that it is clearly not a min-max theorem. Indeed if one considers the digraph which is just a directed circuit, its vertices can be covered by only one circuit but its stability can be as large as half the number of vertices. The idea was thus to find a min-max result that would lie between these quantities : the minimum number of circuit needed to cover the vertices, and the maximal size of a stable set.

3.2 Cyclic Orders

We define a *linear order* of a (di)graph G as an order $O = v_1, v_2, \dots, v_n$ of its vertices, that is a bijection from $V(G)$ onto $\{1, \dots, n\}$. A *circular order* is an equivalence class of linear orders for the relation : there exists $1 \leq i \leq n$ such that $O = v_1, v_2, \dots, v_n$ and $O' = v_i, \dots, v_n, v_1, \dots, v_{i-1}$.

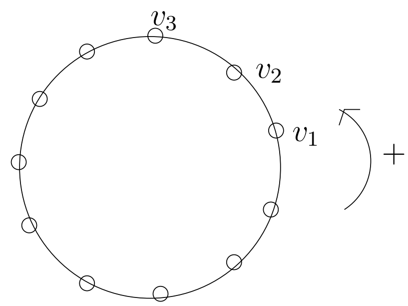


Figure 3.1: A circular order given with an orientation

forward arc
backward arc
Ret(O)

In a linear order $O = v_1, \ldots, v_n$, an arc v_iv_j with $i < j$ is called a *forward arc* of O, otherwise it is called a *backward arc* . We will denote by $Ret(O)$ the set of backward arcs of O.

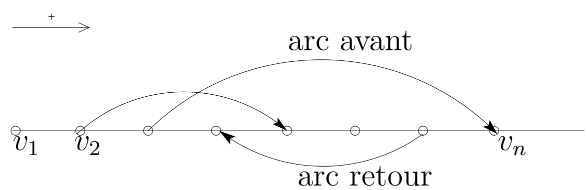


Figure 3.2: Forward and Backward arcs in a linear order

Here we want to state an easy lemma about backward arcs that we will often use in this thesis. Note first that set of backward arcs in a given linear order constitutes a feedback arc set of the digraph. And conversely :

Lemma 3.1 *If D is an acyclic digraph, there exists a linear order on its vertices such that all arcs are forward arcs. Equivalently, if F is a feedback arc set of a digraph D, there exists a linear order O such that $Ret(O) \subset F$.*

Proof
We will prove that it is possible to represent an acyclic digraph with only forward arcs. We proceed by induction on the number of vertices by removing some vertex of outdegree 0 (there must be one since the digraph is acyclic) and put it at the beginning of the order given by the induction hypothesis.
■

We will come back in section 3.5 on the links between feedback arc sets and backward arcs.

If $O = v_1, \ldots, v_n$ is a linear order on the vertices of D, the *length* of an arc $a = v_i v_j$ is defined as $(j-i)/n$ if a is a forward arc and $(n+j-i)/n$ if it is a backward arc. It is simply the length of the portion of circle from v_i to v_j when the vertices are represented on the circle of perimeter 1. length

If a digraph D is endowed with a linear order, and C is a directed circuit of D, we can define the central notion of this chapter, that is the *winding number* (or *index*) of C with respect to O, denoted by $\mathrm{ind}_O(C)$. Basically, we represent the digraph on a circle with a positive orientation and count the number of times the circuit goes around the circle. winding number index

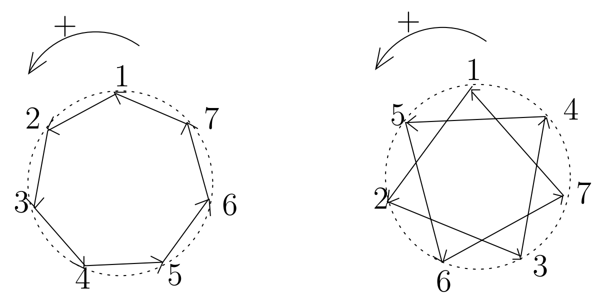


Figure 3.3: A circuit of length 7, with winding number 1 on the left and 2 on the right

We can formalize this in different equivalent ways. We can define it at the number of backward arcs of the circuit in the given linear order. We can also remark that this corresponds to the sum of the length of the arcs. Depending on the context, we will use the first or the second point of view.

It is easy then to see that the winding number of a circuit is identical in two linear orders that belong to the same circular order. In fact, we can say more. If one starts with a linear order O and switches two consecutive non adjacent vertices on the order, then the linear order O' obtained satisfies also $\mathrm{ind}_O(C) = \mathrm{ind}_{O'}(C)$ for every circuit. This leads to the following definition : a linear order $O = v_1, \ldots, v_n$ is elementary equivalent to O' if one of the two following properties holds : either $O' = v_n, v_1, \ldots, v_{n-1}$ or $O' = v_2, v_1, \ldots, v_n$

with v_1 and v_2 non adjacent. Two linear orders O and O' are then said to be *equivalents* (and we denote it by $O \sim O'$) if there exists a sequence $O = O_1, \ldots, O_k = O'$ such that O_i and O_{i+1} are elementary equivalent for $i = 1, \ldots, k-1$.

If O and O' are linear orders, then the previous remark can be stated as :

$$O \sim O' \Longrightarrow \forall C \textit{ circuit} \quad \mathrm{ind}_O(C) = \mathrm{ind}_{O'}(C). \tag{3.1}$$

The equivalence classes for this relation are called *cyclic orders*. It is thus possible to speak about the winding number of a circuit with respect to a cyclic order.

A natural question is to know whether 3.1 is an equivalence. In other words, if two linear orders have the same winding number for each circuit, are they equivalent? This is the topic of the next section.

3.3 Equivalence of cyclic orders

Before stating the main theorem of this section, we make the following remark. The definitions of linear orders or cyclic orders do not depend of the orientation of the arcs, in fact these are definitions valid for any non-oriented graph. And similarly, as long as one chooses one of the two possible orientations, the winding number of a circuit is also well defined in the context of non-oriented graphs. In fact, it suffices to see a non oriented graph as the digraph obtained by replacing each edge by two opposite arcs. So the question stated at the end of the previous section holds for non oriented graphs as well. This is the one we answer first.

Theorem 3.2 *Let $G = (V, E)$ be a non-oriented graph and O_1 and O_2 two linear orders of V. The two following assertions are equivalent :*

(i) $O_1 \sim O_2$

(ii) The winding number of every circuit (after choosing one orientation) is the same with respect to O_1 and O_2.

It follows for instance that *for forests any two cyclic orders are equivalent* (which is easy to check directly). Note that this theorem provides a good characterization (a linear $NP \cap coNP$ certificate) for two orders to be equivalent. It is not surprising that this certificate depends only on the underlying undirected graph: the elementary operations depend only on this graph.

Note that this theorem has no condition on G. As we will see later, for strongly connected digraphs the condition on directed cycles will turn out to imply the condition for undirected cycles.

Proof

Since an elementary operation does clearly not change the index of a circuit or of a closed walk, (i) implies (ii).

Let us prove now the essential statement "(ii) implies (i)", by induction on the number of edges. Let $e = xy \in E$ $(x, y \in V)$ be an arbitrary edge.

By the induction hypothesis the statement is true for $G - e$, that is, there exists a sequence $\pi_1, \ldots, \pi_k$ of elementary operations that brings the order O_1 to the order O_2 in $G - e$. Every elementary operation on $G - e$ is also an elementary operation of G, except for the permutation of x and y. If this operation does not occur, we are done : we have a sequence of elementary operations that brings O_1 to O_2.

Claim: If the permutation of x and y does occur among $\pi_1, \ldots, \pi_k$, then there exist O_1', O_2', $O_1' \sim O_1$, $O_2' \sim O_2$ such that x is followed by y in both O_1' and O_2', or y is followed by x in both.

This Claim finishes the proof of the theorem: since e joins neighboring vertices in both O_1' and O_2', and in the same order, these orders obviously define orders O_1'', O_2'' of G/e (the graph obtained after contraction of e); furthermore, since (i) implies (ii) (and this is already proven), the condition (ii) is still satisfied for O_1' and O_2', and therefore for O_1'' and O_2'' as well. Since G/e has less edges than G, by the induction hypothesis $O_1'' \sim O_2''$, and the elementary operations of G/e correspond obviously to one or two elementary operations of G.

In order to prove the Claim let i, j, $1 \leq i \leq j \leq k$ be the first and the last index where the permutation of x and y occurs. Let O_1' be the cyclic order we get from O_1 if we stop before executing π_i, and O_2' the order we get by executing the permutations in reverse order from O_2 and stopping before executing π_j. Clearly, $O_1' \sim O_1$, $O_2' \sim O_2$, and therefore O_1', O_2' satisfy (ii).

In both O_1' and O_2' x and y are consecutive by definition. If they follow one another in the same order in O_1' and O_2', then we are done. If not, suppose without loss of generality (by possibly interchanging the notation x, y) that x precedes y in O_1', and y precedes x in O_2'.

Take a shift in O_2' so that x is the first, and y the last element. *There is no forward path now from x to y in $G - e$*, because if there was such a path $P = (x = x_0, x_1, \dots, x_p = y)$, then $p \geq 2$, and with the edge yx, P is in fact a cycle of index 1. On the other hand taking an opening of O_1' different from (x, y) we see that yx is a backward arc in P, and there must be another backward arc since otherwise $p = 1$. Therefore the index of the cycle P in O_1' is at least 2, while it is 1 in O_2', contradicting (ii).

It follows that the set X of vertices that can be reached from x with a forward path (in O_2') have no forward neighbour after the last element of X, and therefore X can be placed after y by a sequence of elementary changes. The vertices in $Y := V \setminus (X \cup \{x, y\})$ have no backward neighbour in X, so similarly, they can be placed before x. Therefore $O_2'' := Y, x, y, X$ is an equivalent order, and y follows x as in O_1. So O_1' and O_2'' are as claimed. ■

What can we say now about directed graphs? The previous theorem thus implies that in order to guarantee the equivalence of two cyclic orders, we need to have the same winding numbers for every non-oriented circuits in the underlying non-oriented graph. In fact, for strongly connected graphs, we have the following result.

Theorem 3.3 *Let $D = (V, A)$ be a strongly connected digraph, $\mathcal{O}_1$ and $\mathcal{O}_2$ be two cyclic orders of V. The two following assertions are equivalent. :*

(i) $\mathcal{O}_1 = \mathcal{O}_2$

(ii) The winding number of every directed circuit of D is the same with respect to $\mathcal{O}_1$ and $\mathcal{O}_2$.

This sharpening follows by simple linear properties of circuits – roughly, the circuits of a strongly connected digraph "generate" all the undirected circuits of the underlying graph. If $D = (V, A)$ is a directed graph, then each cycle of the underlying graph can be represented as a vector in $\{-1, 0, 1\}^A$ in the following usual way :

Let C be an undirected circuit (with one of the two orderings fixed for reference), and define the vector $\overrightarrow{C} \in \{-1, 0, 1\}^A$ as follows: $\overrightarrow{C}(a) = 1$ if $a \in C$ is oriented in the sense of the orientation of C, $\overrightarrow{C}(a) = -1$ if it is oriented in the opposite sense, and $\overrightarrow{C}(a) = 0$ if $a \notin C$.

$$\mathcal{C}(D) := \text{Vect}\{\overrightarrow{C} : C \text{ is a circuit of } G(D)\}.$$

Note that the definition of $\mathcal{C}(D)$ does not depend on which of the two orientations of the circuits we chose, since the vector defined by the opposite orientation is just $-\overrightarrow{C}$. $\mathcal{C}(D)$ is the set of circulations.

Lemma 3.2 *A 2-edge-connected digraph $D = (V, A)$ is strongly connected if and only if $\mathcal{C}(D)$ is spanned (linearly) by the (directed) circuits of D (as vectors in $\{0,1\}^A$).*

Proof
Indeed, if D is not strongly connected, let e be a an edge not contained in a directed circuit. Then since the underlying undirected graph is 2-edge connected, there exists an undirected circuit C in $G(D)$, $e \in E(C)$; since e is not contained in any directed circuit, C is not generated by circuits of D.

Conversely suppose that D is strongly connected. Then any circulation f is generated by directed circuits: indeed, for each of the negative coordinates $e_1, \ldots, e_p$ of f choose a circuit C_i containing e_i $(i = 1, \ldots, p)$; $f - \sum_{i=1}^{p} f(e_i)C_i$ is a nonnegative circulation, which is obviously a (nonnegative) combination of directed circuits, and then so is f. ■

Proof of Theorem 3.3
We have to prove only that the condition is implied for every undirected circuit of the underlying graph, because then Theorem 3.2 implies the assertion. Denote by $\operatorname{ind}_i(C)$ the index of circuit C according to O_i, $(i = 1, 2)$.

Let L_1 and L_2 be two linear orders that belong respectively to $\mathcal{O}_1$ and $\mathcal{O}_2$. Define the vectors $w_1, w_2 \in \{1, -1\}^A$ to be -1 on backward arcs and 1 on forward arcs.

Note first that $w_i(C) = (|E(C)| - \operatorname{ind}_i(C)) - \operatorname{ind}_i(C) = |E(C)| - 2\operatorname{ind}_i(C)$ for every cycle $(i = 1, 2)$. So *the assumption on the equality of indices according to the two orders is equivalent to $w_1(C) = w_2(C)$ for every circuit C.*

The equation

$$w_i^T \overrightarrow{C} = |E(C)| - 2\operatorname{ind}_i(C)$$

holds for all the circuits of $G(D)$. Indeed, in the inner product $w_i^T \overrightarrow{C}$ we have four kinds of terms: $1 \cdot 1$, $1 \cdot (-1)$, $(-1) \cdot 1$, $(-1) \cdot (-1)$, and it is clear that the result is 1 if the corresponding edge goes forward in $\overrightarrow{C}$, and -1 if it goes backward, and the difference of the forward and backward edges is $|E(C)| - 2\operatorname{ind}_i(C)$.

So for checking that the condition (ii) of Theorem 3.2 holds, it is sufficient to prove $w_1^T\overrightarrow{C} = w_2^T\overrightarrow{C}$ for every circuit C of $G(D)$. However, since we know that this holds for directed circuits, and by Lemma 3.2 the directed circuits generate $\mathcal{C}(D)$, it follows for every undirected circuit C of $G(D)$, as expected.
■

3.4 Coherent Orders and Gallai's conjecture

3.4.1 Coherent cyclic orders

As we explained in the introduction of this chapter, the notion of cyclic order was introduced by S. Bessy and S. Thomassé in order to prove Theorem 3.1. The idea is, for a family of circuits covering the vertices, to bound the number of circuits by the sum of the winding numbers (since the winding number is at least 1). To bound this by the stability of the digraph, we have to choose a "good" cyclic order, that is one in which the winding numbers are as small as possible. They gave the following definition :

coherent

A cyclic order of a digraph D is *coherent* if every arc of D belongs to a circuit of index 1.

The main result is that such an order exists for every strongly connected digraph. In order to prove this, we will explain why their definition corresponds to the intuitive idea of a "good" cyclic order described above.

We define the following reflexive and transitive relation on cyclic orders :

$$O_1 \leq O_2 \text{ if for each circuit } C \text{ of } D,\ ind_{O_1}(C) \leq ind_{O_2}(C) \qquad (3.2)$$

Theorem 3.3 is thus equivalent to say that for strongly connected digraphs this relation is a partial order on cyclic orders. We can now prove the existence of a coherent cyclic order by the following proposition

Proposition 3.1 *Let D be a strongly connected digraph. A cyclic order of D is coherent if and only if it is minimal with respect to relation $\leq$ described in 3.2.*

Proof
Let $\mathcal{O}$ be a non coherent cyclic order and prove that it is not minimal. Let

$a \in A$ an arc that is not contained in a circuit of winding number 1. It is possible to choose a linear O in $\mathcal{O}$ such that $a \in B$, where $B = \text{Ret}(O)$ is the set of backward arcs in O. The set $A \setminus (B \setminus a)$ does not contain a cycle – since every cycle has a backward arc and not only a – so by Lemma 3.1 there exists a linear order O' in which every arc in $A \setminus (B \setminus a)$ is a forward arc. In this order the set of backward arcs $B' = \text{Ret}(O')$ satisfies $B' \subseteq B \setminus e$. Clearly, for every circuit C:

$$\text{ind}O'(C) = |C \cap B'| \leq |C \cap B| = \text{ind}_O(C)$$

Since D is strongly connected, a is contained in a circuit C, and for this circuit strong inequality holds, proving that O is not minimal with respect to $\leq$.

Conversely, let $\mathcal{O}$ be a coherent cyclic order, and $\mathcal{O}'$ be a cyclic order such that $\mathcal{O}' \leq \mathcal{O}$. Let us prove that $\mathcal{O}' = \mathcal{O}$. By Theorem 3.3, it suffices to prove that the winding numbers are the same for every circuit. So let C be a circuit of D. As D is strongly connected, and $\mathcal{O}$ is coherent, every arc a of C is contained in a circuit C_a of winding number 1. By identifying circuits with vectors in $\{0, 1\}^A$, we can write :

$$\sum_{a \in C} C_a = C + \sum_{i=1}^{p} C_i$$

where C_i are circuits of D, since the sum of the C_a is an eulerian digraph containing C.
As the circuits C_a ave winding number 1 in $\mathcal{O}$ they must also have winding number 1 in $\mathcal{O}'$ so the winding number of the sum of circuits C_a is equal to $|C|$ in both $\mathcal{O}$ and $\mathcal{O}'$. Thus the right side of the equation has also same winding number, and since the winding numbers of every circuits is less in $\mathcal{O}'$, this implies equality for each circuit. We have proved that $\text{ind}_{\mathcal{O}}(C) = \text{ind}_{\mathcal{O}'}(C)$ for every circuit, which concludes the proof. ■

3.4.2 Index Bounded Weightings and Gallai's Conjecture

In this section we will see how the existence of a coherent cyclic order proved in 3.1, along with the theorem of duality on linear programming, give a very simple proof of Gallai's Conjecture proved by Bessy and Thomassé. This proof was obtained in collaboration with Adrian Bondy who suggested this purely linear programming approach (see [11]).

A *weighting* of a digraph D is a function $w : V \to \mathbb{N}$. The *weight* of a vertex v of D is the value $w(v)$. By extension, the *weight* of a subgraph of D is the sum of the weights of its vertices. If D is endowed with a cyclic order $\mathcal{O}$, and if $w(C) \leq \text{ind}(C)$ for every circuit C of D, we say that the weighting

index-bounded

w is *index-bounded* (with respect to $\mathcal{O}$). We prove the following result.

Theorem 3.4 *Let D be a digraph in which every vertex belongs to a circuit (that is each connected component is strongly connected) and $\mathcal{O}$ be a cyclic order on the vertices of D. Then*

$$\begin{aligned} & \min \quad \{ \, ind(\mathcal{C}) : \; \mathcal{C} \text{ a family of circuits covering the vertices of } D\} \\ = & \max \quad \{ \, w(V) : \; w \text{ an index-bounded weighting}\}. \end{aligned}$$

Gallai's conjecture can now be easily deduced by applying this theorem to a coherent cyclic order:

- for every family $\mathcal{C}$ of circuits of D, $|\mathcal{C}| \leq \text{ind}(\mathcal{C})$;
- since each vertex is the endpoint of an arc and the order is coherent, it is also contained in a circuit of index 1, and therefore an index-bounded weighting of D is necessarily $(0,1)$-valued;
- There is no arc $a = v_i v_j \in A$ such that $w(i) = w(j) = 1$, because the circuit C of index 1 containing w satisfies $w(C) \leq 1$.

Therefore, for a coherent cyclic order, the support of an index bounded weighting w is a stable set, and thus $w(D) \leq \alpha(D)$ follows. These specific stable sets have been studied by Bessy and Thomassé under the name cyclic stable set. We will return on this notion in paragraph 3.5.3.

Proof of Theorem 3.4
One inequality is easy. Let $\mathcal{C}$ be a family of circuits covering the vertices, and w any index bounded weighting with respect to $\mathcal{O}$. Then

$$\text{ind}(\mathcal{C}) = \sum_{C \in \mathcal{C}} \text{ind}(C) \geq \sum_{C \in \mathcal{C}} w(C) = \sum_{v \in V} |\{C \in \mathcal{C} \, : \, v \in C\}| w(v) \geq w(V).$$

Thus the max of the Theorem is no greater than the min. What remains is to find a family of circuits covering V and an index-bounded weighting for which equality holds.

We want to write this problem as a linear program. The relaxation of a set of circuits is a circulation, that is a weighting of the arcs of the digraph such that the sum of the weights or the arcs entering a vertex is equal to the sum of the weights of the arcs leaving it. Then, asking that every vertex is covered is the same as asking that this sum of weights is non zero, for example greater than 1. Then, we will find a way to express the index as the linear function to minimize. Let us write this more formally.

Let V be the set of vertices and $O = (v_1, \dots, v_n)$ a linear order belonging to the cyclic order O. The arc set is denoted by $A = \{a_1, \dots, a_m\}$. An arc (v_i, v_j) is thus forward if $i < j$, and backward otherwise. We will now use the matrix $U(D)$ introduced in section 2.3. Recall that it is defined by

$$\mathbf{U} = \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$$

where $\mathbf{M} = (m_{ij})$ is the $n \times m$ incidence matrix of D:

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } a_j \\ -1 & \text{if } v_i \text{ is the head } a_j \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{N} = (n_{ij})$ is the $n \times m$ matrix defined by :

$$n_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } a_j \\ 0 & \text{otherwise} \end{cases}$$

Then we define the vectors $\mathbf{b} = (b_1, \dots, b_{2n})$ and $\mathbf{c} = (c_1, \dots, c_m)$ as follows :

$$b_i = \begin{cases} 0 & \text{if } 1 \leq i \leq n \\ 1 & \text{otherwise} \end{cases}$$

and

$$c_j = \begin{cases} 1 & \text{if } a_j \text{ is a back arc} \\ 0 & \text{otherwise} \end{cases}$$

Let us now consider the linear program (P):

$$\begin{array}{rrcl} \textit{minimize} & \mathbf{cx} & & \\ \textit{subject to the constraints} & \mathbf{Ux} & \geqq & \mathbf{b} \\ & \mathbf{x} & \geqq & \mathbf{0} \end{array}$$

Let x be a vector with non-negative entries that satisfies the constraints $\mathbf{Ux} \geq \mathbf{b}$, that is constraints $\mathbf{Mx} \geq \mathbf{0}$ and $\mathbf{Nx} \geq \mathbf{1}$, where $\mathbf{0}$ and $\mathbf{1}$ are vectors 0's and 1's respectively. As the columns of $\mathbf{M}$ sum up to 0, the entries $\mathbf{Mx}$ also do, which implies that $\mathbf{Mx} = \mathbf{0}$ since the entries of x are non-negative. Thus such a vector x is a non-negative circulation, that is a weighted sum of circuits of D. Furthermore, the condition $\mathbf{Nx} \geqq \mathbf{1}$ implies that this set of circuits is a vertex covering. Since by assumption D has at least one circuit cover, these constraints are feasible.

The problem (P) being bounded since $\mathbf{c}$ is non-negative, it has optimal solutions. We have seen in Lemma 2.1 that the matrix of constraints is totally unimodular, so Proposition 2.1 implies the existence of integer optimal solutions, and these are union of circuits which cover the vertices of D. Eventually, for such a solution, $\mathbf{cx}$ represents exactly the sum of the indices of the circuits. Thus, we have proved that the optimum of problem (P) is exactly the min of the Theorem.

We now study the dual (P^*) of (P):

$$\begin{array}{rrcl} \textit{maximize} & \mathbf{yb} & & \\ \textit{subject to} & \mathbf{yA} & \leq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

Let us write $\mathbf{y} := (z_1, \ldots, z_n, w_1, \ldots, w_n)$. Then (P^*) is the problem of maximizing $\sum_{i=1}^{n} w_i$ subject to the constraints:

$$z_i - z_k + w_i \leq \begin{cases} 1 & \text{if } a_j := (v_i, v_k) \text{ is a back arc} \\ 0 & \text{if } a_j \text{ is a forward arc} \end{cases}$$

Consider an integral optimal solution to (P^*). If we sum the above constraints over the arc set of a circuit C of D, we obtain the inequality

$$\sum_{v_i \in V(C)} w_i \leq i(C).$$

In other words, the function w defined by $w(v_i) := w_i$, $1 \leq i \leq n$, is an index-bounded weighting, and the optimal value is the weight $w(D)$ of D.

But this means that every dual solution represent an index-bounded weighting, and by strong duality, there exists one optimal solution such that the objective value is equal to the min of the primal. Thanks to the remark done at the beginning of the proof, this concludes the proof of Theorem 3.4.

■

3.5 Feedback and Cyclic Stable Sets

3.5.1 Cyclic Feedback Arc Sets

The notion of feedback arc set is natural when one studies cyclic orders. Indeed, if a digraph is given with a linear order on its vertices, and if we consider the set of backward arcs, then this set is clearly a feedback arc set. And conversely, it is easy to check that if we are given a digraph with a feedback arc set, then there exists a linear order in which all backward arcs are in this feedback arc set - there is not always equality here since a feedback arc set can contain a circuit. Meanwhile, if we are interested in feedback arc sets that are minimal with respect to inclusion (we denote these by min FAS) there is equivalence.

Proposition 3.2 *Let $D = (V, A)$ be a digraph, and $F \subset A$.*

(i) *F is a min FAS $\Rightarrow$ there exists a linear order O such that $F =$ Ret(O).*

(ii) *there exists a linear order O belonging to a coherent cyclic order such that $F =$ Ret(O) $\Rightarrow$ F is a min FAS*

Proof
(i) Since F is a min FAS, there exists a linear order O such that all backward arcs are in F. If $a \in F$ is forward, then $F \setminus A$ still contains all backward arcs and thus it is a feedback arc set which contradicts the minimality of F.
(ii) The assumption implies that F is a feedback arc set. If F' is a minimal feedback arc set and is a strict subset of F then let us consider a linear order O' in which all its arcs are backward. The winding number of a circuit C in this order is given by $C \cap F'$, and thus the cyclic order to which it belongs,

is strictly less than the one containing O for the partial order defined in 3.2, which contradicts the coherence by Proposition 3.1 ■

If $\mathcal{O}$ is a cyclic order, let us define $\text{Ret}(\mathcal{O}) = \{\text{Ret}(O), O \in \mathcal{O}\}$, that is the sets of arcs that are backward in some linear order belonging to $\mathcal{O}$.

cyclic feedback arc set

In fact, a set of backward arcs satisfies the stronger property of intersecting each circuit as many times as its index. This leads to the following definition : a set of arcs F in a digraph D is a *cyclic feedback arc set* if there exists a cyclic order $\mathcal{O}$ on the vertices of D such that for every circuit C, $|F \cap C| \geq \text{ind}_{\mathcal{O}}(C)$. Then we define the set $\text{CycFAS}(\mathcal{O})$ which contains all set or arcs that are cyclic feedback arc sets for $\mathcal{O}$. Thus this gives for any cyclic order $\mathcal{O}$ the following chain of inclusions (which can be strict)

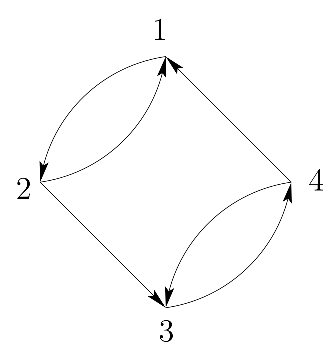
$$\text{Ret}(\mathcal{O}) \subset \text{CycFAS}(\mathcal{O}) \subset \text{FAS}(D)$$

The fact that for every feedback arc set F, there is a linear order in which all backward arcs are in F, associated with the first inclusion implies that :

$$\text{FAS}(D) = \bigcup_{\mathcal{O}} \text{CycFAS}(\mathcal{O})$$

where the union is taken over all cyclic orders.

To see that an element of $\text{Ret}(\mathcal{O})$, even if it is minimal with respect to inclusion, does not necessarily belongs to $\text{CycFAS}(\mathcal{O})$, it suffices to consider the following digraph :



$\{12, 34\}$ is a cyclic feedback arc set that is minimal with respect to inclusion but these two arcs cannot be simultaneously backward in any equivalent order since this representation is unique (up to rotations).

Meanwhile, one can say when equivalence holds between cyclic feedback arc set minimal with respect to inclusion and set of backward arcs. The previous example is a coherent cyclic order, so we see that it is not the good condition. In fact, the good orders are called anticoherent, let us define this notion now. If $\mathcal{O}$ is a cyclic order that contains the linear order $(v_1, \dots, v_n)$, the *inverse order* $\overline{\mathcal{O}}$ is simply defined as the cyclic order containing the linear order $(v_n, \dots, v_1)$. A cyclic order is then said to be *anticoherent* if its inverse is coherent. Given a cyclic order $\mathcal{O}$, since the winding number of a circuit is equal to the number of its arcs that are backward in any linear order belonging to $\mathcal{O}$, the following holds for every circuit C, $\mathrm{ind}_{\mathcal{O}}(C) = |C| - \mathrm{ind}_{\overline{\mathcal{O}}}(C)$. The anticoherent cyclic orders are thus the maximal ones with respect to the partial order defined in 3.2.

inverse order
anticoherent

Proposition 3.3 *Let $D = (V, A)$ be a digraph endowed with a anticoherent cyclic order $\mathcal{O}$ and $F \subset A$. The two following assertions are equivalent.*

i) *F is a cyclic feedback arc set*

ii) *There exists a linear order O belonging to $\mathcal{O}$such that $Ret(O) \subset F$,*

equality in **ii)** *corresponding to feedback arc sets that are minimal with respect to inclusion.*

In order to prove this, we are going to choose the completely equivalent but more natural point of view of coherent cyclic orders. Note that F is a cyclic feedback arc set for $\mathcal{O}$ if and only if its complement $F' = A \setminus F$ satisfies $|F' \cap C| \leq \mathrm{ind}_{\overline{\mathcal{O}}}(C)$ for every circuit C. We are proving the following equivalent proposition.

Proposition 3.4 *Let $D = (V, A)$ be a digraph endowed with a coherent cyclic order $\mathcal{O}$ and $F \subset A$. The two following assertions are equivalent.*

i) *For every circuit C of D, $|F \cap C| \leq ind(C)$*

ii) *There exists a linear order O belonging to $\mathcal{O}$ such that $F \subset Ret(O)$,*

equality in **ii)** *corresponding to subsets F that are maximal with respect to inclusion for the property* **i)**.

Proof

ii)$\Rightarrow$ **i)** is straightforward, so we need only to prove **i)** $\Rightarrow$ **ii)**. To achieve this, it suffices to study the case where F is maximal with respect to inclusion of property **i)**. We begin to prove that in that case F is a feedback arc set, that is F intersects every circuit. Assume by contradiction, that there exists a circuit C a circuit such that $C \cap F = \emptyset$. Then by maximality of F, every arc a of C is contained in some circuit C_a such that $\text{ind}(C_a) = F \cap C_a$ (otherwise $F \cup \{a\}$ would satisfy condition **i)**). Now, if we consider the circulation C' given by $C' = \sum_{a \in C} C_a - C$ (we again use the notation already used in section 3.3 which consists in identifying circuits with the corresponding vector in $\{0,1\}^A$), then $\text{ind}(C') < \sum_{a\in A} \text{ind}(C_a) = \sum_{a \in A} |F \cap C_a| = \sum_{a\in A} |F \cap (C_a \setminus \{a\}|$. And this contradicts **i)** for this sum of circuits C'. Therefore $A \setminus F$ induces an acyclic digraph, and there exists a linear order O in which all backward arcs belong to F. Since the winding number of a circuit in O is the number of its backward arcs, then the winding number of every circuit is less than the size of its intersection with F and thus it is less than the winding number in $\mathcal{O}$. Since $\mathcal{O}$ is coherent, by Proposition 3.1, this implies that O belongs to $\mathcal{O}$ and that the winding number of every circuit is equal to the number of its arcs that belong to F. It is now clear that this implies $F = \text{Ret}(O)$. ∎

To conclude this paragraph, we want to state a min-max theorem involving cyclic feedback arc sets, but we need in order to do this to introduce the relaxation of this notion by defining a *cyclic feedback weighting of the arcs* as a function $f : A \to \mathbb{N}$ such that for every circuit C, the following inequality holds : $f(C) \geq \text{ind}_{\mathcal{O}}(C)$. This result can be seen as some sort of cyclic Erdős-Posà property.

cyclic feedback weighting of the arcs

Theorem 3.5 *Let $D = (V, A)$ be a digraph endowed with a cyclic order $\mathcal{O}$. Then*

$$\begin{aligned} & \max \quad \{i(\mathcal{C}) : \ \mathcal{C} \text{ collection of arc-disjoint circuits } D\} \\ = & \min \quad \{f(A) : \ f \text{ cyclic feedback weighting of the arcs of } D\}. \end{aligned}$$

Proof

Let us define $n = |V|$ and $m = |A|$. We will not give the details of this proof, since it is identical to the proof of Theorem 3.4. It suffices to replace the optimization problem by the following :

$$\begin{array}{rrcl} \textit{maximize} & \mathbf{cx} & & \\ \textit{subject to the constraints} & \mathbf{Tx} & \leq & \mathbf{d} \\ & \mathbf{x} & \geqq & \mathbf{0} \end{array}$$

where $\mathbf{T}$ is the totally unimodular matrix with $(n+m)$ rows and m columns defined by :

$$\mathbf{T} = \begin{bmatrix} \mathbf{M} \\ \mathbf{I_m} \end{bmatrix}$$

$\mathbf{M}$ denoting as before the vertex-arcs incidence matrix of D and $\mathbf{I_m}$ the identity matrix of size m.
As in the proof of Theorem 3.4, c is the characteristic vector of the backward arcs of some linear order belonging to $\mathcal{O}$ and $\mathbf{d} = (d_1, \ldots, b_{n+m})$ is defined by :

$$d_i = \begin{cases} 0 & \text{if } 1 \leq i \leq n \\ 1 & \text{otherwise} \end{cases}$$

■

3.5.2 Index-Bounded Weightings of the Arcs

In a similar way as we did for index-bounded weightings of the vertices, we define *index-bounded weightings of the arcs* of $D = (V, A)$ as functions $\omega : A \rightarrow \mathbb{N}$ such that for every circuit C, $\omega(C) \leq \text{ind}(C)$. We can state the following min-max result.

index-bounded weightings of the arcs

Theorem 3.6 *Let D be a strongly connected digraph and $\mathcal{O}$ a cyclic order on its vertices. Then*

$$\begin{array}{rl} & \min \; \{i(\mathcal{C}) : \; \mathcal{C} \textit{ a collection of circuits covering of the arcs of } D\} \\ = & \max \; \{w(A) : \; w \textit{ index-bounded weighting of the arcs of } D\}. \end{array}$$

Proof The proof is identical to the one of Theorem 3.5 if one replaces the optimization problem by the following (with the same notations) :

$$\begin{array}{rrcl} \textit{minimize} & \mathbf{cx} & & \\ \textit{subject to the constraints} & \mathbf{Tx} & \geqq & \mathbf{d} \\ & \mathbf{x} & \geqq & \mathbf{0} \end{array}$$

■

For index-bounded weightings of the vertices, we have seen before that if the order is coherent, it is possible to describe things more precisely, that was how we deduced Gallai Conjecture from Theorem 3.4. In the arc weighting case, since every arc is contained in a circuit of index 1, being in a coherent cyclic order simply means that the weighting takes its values in $\{0, 1\}$. Therefore it is just a subset of arcs F satisfying :

$$\text{for every circuit } C,\ |F \cap C| \leq \text{ind}(C) \tag{3.3}$$

This is exactly the set of arcs involved in Proposition 3.4, the maximal subsets satisfying 3.3 being exactly the set of backward arcs. Eventually, this allows us to deduce from Theorem 3.6 the following corollary that was already stated by Bessy and Thomassé in [15]):

Corollary 3.1 *Let D be a strongly connected digraph and $\mathcal{O}$ a coherent cyclic order on its vertices. Then*

$$\begin{aligned} &\min \ \{i(\mathcal{C}) :\ \mathcal{C} \text{ a collection of circuits covering the arcs of } D\} \\ = &\max \ \{|F| :\ F = Ret(O) \text{ for some linear order } O \text{ belonging to } \mathcal{O}\}. \end{aligned}$$

3.5.3 Cyclic Stability

cyclic stable set

We want in this paragraph to come back on the notion of index-bounded weighting introduced in the proof of Gallai's conjecture exposed in last section. We have seen that this notion, defined for any cyclic order, had a particular interpretation in the case of a coherent one. In that case, an index-bounded weighting is simply a stable set of the digraph. In [15], the authors speak about *cyclic stable set* for that notion. In fact they do not define it that way, for them a cyclic stable set being a stable set such that there exists some linear order in the given cyclic order, in which all of its vertices are consecutive. In the case of a coherent cyclic order, it is equivalent to being the support of an index-bounded weighting. This result was proven by A. Sebő in [44] but the proof given here is different.

Proposition 3.5 *Soit $D = (V, A)$ be a digraph endowed with a coherent cyclic order $\mathcal{O}$ and $S \subset V$ a stable set. The two following assertions are equivalent.*

i) *There exists a linear order in the cyclic order $\mathcal{O}$ in which the vertices of S are in consecutive order.*

ii) *For every circuit C, $|C \cap S| \leq ind_{\mathcal{O}}(C)$.*

Proof
i) ⇒ ii) is the easy part. The vertices of S are consecutive and constitute a stable set, thus a circuit has to turn once around the circle to go from one vertex of S to another.
ii) ⇒ i). Let S be a stable set satisfying **ii)**. We begin to note that this implies that the set F of arcs whose end is in S satisfies $|F \cap C| \leq \mathrm{ind}_{\mathcal{O}}(C)$ for every circuit C. Thus, Proposition 3.4 implies the existence of a linear order O in $\mathcal{O}$ such that the arcs of F are all backwards. Let $s \in S$, assume that there is an arc linking s to one of the vertices that are preceding s in the order. By what we have just said this can only be an arc sx going out of s. But by coherence, this arc belongs to a circuit of winding number 1, so there is a path from x to s with only forward arcs, and thus there is one forward arc entering s, which gives a contradiction. Therefore, there is no arc between s and any of its predecessors. Hence, it is possible by elementary switches to bring s to the beginning of the order while staying in the same cyclic order and we have proved that S satisfies **i)**. ■

In the same way as we did for arcs, we can define a cyclic feedback vertex set and state a min-max theorem similar to Theorem 3.5.

Theorem 3.7 *Let $D = (V, A)$ a digraph endowed with a cyclic order $\mathcal{O}$ on its vertices. Then*

$$\begin{aligned} & \max \quad \{i(\mathcal{C}) : \ \mathcal{C} \textit{ a set of vertex-disjoints circuits of } D\} \\ = & \min \quad \{f(D) : \ f : V \to \mathbb{N} \textit{ such that } f(C) \geq ind(C) \textit{ for every circuit } C\}. \end{aligned}$$

The proof is again identical to the previous ones.

It is possible to state other min-max results as well as generalizations of these. For more details, we invite the readers to refer to [44].

Chapter 4

Cyclic Length and Embeddings

4.1 Introduction

Let $D = (V, A)$ be a digraph endowed with a cyclic order $\mathcal{O}$. We define the *cyclic length* of a circuit C of D with respect to $\mathcal{O}$ as $l_O(C) := |C|/\operatorname{ind}(C)$, where $|C|$ denotes the number of arcs of C, that is the usual length. In this section we will focus on different min-max theorems involving that notion. We will start to study the minimum cyclic length of a circuit. This study is motivated by the Caccetta-Häggkvist Conjecture, which can be stated as follows : if δ^+ denotes the minimum outdegree of a digraph, and g its girth, that is the minimum size of a directed circuit, then $g \leq \lceil n/\delta^+ \rceil$. We will see how this translates when one considers the cyclic version of these notions. cyclic length

Before starting the chapter, we want to make a first trivial remark about the cyclic length. Here, it is equivalent to study the maximum cyclic length or the minimum one, which is totally different from the usual length. Indeed, if $(v_1, \ldots, v_n)$ belongs to the cyclic order $\mathcal{O}$ and if we denote by $\overline{\mathcal{O}}$ the inverse cyclic order (that is the one containing the linear order $(v_n, \ldots, v_1)$) then for every circuit C of D, $1/l_{\mathcal{O}}(C) = 1 - 1/l_{\overline{\mathcal{O}}}(C)$. Therefore, studying the minimum cyclic length in a cyclic order is the same as studying the maximum in its inverse. Thus, all the results we are exposing concerning this notion can be translated from max to min or vice-versa depending the point of view we have chosen. Nevertheless, everything is not strictly equivalent as long as we make assumptions on the cyclic order, since for example the inverse of a coherent cyclic order is usually not coherent. In fact we will in this chapter go from one point of view to another, in order for instance to give a simplified proof of a result first proven by Bessy and Thomassé concerning the maximum cyclic length of a circuit in a coherent cyclic order, this being related to the notion of circular colouring (see section 4.3).

All through this chapter, the digraphs are assumed to be strongly connected.

4.2 Cyclic Girth

cyclic girth
$g_c(\mathcal{O})$

As we explained it in the introduction to this chapter, we want in this section to study the *cyclic girth* of a digraph D endowed with a cyclic order $\mathcal{O}$. We use the notation $g_c(\mathcal{O})$, and it is defined as the minimum cyclic length over directed circuits.

First, we want to link this notion to the Caccetta-Häggkvist Conjecture. Assume that $(x_1, \dots, x_n)$ is a linear order that belongs to some cyclic order $\mathcal{O}$ of D. To every vertex x_i, associate the vertex $\phi(x_i)$ which is as far as possible from x_i in the cyclic order. Observe that $]x_i, \dots, \phi(x_i)]$ has at least δ^+ vertices. Consider now a circuit $C = (x, \phi(x), \phi^2(x), \phi^k(x) = x)$. By the observation, we directly have that the index of C is at least $k\delta^+/n$. In particular we have $l_{\mathcal{O}}(C).\delta^+ \leq n$. Thus $g_c(\mathcal{O}) \leq \frac{n}{\delta^+}$ and the Caccetta-Häggkvist conjecture holds if one replace 'girth' by 'cyclic girth'. But of course, the inequality is in the wrong direction.

We want to get a duality result involving g_c. Let $O = (v_1, \dots, v_n)$ be a linear order that belongs to the cyclic order $\mathcal{O}$ on V. Recall that for an arc (v_i, v_j), the length is defined as $(j-i)/n$ if $j > i$ and $(n+j-i)/n$ otherwise. Thus, if O is such that the length of each arc is no more than r then by summing the lengths over a circuit C, we have $\text{ind}_{\mathcal{O}}(C) \leq r|C|$ and thus $1/l_c(C) \leq r$.
Therefore, we have the inequality

$$\frac{1}{g_c(\mathcal{O})} \leq \min\{r \ : \ \exists O \in \mathcal{O} \text{ in which every arc has length at most } r\} \quad (4.1)$$

One could hope that equality holds. Unfortunately, the example on Figure 4.1, where the cyclic order is represented counter clockwise, disproves this.

Indeed in this case since there are arcs for every pair of consecutive vertices, this is the only enumeration in the cyclic order (apart from the ones obtained by a rotation), and the central arc has length 1/2, whereas the cyclic girth is 1/3.

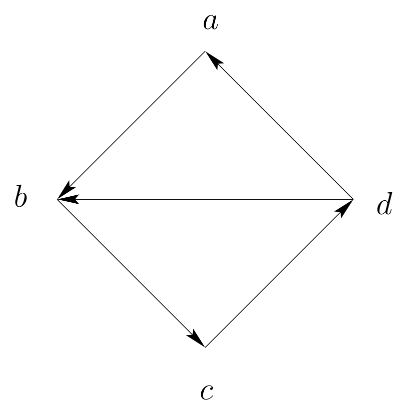


Figure 4.1: A first counter-example

Here we see that the problem is the "rigidity" of a cyclic order as an embedding of a digraph onto the circle. Thus, we want to consider more general embeddings, given by functions $f : V \longrightarrow [0, 1[$. For such a function, an arc $a = xy$ such that $f(x) < f(y)$ is again called a *forward arc*, otherwise it is called a *backward arc*.
Furthermore, the *length* of an arc $a = xy$ with respect to f is naturally defined as

$$l_f(a) = \begin{cases} f(y) - f(x) & \text{if } a \text{ is forward} \\ 1 + f(y) - f(x) & \text{if } a \text{ is backward} \end{cases}$$

Again the *index* of a circuit with respect to such an embedding is simply the sum of the lengths of its arcs.
We say that two embeddings are *equivalent* if for every circuit, the index is the same with respect to the two embeddings. Note that a linear order $O = (v_1, \ldots, v_n)$ that belongs to a cyclic order $\mathcal{O}$ can be seen as an embedding f by $f(v_i) = (i-1)/n$. By extension, we say that an embedding is equivalent to a cyclic order $\mathcal{O}$, if it is equivalent to any of these canonical embeddings of $\mathcal{O}$.

It is easy to check that the analog of inequality 4.1 remains valid, that is :

$$\frac{1}{g_c(\mathcal{O})} \leq \min\{r \,:\, \exists f \text{ equivalent to } \mathcal{O} \text{ in which every arc has length at most } r\} \tag{4.2}$$

Again, one could ask about equality. The answer is again no. Before giving a counterexample, we will make a remark about these embeddings. Observe that the mapping $l_f : A \rightarrow]0,1]$ is such that the index of every circuit is integer-valued. The following lemma gives the equivalence:

Lemma 4.1 *If $\omega : A \rightarrow]0,1]$ is such that $\omega(C)$ is an integer for every circuit C, then there exists $f : V \longrightarrow [0,1[$ such that $\omega = l_f$.*

Proof

Let us fix a vertex r of D, and consider a spanning out-branching T rooted at r. Fix now $f(x) := \omega(P_{rx}) \bmod 1$ where P_{rx} is the unique directed (r,x) path of T and f has values in $[0,1[$. We claim that $\omega = l_f$. To see this, observe that since D is strongly connected, every directed path can be completed in a directed closed walk with integer weight. Thus, if P and P' are both (x,y) directed paths, $\omega(P) \equiv \omega(P') \bmod 1$. Then if $xy \in A$, $f(y) - f(x) \equiv \omega(P_{ry}) - \omega(P_{rx}) \bmod 1 \equiv \omega(xy)$ since P_{ry} and $P_{rx} \cup xy$ are (x,y) paths. Since $\omega(xy)$ belongs to $]0,1]$ and $(f(y) - f(x))$ belongs to $]-1,1[$, we have the equality $l_f(xy) = \omega(xy)$.

■

Thus, considering an embedding of the vertices is the same as considering the weighting of the arcs given by their length. We can rewrite the inequality 4.2 this way :

$$\frac{1}{g_c(\mathcal{O})} \leq \min\{\max_{a\in A} \omega(a) \;:\; \omega :\, A \rightarrow]0,1] \text{ equivalent to } \mathcal{O}\} \tag{4.3}$$

where ω is said to be equivalent to $\mathcal{O}$ if for every circuit $\omega(C) = \mathrm{ind}_{\mathcal{O}}(C)$.

The example given on Figure 4.2 proves that this is not an equality. Here the cyclic girth is equal to 3 and suppose we are able to find ω bounded by $1/3$ equivalent to this order. Since the two triangles have index 1, this implies that the arcs $(a,b), (b,c), (c,d), (d,e)$ have weight precisely $1/3$. But it is then impossible to find a non-negative weight on (e,a) such that the

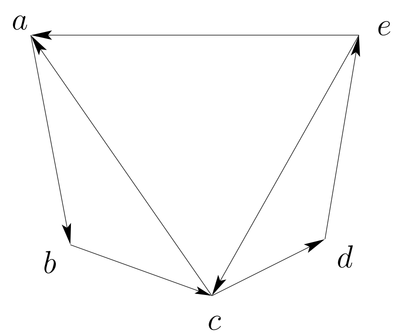


Figure 4.2: A second counter-example

circuit *abcde* has weight 1.
However, if we allow ω to take negative values we see that it is possible here to find a good ω (just take $-1/3$ for (e, a) and $1/3$ for all other arcs).

Thus, we can ask if the following holds

$$\frac{1}{g_c(\mathcal{O})} = \min\{\max_{a \in A} \omega(a) \ : \ \omega : A \rightarrow]-\infty, 1] \text{ equivalent to } \mathcal{O}\} \quad ? \qquad (4.4)$$

Now the answer is yes. Let us define the *cyclic maximum degree* of a cyclic order $\mathcal{O}$ of a graph by :

$$\delta_c(\mathcal{O}) = \min\{\max_{a \in A} \omega(a) \ : \ \omega : A \rightarrow]-\infty, 1] \text{ equivalent to } \mathcal{O}\}$$

Theorem 4.1 *Let D be a digraph, and $\mathcal{O}$ a cyclic order on its vertices.*

$$\delta_c(\mathcal{O}) g_c(\mathcal{O}) = 1.$$

We will use here a simple lemma that is a straightforward consequence of Farkas Lemma.

Lemma 4.2 *Let $p : A \longmapsto \mathbb{R}$ be a weighting of the arcs of a digraph $D = (V, A)$ such that for every circuit C of D, $\sum_{a \in C} p(a) \geqq 0$. then there exists $z : V \longmapsto \mathbb{R}$ such that*

$$\forall\, a = (x, y) \in A \quad z(y) - z(x) \leq p(a).$$

Proof of Theorem 4.1
The inequality $\delta_c(\mathcal{O})g_c(\mathcal{O}) \geq 1$ follows from the remark made at the beginning of the section. Indeed, if ω is equivalent to $\mathcal{O}$ and satisfies $\omega(a) \leq \delta_c(\mathcal{O})$ for each arc, then for every circuit $\operatorname{ind}(C) \leq \delta_c(\mathcal{O})|C|$ and thus $1/g_c(\mathcal{O}) \leq \delta_c(\mathcal{O})$.

Let us prove the opposite inequality. Let $E = (v_1, \dots, v_n)$ be an enumeration of $\mathcal{O}$, and $\varepsilon : A \longmapsto \{0, 1\}$ defined by $\varepsilon(a) = 1$ if a is backward in E and 0 otherwise. Since for every circuit C, $g_c(\mathcal{O}) \leq |C|/\operatorname{ind}_{\mathcal{O}}(C)$, if we define $p(a) = 1/g_c(\mathcal{O}) - \varepsilon(a)$, then p is non-negative on all circuits. Therefore, Lemma 4.2 implies the existence of $f : V \longmapsto \mathbb{R}$ such that if we define $\omega : A \longmapsto \mathbb{R}$ by $\omega(a) = f(y) - f(x) + \varepsilon(a)$ for every arc $a = (x, y)$, then $\omega(a) \leq 1/g_c(\mathcal{O})$. Furthermore, for every circuit C :

$$\sum_{a \in C} \omega(a) = \sum_{a=(x,y) \in C} (f(y) - f(x)) + \sum_{a \in C} \varepsilon(a) = \operatorname{ind}_{\mathcal{O}}(C)$$

and ω is equivalent to $\mathcal{O}$. ■

We have just seen that 4.2 (or 4.3) was not valid for any cyclic order and that we had to replace $[0, 1]$-valued weightings by $]-\infty, 1]$-valued ones. Meanwhile, we can describe cases where 4.2 holds. These are orders we have already seen, the anticoherent orders, that are inverse of coherent ones. This will be the subject of next section. But we will change the point of view for this and prove the result in terms of maximum cyclic length, what we have seen in the introduction to this chapter to be equivalent. We have chosen to do this for several reasons :

- First, because this result has already been proved by Bessy and Thomassé in [15] with a purely combinatorial approach, and similarly to the proof of Gallai Conjecture, we are exposing here a simpler proof using duality in linear optimization.
- Then, because it is interesting to show that in this case these notions are related to the more classical one of circular chromatic number.
- Last, because if we had chosen to focus on minimum cyclic length at the beginning of this chapter in order to make a connection with Caccetta-Häggkvist Conjecture, it seemed that now that we have to study particular cases, it would be more natural to consider coherent cyclic orders instead of anticoherent ones.

4.3 Maximum Cyclic Length and Cyclic Colourings

As we explained it before, we are in this section studying the maximum cyclic length over all circuits of a given digraph endowed with a cyclic order. When we studied the minimum cyclic length, we put this quantity in duality with embeddings in which the length of an arc was no more than r, while trying to minimize r. Now that we are working on the inverse cyclic order, we are naturally going to consider embeddings for which every arc has a length at least r and try to maximize r.

In fact, we will study the equivalent problem stated below. We want to embed the vertices of a digraph on a circle with the smallest perimeter r such that each arc has length at least 1. As we are in the coherent case, this implies also that each arc has length at most $r-1$. Indeed it is contained in a circuit of winding number 1 and the arc following it must have also length at most 1 by assumption. In other words, if we are in the coherent case, assuming that the arcs have length at least one is the same as assuming that in the underlying non-oriented graph, the endpoints of every edge are at distance at least one on the circle. So this is completely a problem of non-oriented graphs, and a known one since we can recognize here the notion of circular coloration (this is in order to fit with its usual definition that we have chosen to present the problem in this equivalent context - instead of fixing the perimeter of the circle and trying to maximize the length of the arcs, we fix the length of the arcs and try to minimize the perimeter of the circle).

Let $G=(V,E)$ be a non-oriented graph. In [49], Zhu defines a r-*circular coloration* of G as a function $f: V \longrightarrow [0,r) = (\{x \in \mathbb{R} : 0 \leq x < r\})$, satisfying for every edge $(x,y) \in E$: $\mathrm{dist}(x,y) \geq 1$, where $\mathrm{dist}(x,y) := \mathrm{dist}_{f,r}(x,y) := \min\{|f(x)-f(y)|, r-|f(x)-f(y)|\}$ – the *distance* between x and y on the circle of perimeter r. circular coloration

If the graph is endowed with a cyclic order $\mathcal{O}$, then a r-circular coloration is called a r-*cyclic coloration* (with respect to $\mathcal{O}$) if in addition the order $(v_1, v_2, \ldots, v_n)$, given by $0 \leq f(v_1) \leq f(v_2) \leq \ldots \leq f(v_n) < r$ belongs to $\mathcal{O}$. We will also use these notions for digraphs as soon as these properties are satisfied for the underlying non-oriented graph. cyclic coloration

circular chromatic number
cyclic chromatic number
$\chi_c(\mathcal{O})$

The minimum real number $r > 0$ such that G admits a r-circular coloration is called the *circular chromatic number* , and is denoted by $\chi_{\text{circ}}(D)$. Similarly the minimum real number $r > 0$ for which there exists a r-cyclic coloration with respect to $\mathcal{O}$ is called the *cyclic chromatic number* and is denoted by $\chi_c(\mathcal{O})$. It is easy to check that $\chi_{\text{circ}} \leq \chi \leq \lceil \chi_{\text{circ}} \rceil \leq \lceil \chi_{\mathcal{O}} \rceil$, where $\chi = \chi(D)$ is the usual chromatic number.

It is clear that $\chi_{\mathcal{O}}$ is bounded below by the cyclic length of any circuit. Indeed, if we are given a r cyclic coloration and if C is a circuit, then $r \text{ ind}_{\mathcal{O}}(C) = \sum_{xy \in C} l_f(xy) \geq |C|$. The following theorem establishes the duality between these two quantities.

Theorem 4.2 (Bessy-Thomassé) *Let $D = (V, A)$ be a strongly connected digraph given with a coherent cyclic order $\mathcal{O}$ of V. Then*

$$\chi_{\mathcal{O}}(D) = \max\{l_{\mathcal{O}}(C) : \ C \textit{ circuit of } D\}.$$

Remarques :

- Apart from the change of context (min becomes max), the first step of this proof is contained into the proof of Theorem 4.1. It is in the proof of Claims 2 and 3 that we will really use coherence and where new ideas will appear. Nevertheless we have chosen to write here the whole argument in order to have an independent proof of this interesting result.
- As it was noted by Bessy and Thomassé in [15], since for every circuit C, $|C| \geq l_c(C)$, and since $\chi(D) \leq \chi_{\mathcal{O}}(D)$, this result implies a theorem of 1971 due to Bondy ([12]) which asserts that in every strongly connected digraph there is a circuit of length at least the chromatic number.

Proof :
Let $O = (v_1, \ldots, v_n)$ be a linear order of V, that belongs to the cyclic order $\mathcal{O}$. We denote the arc set of D by $A = \{a_1, \ldots, a_m\}$.

Consider the following linear program (P) :

$$\textit{maximize} \quad \mathbf{1.x} = \sum_{i=1}^{m} x_i$$

$$\textit{subject to the constraints}\quad \begin{aligned}\mathbf{Mx} &\leq \mathbf{0}\\ \mathbf{cx} &\leq 1\\ \mathbf{x} &\geq \mathbf{0}\end{aligned}$$

where, as in the proof of Theorem 3.4, M denotes the $n \times m$ incidence matrix of D :

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } a_j \\ -1 & \text{if } v_i \text{ is the head of } a_j \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{c} = (c_1, \ldots, c_m)$ the vector of backward arcs :

$$c_j = \begin{cases} 1 & \text{if } a_j \text{ is backward} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, this linear program is feasible and bounded.

Claim 1: The primal optimum of (P) is equal to the right hand side of the theorem.

Indeed, again, $\mathbf{Mx} \leq \mathbf{0}$ implies $\mathbf{Mx} = \mathbf{0}$, so primal solutions are non-negative circulations $\mathbf{x}$, that is linear combinations of circuits. We write $\mathbf{x} = \sum_{C \in \mathcal{C}} \lambda_C C$, $(\lambda_C \geq 0)$, for some set $\mathcal{C}$ of circuits. The constraint $\mathbf{c}.\mathbf{x} \leq 1$ is equivalent to $\sum_{C \in \mathcal{C}} \lambda_C \operatorname{ind}(C) \leq 1$ and therefore :

$$\mathbf{1}.\mathbf{x} = \sum_{C \in \mathcal{C}} \lambda(C)|C| = \sum_{C \in \mathcal{C}} \lambda(C) \operatorname{ind}(C) l_O(C) \leq \max\{l_O(C) :\ C \text{ a circuit of } D\}$$

Conversely, it is clear that any circuit C of D is represented by the solution $C/\operatorname{ind}(C)$ of this problem, and the claim is now proven.

Note that Claim 1 does not use that the given order is coherent. This will be exploited for Claim 2.

Fix $(\pi_1, \ldots, \pi_n, r)$ to be a dual optimum solution. Starting with this vector we construct a cyclic colouring.

According to Claim 1, $r = \max\{l_O(C) :\ C \text{ a circuit of } D\ \} > 1$.

Claim 2: For every forward arc uv, $1 \leq \pi_v - \pi_u \leq r - 1$.
For every backward arc uv, $1 \leq \pi_u - \pi_v \leq r - 1$.

First, $(\pi_1, \ldots, \pi_n, r)$ satisfies the dual constraints for each $a = uv \in A$, that is :

$$\pi_v - \pi_u \geq \begin{cases} 1 & \text{if } uv \text{ is a forward arc} \\ 1 - r & \text{if } uv \text{ is a back arc} \end{cases} \tag{4.5}$$

Furthermore, if uv is a backward arc, by coherence, there exists a forward path P between v and u, and adding up the inequalities concerning the arcs of this path: $\pi_u - \pi_v \geq |P| - 1 \geq 1$.
Likewise, if uv is a forward arc, uv lies in a circuit C on index 1. Let $u'v'$ be the unique backward arc of C. Then $\pi_{v'} \leq \pi_u \leq \pi_v \leq \pi_{u'}$, and therefore $|\pi_u - \pi_v| \leq |\pi_{u'} - \pi_{v'}| \leq r - 1$. This finishes the proof of Claim 2.

For any dual solution $(\pi_1, \ldots, \pi_n, r)$ of (P) define $f : V(D) \longrightarrow [0, r)$ with $\pi_i =: p(v_i)r + f(v_i)$, that is, $f(v_i)$ is the remainder of π_i modulo r. It is straightforward to check that Claim 2 implies that f is a circular r-coloration.

Claim 3: The function f is a cyclic r-coloration with respect to O.

In addition to Claim 2 we have to check that the increasing order on $f(v)$, $v \in V$ defines a linear order *equivalent to* O. First note that it is easy to see from Claim 2 that the order defined by π is equivalent to O, since these two orders have the same backward arcs and we can conclude with the use of Theorem 3.3. So what we have to prove is that f defines an order equivalent to the one given by π. We will show here two proofs of this fact. We have chosen to write these two proofs since they do not use the same tools.

Method 1 :

We will prove directly that the order O_f defined by increasing order of f is equivalent to the one defined by π. In order to clarify matters, if $0 \leq \pi_{j_1} \leq \pi_{j_2} \leq \ldots \pi_{j_n}$ we use the notation $v_{j_k} = w_k$ and $\pi_{j_k} = z_k$.

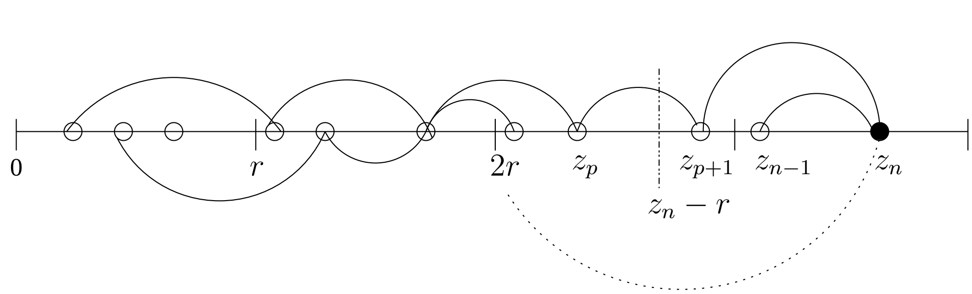


Figure 4.3: Method 1 : The vertex with affix z_n is not linked by any arc to the vertices of affixes less than $z_n - r$ ...

Suppose that $r \leq z_n$ and $0 \leq z_1 \leq z_2 \leq \ldots \leq z_p \leq z_n - r \leq z_{p+1} \leq \ldots \leq z_n$. Then $(w_1, w_2, \ldots, w_p, w_n, w_{p+1}, \ldots, w_{n-1})$ is an ordering equivalent

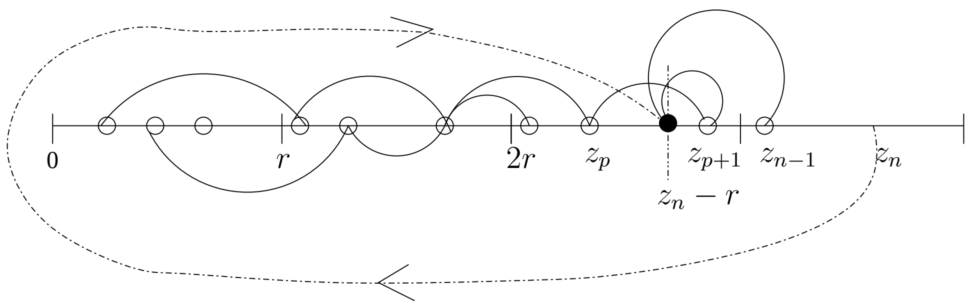


Figure 4.4: ... so we can switch it with these vertices and bring it into position $z_n - r$ while staying in the same cyclic order.

to $O_\pi = (w_1, \dots, w_n)$. Indeed, by assumption, there is no arc between w_n and any of the vertices w_i such that $z_i \leq z_n - r$, so we can switch w_n with w_1, then with w_2, and so on, until w_n has reached the position given by $z_n - r$ (see figure 4.3). Then we can apply the same operation to the vertex of maximal affix (which can still be w_n if $p = n-1$), and so on, until no z_i is greater than or equal to r. When we have reached this stage, it is clear that the position of the vertex w_i is precisely the one given by f. Thus the order defined by f is equivalent the one given by π and we are done.

Method 2 :
According to Theorem 3.2, it is sufficient to check that in the linear order O_f where the vertices are in increasing order of f, the winding numbers of the circuits are the same as in O_π.
Let C be an arbitrary circuit. Thanks to Claim 2, we know that for an arc uv, $p(u) - p(v)$ equals either -1, 0, or 1. Clearly, arcs uv with $p(u) = p(v)$ are forward arcs or backward arcs in both O_π and O_q; we also see from Claim 2 that in case $p(u) - p(v) = 1$, uv is a backward arc in O_π, and it is a forward arc in O_f; similarly, if $p(u) - p(v) = -1$, then uv is a forward arc in O_π, and a backward arc in O_f; since $\sum_{uv \in C} p(u) - p(v) = 0$, we have :

$$|\{uv \in C : p(u) - p(v) = 1\}| = |\{uv \in C : p(u) - p(v) = -1\}|,$$

that is, the number of backward arcs remains the same in O_f and O_π in every circuit, which is what we wanted to prove.

Claims 1 and 3 assert that $\max\{l_O(C) : \ C \text{ a circuit of } D\} = r \geq \chi_{\mathcal{O}}(D)$. The remark made before the statement of the theorem gives the other inequality and this concludes the proof of the theorem ∎

4.4 Back to Cyclic Girth and Study of the General Case

The result we just proved in last section can be stated in terms of cyclic girth in the following way.

Theorem 4.3 *Let $D = (V, A)$ be a strongly connected digraph endowed with an anticoherent cyclic order. Then there is an embedding f of V into the circle of perimeter 1, equivalent to $\mathcal{O}$, such that for every arc $a \in A$, $l_f(a) \leq 1/g_c(\mathcal{O})$.*

Hence in this case we have managed to establish a connection between the cyclic girth and some geometric representation of the digraph, which was not given by Theorem 4.1. The problem if we try to deduce something from this about the usual girth is that in the case of an anticoherent cyclic orders, the circuits will have large winding numbers and thus the usual girth is much smaller than the cyclic one. So we want to see if, in addition to Theorem 4.1, it is possible to obtain in the case of an arbitrary cyclic order a result that will translate the cyclic girth in terms of embeddings.

Corollary 4.1 *Let $D = (V, A)$ be a strongly connected digraph endowed with a cyclic order $\mathcal{O}$. Then there exist functions $f : V \longmapsto [0, 1[$ and $p : A \longmapsto]-\infty, 1] \cap \mathbb{Z}$ such that p is a weighting of the arcs equivalent to $\mathcal{O}$ and :*

$$\forall\, a = (x, y) \in A \quad f(y) - f(x) \leq 1/g_c(O) - p(a)$$

Proof :
Let ω be given by Theorem 4.1. By Lemma 4.1, there exists $f : V \longmapsto [0, 1[$ such that $\forall a \in A$, $l_f(a) = \omega(a) - \lceil \omega(a) - 1 \rceil$. Let $\varepsilon : A \longmapsto \{0, 1\}$ be the function defined by $\varepsilon_f(a) = 1$ if a is backward in f and 0 otherwise. Then

$$f(y) - f(x) = \omega(a) - \lceil \omega(a) - 1 \rceil - \varepsilon_f(a) \leq \delta_c(\mathcal{O}) - p(a)$$

where $p(a) = \lceil \omega(a) - 1 \rceil + \varepsilon_f(a)$ has values in the desired set. Moreover, the first equality implies that p is equivalent to $\mathcal{O}$.

■

$\delta_c(\mathcal{O})$-embedding

Such a pair (f, p) is called a $\delta_c(\mathcal{O})$-*embedding* of $(D, \mathcal{O})$.

Remark 4.1 *If $p(a) = 1$, then $f(y) < f(x)$ since $1/g_c(\mathcal{O}) < 1$. Hence for every circuit C :*

$$ind_{\mathcal{O}}(C) = \sum_{a \in C} p(a) \leq |\{a \in C \,:\, p(a) = 1\}| \leq \; ind_f(C).$$

Therefore, the embedding f that we get from this corollary represents a cyclic order greater that $\mathcal{O}$ for the partial order defined in 3.2. For example, since anticoherent orders are maximal elements for this order, it means that in this case f is equivalent to $\mathcal{O}$.

Let (f, p) given by the previous corollary. What properties can we deduce from this embedding? Before going further we need a definition:
For every $\alpha \in \mathbb{R}$, we define $g : V \longrightarrow [0,1[$ by $g(x) \equiv f(x) + \alpha \bmod 1$ and p' : $A \longrightarrow \mathbb{R}$ by $p'(a) = p(a) + \epsilon$, where $\epsilon = \begin{cases} 1 & \text{if } a \text{ is forward in } f \text{ and backward in } g \\ -1 & \text{if } a \text{ is backward in } f \text{ and forward in } g \\ 0 & \text{otherwise} \end{cases}$.
(g, p') is called an α-shift of (f, p). Observe that (g, p') is also a $\delta_c(\mathcal{O})$-embedding of $(D, \mathcal{O})$.

The remark that we want to make is the following :

Proposition 4.1 *Let (f, p) be a $\delta_c(\mathcal{O})$-embedding of $(D, \mathcal{O})$. For every interval $I = [a, b[$ of $[0, 1[$ of length at least δ_c, the set $f(V) \cap I$ is a feedback vertex set, that is $D \setminus X$ is acyclic.*

Proof
Free to consider a $(-a)$-shift of f, we can assume that $I = [0, d[$, where $d \geq \delta_c(\mathcal{O})$. Assume for contradiction that C is a circuit of $D \setminus X$. For every arc $a = (x, y)$ of C, we have $f(y) - f(x) > \delta_c(\mathcal{O}) - 1$. By definition of f, we also have $f(y) - f(x) \leq \delta_c(\mathcal{O}) - p(a)$. In particular, $p(a) < 1$. Since p is integer-valued, we have $p(a) \leq 0$ for every arc of C, which is impossible since $p(C) = \text{ind}(C) \geq 1$.

■

This permits us to prove the following theorem.

Theorem 4.4 *Let $D = (V, A)$ be a digraph. The two following assertions are equivalent.*

1. *There exists a linear order of the vertices of D such that for every circuit* $|\{\textit{forward arcs of } C\}| \geq |C|/k$.

2. *There exists a partition of the vertices of D into k acyclic digraphs.*

Proof
Suppose that (1) is true. It means that there is an order O on the vertices of D such that $g_c(\mathcal{O}) \geq \frac{k}{k-1}$ and thus $1-\delta_c(O) \geq \frac{1}{k}$. If so, let us consider a $\delta_c(\mathcal{O})$-embedding (f,p) of $(D,\mathcal{O})$. By Proposition 4.1, $f([0,1/k[)$, $f([1/k,2/k[)$, $\ldots$, $f([1-1/k,1[)$ gives a partition of V into acyclic subgraphs.
Conversely, assume that (2) is true. For an acyclic digraph,we know that we can find an order on the vertices such that all arcs are forward arcs. If we partition D into $(D_1,\ldots,D_k)$ acyclic subgraphs, let us call O_i these orders for the subgraphs D_i, and define an order O on the vertices of D by $O = O_1, O_2, \ldots, O_n$. In this order backward arcs are only arcs (x,y) where $x \in D_i$ and $y \in D_j$, with $i > j$. Thus O satisfies (1). ■

For example, in the case $k = 2$ we get that the existence of a cyclic order $\mathcal{O}$ with $g_c(\mathcal{O}) \geq 2$ is equivalent to the ability to partition the vertex set into two acyclic subgraphs. This could be a way of proving the following conjecture :

Conjecture 4.1 (V. Neumann-Lara) *If D is a planar digraph, there is a partition of its vertex set into two acyclic subgraphs.*

4.5 Inversion of circuits

To finish this chapter, we will prove a result related to the same subject. We are interested into digraphs that can be obtained through a sequence of circuits inversions, that is taking a directed circuit and reversing the orientation of its arcs. We prove the following result :

Theorem 4.5 *For every digraph, there exists a sequence of circuit inversions such that the digraph obtained admits a partition into two acyclic subgraphs.*

Proof
Let $D = (V,A)$ be a digraph, we use the following notations : $n = |V|$ and $m = |A|$.
Using Theorem 4.4, if we are given a cyclic order on the vertices, it suffices

to prove that after a sequence of circuits inversions we can get a digraph in which the cyclic girth in this order is at least 2. If $\mathcal{O}$ is cyclic order on the vertices of D, consider the function of $\mathcal{O}$ given by $\sum_{a \in A} l(a)$, where $l(a)$ is the length of arc a in this order. When one reverses the orientation of a circuit C with cyclic length at most 2, then this quantity decreases by at least $1/n$. Indeed, reversing an arc means replacing its length $l(a)$ by $(1 - l(a))$ in the above sum and thus

$$\sum_{a \in C}(1 - l(a)) = |C| - ind(C) < ind(C) - 1/|C| \leq \sum_{a \in C} l(a) - 1/n$$

Therefore, if one starts with a digraph D given with an arbitrary cyclic order $\mathcal{O}$, either $g_c(\mathcal{O}) \geq 2$ in which case we are done, or there must be a circuit of cyclic length less than 2. In that case, we reverse the orientation of its arcs, and the function described above strictly decreases (at least by $1/n$). Since this quantity is bounded below (by m/n) this means that after a sequence of circuit inversions, we obtain a digraph that admits a partition into two acyclic digraphs.

■

Chapter 5

Caccetta-Häggkvist Conjecture

5.1 Introduction

In this chapter we want to discuss a conjecture of graph theory, that we already mentioned in the previous chapter. This conjecture can be stated as follows :

Conjecture 5.1 (Caccetta-Häggkvist -1978) *Let D be a digraph on n vertices. If for every vertex x $|x^+| \geq n/k$, where k is an integer, then D contains a directed circuit of length at most k.*

We can also write that if a digraph on n vertices has minimum outdegree r, then it contains a circuit of length at most $\lceil n/r \rceil$. This conjecture was proven for $r = 2$ by Caccetta and Häggkvist, for $r = 3$ by Hamidoune and for $r = 4, 5$ by Hoáng and Reed.

The cases we are interested in here are small values of k in the statement 5.1, the case $k = 3$ being still open. In this case, the first bound one can obtain is the following :

Proposition 5.1 *Let $D = (V, A)$ be a digraph on n vertices. If $\forall x \in V$ $|x^+| \geq pn$, where $p \geq \frac{3-\sqrt{5}}{2} \approx 0.382$, then D contains a circuit of length at most 3.*

Proof
This is a proof by induction on n. Assume by contradiction that there is non circuit of length less than 4. By double counting, it is easy to see that there exists a vertex x such that $|x^-| \geq pn$. Denote by A its in-neighbourhood,

by B its out-neighbourhood, and by C its second out-neighbourhood, that is vertices that are out-neighbours of B but not in B. So by assumption these sets are disjoint and $|A| \geq n/3$ and $B \geq n/3$. By the induction hypothesis applied to B, we know that there is a vertex in B such that its out-degree in B is strictly less than p^2n. But this implies that its remaining Since we must have $|A|+|B|+|C| < n$, this implies $p^2-3p+1 < 0$, that is $p < (3-\sqrt{5})/2$. ■

This proof suggests that it is interesting to have information about the second neighbourhood x^{++} of a vertex x. Seymour's second neighbourhood conjecture is the following :

Conjecture 5.2 (Seymour) *Let D be a simple digraph. Then there exists a vertex V such that $|x^{++}| \geq |x^+|$.*

This conjecture implies the weaker case of Conjecture 5.1 where both out-degrees and in-degrees are supposed to be larger then $n/3$.

One could also be interested in the number of counterexamples to these conjectures. Shen, in [46], proved that for each given $r = n/k$, the number of counterexamples to the conjecture 5.1 is finite. More precisely, he showed that the conjecture is true when $n \geq 2r^2 - 3r + 1$, which implies the result if $r \leq \sqrt{n/2}$.

Before going further, we want to present the classical families of extremal digraphs that prove that if the conjecture is true then it is best possible.
The first family has somehow already been described and can be constructed the following way. We put $n = 3k + 1$ points at regular distance on a circle, and add an arc from x to the k vertices that follow it in this circular order. We see that these examples are digraphs in which the length described in the previous chapter is less than 1/3. The out-degree of each vertex is $n/3 - \varepsilon$ and since every arc has a length strictly less than 1/3 of the total perimeter it is clear that they do not contain any circuit of length at most 3.

The second family is a little bit different. We will construct it by induction. We start from the circuit C_4 on 4 vertices. Then we replace each vertex by another C_4 and add all arcs between the vertices corresponding to the exploded vertices for which an arc existed before. The k-th digraph constructed by repeating this operation has 4^k vertices and it is easy to prove,

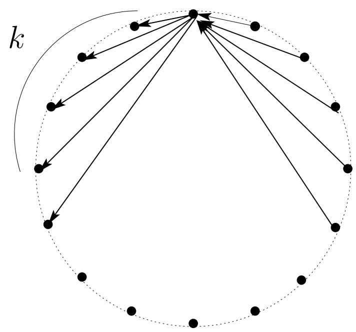


Figure 5.1: An example of the first family of extremal graphs for $k = 5$

by induction, that the out-degree of each vertex is equal to $(4^k - 1)/3$. Moreover, the girth cannot decrease on each step and stays 4 for each of these digraphs. We will comeback on these families in the following sections. the

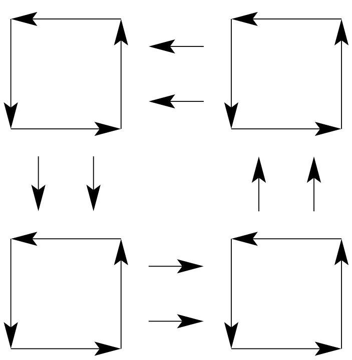


Figure 5.2: An example of the second family for $k = 2$

aim of this chapter is to show some ideas and results about this problem that I obtained during my thesis, in particular a lot of equivalent statements of this conjecture and also in the last sections to make a short kind of survey of some methods investigated by other people.

In the following, we will denote by $\mathcal{D}_k$ the set of all digraphs with girth strictly greater than k. $\mathcal{D}_k$

5.2 Toric Embeddings

In this section, we will try to extend the notion of cyclic length to other kind of embeddings by starting to construct a natural family of digraphs included in D_k. In order to achieve this, we will return on the notion of length that we have defined in the previous chapters. The simple remark that we already made is the following : if we are given a digraph embedded in the circle of perimeter 1 in such a way that each arc has length at most $1/k$, then this digraph does not contain any circuit of length at most k. In this case it is easy to check that Caccetta-Häggkvist conjecture holds and prove that there is a vertex of outdegree strictly less than n/k. Indeed, by double counting there exists an open interval of length $1/k$ on the circle that contains strictly less than n/k points. The first vertex before this interval will have its out-neighbourhood included in the interval and thus of size less than n/k.
Unfortunately, it is clear that not all digraphs in $\mathcal{D}_k$ can be represented this way. Indeed, for such a digraph, the out-neighbourhood induce acyclic digraphs, which is of course not the case of all digraphs in $\mathcal{D}_k$.

We want to generalize this remark to combinations of cyclic orders.
Let f_1 and f_2 be two embeddings of the vertex set $V = \{1, \ldots, n\}$ of a digraph into the circle of perimeter 1. We define the average length of an ordered pair (i, j) by :

$$l_{f_1,f_2}(i,j) = \frac{l_{f_1}(i,j) + l_{f_2}(i,j)}{2}.$$

And again, it is easy to check that if D is a digraph such that every arc has average length strictly less than $1/k$ then $D \in \mathcal{D}_k$. Indeed, if we denote by ind_i the winding number of a circuit with respect to the embedding f_i, by summing over all arcs we get :

$$|C|/k > \sum_{a \in C} l_{f_1,f_2}(a) = \frac{ind_1(C) + ind_2(C)}{2} \geq 1\,.$$

It is therefore natural to wonder if it is easy for digraphs that can be represented that way to prove the existence of a vertex of outdegree less than n/k.

We will define more precisely the class of digraphs we are interested in. Given the two embeddings f_1 and f_2 of $V = \{1, \ldots, n\}$ into $[0, 1[$, and a real number $\alpha \in]0, 1[$, we define the digraph $D_{f_1,f_2,\alpha}$ with vertex set V and arc set $A = \{(i,j),\ \text{ such that } l_{f_1,f_2}(i,j) < \alpha\}$. Thus, the digraphs that we have considered in the previous paragraph are simply subgraphs of $D_{f_1,f_2,1/k}$. It is easy to draw the out-neighbourhood of a vertex in this graph, if we represent the vertices on the square $[0,1]^2$. It is an isosceles rectangle triangle pointed on the vertex and which equal sides have length 2α. Beware that we are in the torus and the triangle has to be considered modulo 1 The figure 5.3 represents this drawing.

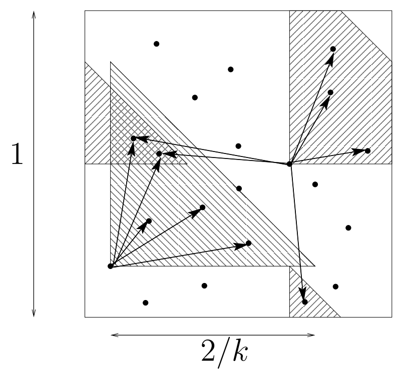


Figure 5.3: An example of two out-neighbourhoods in a toric digraph

Here we are in dimension 2 but it is easy to generalize this in upper dimension by taking the average on n embeddings and we speak of toric digraphs. We consider the class of digraphs $\mathcal{DT}_\alpha^n$ defined by $\mathcal{DT}_\alpha^n$

$$\mathcal{DT}_\alpha^n = \{D \text{ digraph , such that there exists } f_1,\dots,f_n \text{ avec } D = D_{f_1,\dots,f_n,\alpha}\} \subset \mathcal{D}_{\lfloor \frac{1}{\alpha} \rfloor}$$

Unfortunately, even in dimension 2, we could not solve the problem of proving that there exists for toric graphs a vertex of small degree. This remains an interesting question, but which is not as difficult as Caccetta-Häggkvist conjecture, since as we will see later, not all digraphs in $\mathcal{D}_k$ are toric digraphs. Nevertheless this definition of toric digraphs can be used to disprove a conjecture due to Joseph Myers that is related to the Caccetta-Häggkvist conjecture. The question was the following.

Problem 5.1 (Myers) *If D is an oriented graph such that every pair of vertices has a common out-neighbour, then does D contain an induced triangle?*

Before proving that this is wrong, let us explain the link with the Caccetta-Häggkvist conjecture. In order to do this, we have to consider the stronger question :

Question : If D is an oriented graph such that the endpoints of any arc have a common outneighbour, then does D contain an induced triangle?

The reason why this question is related to Caccetta-Häggkvist conjecture is the following. Consider a oriented graph $D = (V, A)$ on n vertices, that satisfies the conditions on degree of the weak version of Caccetta-Häggkvist conjecture, that is every vertex has outdegree and indegree at least $n/3$. Let now xy be an arc of D. As the digraph does not contain any circuit of length 2, it implies that the out and in neighbourhoods x^+ and x^- of x are disjoints. But if the out-neighbourhoods of x and y are also disjoints, and since these three sets x^+, x^- and y^+ shave size at least $n/3$, it implies that x^- et y^+ intersect and thus the digraph contains a circuit of length 3.

Hence, similarly to the second neighbourhood conjecture of Seymour, if this conjecture was true then it would imply the weaker case of Caccetta-Häggkvist conjecture. In fact, it is very easy to answer this question. It suffices to consider the product $C_4 \times C_4$, which is the second element of the second family of extremal examples constructed in the introduction to this chapter. It is the digraph drawn on figure 5.2. In this digraph, every arc is dominated but there is no circuit of length 2 or 3.

Here we are going to disprove the conjecture made by Joseph Myers. To achieve this, let us consider a toric digraph in dimension 4, with $\alpha = 1/3$. As before, if we sum the average lengths (which are less than 1/3 by assumption) of the arcs in a circuit C, we obtain that the average winding number taken on the 4 dimensions is strictly less than $|C|/3$. Thus, such a digraph does not contain a circuit of length at most 3. Hence we will construct such a digraph with in addition the property that each pair of vertices has a common outneighbour.

Let p be a large enough integer, we define the set of vertices V of our digraph by :

$$V = \{\frac{1}{p}(x_1, x_2, x_3, x_4),\ x_i \in [0, p[\cap \mathbb{Z}\}$$

We define $\varepsilon = 1/p$, and will denote by l the average oriented) distance in the this torus of dimension 4. Let us consider two points X and Y in V, we want to find a common outneighbour. We make three remarks :

- Without loss of generality we can assume that $l(X,Y) \leq 2$, since for every points X and Y, $l(X,Y) + l(Y,X) = 4$)
- Free to translate the problem, we can also assume that X has coordinates $(0,0,0,0)$.
- Eventually, free to change the order on the dimensions we can also assume that the coordinates (a,b,c,d) of Y satisfy $a \leq b \leq c \leq d$.

We will define $Z \in X^+ \cap Y^+$. We distinguishes two cases :

i) $c + d > 2/3 + 4\varepsilon$.
This implies that $a + b < 4/3 - 4\varepsilon$ since $a+b+c+d \leq 2$. In that case $Z = (a+\varepsilon, b+\varepsilon, \varepsilon, \varepsilon)$ (where these coordinates are taken modulo 1). Hence we have $l(X,Z) = a+b+4\varepsilon < 4/3$ and $l(Y,Z) = 2+4\varepsilon-(c+d) < 4/3$.

ii) $c + d \leq 2/3 + 4\varepsilon$.
As we assumed that $a \leq b \leq c \leq d$, this implies that $a + b \leq 2/3 + 4\varepsilon$ and $c \leq 1/3+2\varepsilon$. Therefore we choose $Z = (a+\varepsilon, b+\varepsilon, c+\varepsilon, \varepsilon)$ and by choosing ε small enough we have $l(X,Z) = a+b+c+4\varepsilon \leq 1+6\varepsilon < 4/3$ et $l(Y,Z) = 1 + 4\varepsilon - d < 4/3$.

This concludes the construction of the counter example.
We end this section by two questions.

On can wonder if the same construction, with the same value $\alpha = 1/3$ but in higher dimension, could give digraphs without any circuits of length at most 3, which satisfy the property that any set of k vertices has a common outneighbour.
The answer is unfortunately no for $k \geq 3$. The problem comes when we want to find a common out-neighbour for the k points on the diagonal of the cube (the points with coordinates all equal to i/k, $i = 0, \ldots, k-1$). Indeed, for every point x, a rapid calculus proves that the sum of the average distances from these points to x is strictly greater than 1. Hence there is at least one point which does not have x as an out-neighbour. This construction does not

give the answer but the question il still open to know if one can construct a digraph with this property for any value of k.

On the other hand, after this construction, one can also ask the following interesting question : If a digraphs satisfies this property that every pair of vertices has a common outneighbour, does it imply a bound on its girth, or can we construct such graphs with arbitrarily high girth?

5.3 Majority Digraphs

Again, we will start this section by constructing a family of digraphs of girth strictly greater to k.

majority digraph

For $r \in [1/2, 1]$, we say that a digraph $D = (V, A)$ is a r-majority digraph if there exists a family $\mathcal{P}$ of total orders (linear orders) on the set $V = \{1, \ldots, n\}$ such that $(i, j) \in A$ if and only if there are more than $r|\mathcal{P}|$ orders of the family for which i is before j. We will denote by $\mathcal{DM}_r$ the set of r-majority digraphs.

$\mathcal{DM}_r$

To fix the ideas, the circuit of length 4 is a r-majority digraph for every r in $[1/2; 3/4[$, just consider for this the orders 1234, 2341, 3412 and 4123.

Now we want to link these digraphs with the toric digraphs introduced in the previous section. In fact we will prove that the $(1-r)$-majority digraphs are exactly the toric digraphs with parameter r that is :

Proposition 5.2

$$\mathcal{DM}_{1-r} = \bigcup_{d=1}^{\infty} \mathcal{DT}_r^d$$

Proof

Let $D \in \mathcal{DT}_r^d$. To fix the ideas, we begin with the case $d = 1$. Hence we have an embedding of the vertices of D in a circle of perimeter 1, and thus a cyclic order. Assume that we "open" this order to make p linear orders, by cutting it in a regular way (see figure 5.4). By choosing p large enough it is always possible to do so in such a way that the points where we open the circle are not the vertices of the digraph. When p goes to infinity, then what is the proportion of orders in which a given arc is backward? An arc xy is backward if and only if the order has been opened between x and y. Hence, the proportion will have as a limit exactly the length of the arc in

the circular embedding. More precisely, if we denote by $f(a)$ the number of orders in which $a = xy$ is a backward arc, and $l(a)$ the length this arc, then $p\,l(a) \leq f(a) \leq p\,l(a) + 1$. Thus the proportion $m(a) = 1 - f(a)/p$ of orders in which this arc is forward satisfies $1 - l(a) - 1/p \leq m(a) \leq 1 - l(a)$. Since $l(a) < r$ for every a, we can choose p large enough in order to have $m(a) > 1 - r$ for all a.
For $d \geq 1$, we do the same thing on each dimension, with the same value of p. The proportion $m(a)$ we get for each arc is then the average of the proportion in each dimension, and since this time it is the average length that is bounded by r, the same proof gives the conclusion.

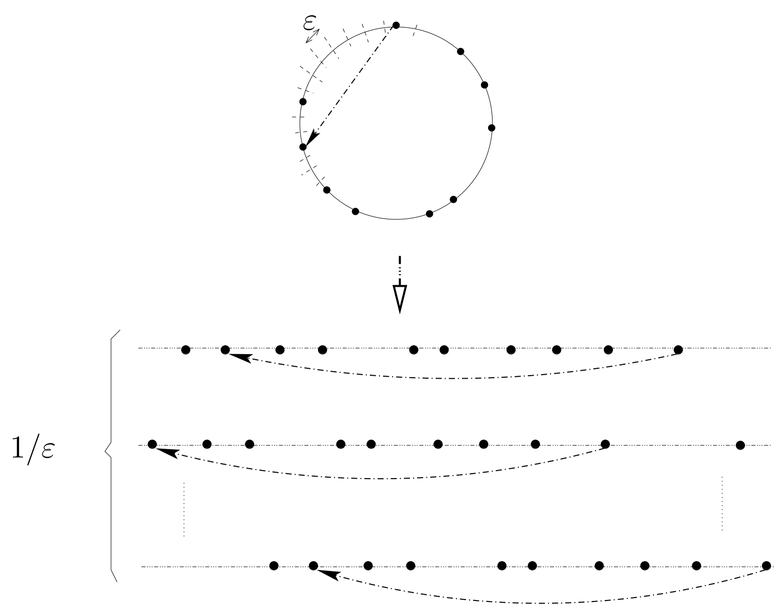


Figure 5.4: To translate a toric digraph into a majority one

Conversely, if $D \in \mathcal{DM}_{1-r}$, we are going to express D as a toric digraph. Let $L_1, \ldots, L_s$ be the linear orders for which D is a $(1-r)$-majority digraph. Let ε (that will be chosen very small) be a positive number and to each $L_i = (v_1, \ldots, v_n)$ we associate the circular embedding f_i given by $f_i(v_j) = \varepsilon j$. In other words, we "compress" the order in an interval of length ε placed at the beginning of the embedding. Thus the length of an arc is no more than ε if the arc is a forward arc, and close to 1 if the arc is backward. If we denote by $m(a) > (1 - r)$ the proportion of orders in which a is a forward arc, the average length of a satisfies the following inequality : $l(a) \leq 1 - m(a) + m(a)\varepsilon$ and by choosing ε small enough, we have $l(a) < r$ for every arc a. ■

Therefore, the $(k-1)/k$-majority digraphs do not contain any circuit of length at most k. We could also say more simply that if there is one, by an easy average argument there must be one of the orders in which all arcs are forward, which is of course impossible.

Starting from that, the natural idea would be to prove that :

i) Show that every digraph with girth more than k is a $(k-1)/k$-majority digraph.

ii) Show that in a $(k-1)/k$-majority digraph, there is a vertex with outdegree less than n/k.

Unfortunately, this can't work. Indeed **i)** is true for $k = 2$ but not for $k \geq 3$.

For $k = 2$, this means we are interested in oriented graphs. We proceed by induction on the number of arcs. Let D be a digraph, we are looking for a family of linear orders on the vertices for which an arc belongs to D if and only if it is a forward arc in strictly more than half of the orders of th e family. By induction hypothesis, for every arc a, there exists a family of orders $\mathcal{P}_a$ that realize the digraph $D \setminus a$ as a $1/2$-majority digraph. Let $\mathcal{P} = \cup_{a \in A} \mathcal{P}_a$, and let us prove that this family is suitable. Indeed, an arc $a = (i,j)$ of D is a forward arc in strictly more than half of the orders in the families $\mathcal{P}_b$, $b \neq a$ and since (i,j) and (j,i) do not belong to $D \setminus a$, this implies that in the family $\mathcal{P}_a$ i is before j in exactly half of the orders. Thus, the arcs of D appear as forward arcs in strictly more than half of the orders in $\mathcal{P}$.
Conversely, if (i,j) is not an arc of D then it is not an arc of $D \setminus a$ for every a and for same reasons it means that $i$is before j in exactly half of the orders of each $\mathcal{P}_a$ and thus in half of th e orders of $\mathcal{P}$, and the proof is finished.

To see that the answer is no in the case $k \geq 3$, we use a probabilistic argument. We are going to see that the majority digraphs are far from being random. Indeed, consider a random orientation of a complete balanced bipartite graph on $2n$ vertices (and denote by m the number of its arcs). This digraph has girth at least 4. Moreover, a feedback arc set of minimum size in such a digraph has size at least $m/2 - o(m)$. This come from the fact that the number of arcs in a feedback arc set follows a binomial law and by Tchebycheff Theorem we easily get that bound. But, if it was a 2/3-majority digraph, by double counting there must be an order of the family in which more than 2/3 of the arcs are forward, which would give a feedback arc set of size at most $m/3$.

Now to fix the ideas, we focus on the case $k = 3$, the bound thus becomes 2/3.
We have seen that all digraphs in $\mathcal{D}_3$ cannot be represented as 2/3-majority digraphs. We want now to make a remark about this construction.
For this, we are representing the digraphs by their adjacency matrices, that is the square matrix A of size the number of vertices defined by $a_{ij} = 1$ if (i, j) is an arc and 0 otherwise. For example, the matrix corresponding to total orders are, up to a permutation of rows and columns, upper triangular matrices.
We begin by taking the convex hull of the matrices of total orders. Then the majority digraph are obtained from this polyhedron the following way. Each entry of a matrix is replaced by 1 if it is strictly greater than 2/3 and by 0 otherwise. Hence we obtain the matrix of an oriented graph and moreover these are precisely the 2/3-majority digraphs. We denote by ϕ this operation on a matrix.

We now study the following transformation : starting from a set of digraphs, represented by their adjacency matrices and

- we take the convex hull
- we apply the function ϕ, to get a larger set of digraphs.

Let us denote by ψ this application, and $\mathcal{TO}$ by the set of total orders. For the reasons we have explained previously, the digraphs we get by iteration ψ are all elements of D_3. Hence we have $\cup_{i=1}^{\infty}\psi^i(\mathcal{TO}) \subset D_3$. And conversely, we can prove that all digraphs of girth greater than 3 are obtained by such operations. Indeed, assume by contradiction that this is not true and consider a minimal counter example D with respect to the number of arcs. By minimality, for every arc a of D, the digraph $D \setminus a$ is obtained by such a construction, so there exists an integer p such that for every arc a, $D \setminus a \in \psi^p(\mathcal{TO})$. Let M be the adjacency matrix of D and M_a the adjacency matrix of $D \setminus a$. Then, if $||D||$ denotes the number of arcs of D, it is easy to check that

$$M = \frac{1}{||D|| - 1} \sum_{a \in A} M_a = \phi(\frac{1}{||D||} \sum_{a \in A} M_a).$$

Therefore $D \in \psi^{p+1}(\mathcal{OT})$ and we have proved :

$$\cup_{i=1}^{\infty} \psi^i(\mathcal{OT}) = D_3$$

Remark 5.1 *It is easy to prove that the same construction can be done for triangle-free non oriented graphs. The only difference is that we have to start with bipartite graphs instead of total orders.*

Let us now come back on the question **ii)** stated in page 66. Since the $(k-1)/k$-majority digraphs are in $\mathcal{D}_k$, if the Caccetta-Häggkvist conjecture is true, then there must be a vertex of outdegree strictly less than n/k. One of the motivations for studying these majority digraphs is that the extremal graphs described in the introduction of this chapter are all $(k-1)/k$-majority digraphs. Proving **ii)** would thus be best possible for these digraphs.

Proposition 5.3 *The two families of extremal graphs described in 5.1 are* $2/3$*-majority digraphs and can be expressed with a family of only* 4 *orders, so that each arc is a forward arc in at least* 3 *of these* 4 *orders.*

Proof

For the first family, proving that they are 2/3-majority digraphs is not difficult if we do not put a constraint on the number of total orders since these arcs are embeddable in the circle with all arcs of length less than 1/3. Nevertheless, here we want to express them with only 4 orders. In order to do this, we begin by constructing 3 orders. Some arcs will be forward arcs in these three orders, so whatever so fourth one we add, these arcs will be in the majority digraph. The others arcs will be backward in one of the orders. Hence we will have to construct a fourth order in such a way that all these arcs are forward. It will be possible because we will see that these arcs induce an acyclic digraph (and we can use the lemma we have used all through this work, that is an acyclic digraph can be represented with all its arcs forward).

Remind that the digraph we study is obtained by placing $3k+1$ vertices on a circle and by adding an arc from a vertex two the k vertices that follow it. The vertices are denoted by $(1, 2, \ldots, 3k+1)$. The first three orders we consider are :

$$\begin{array}{c} 1, \ldots, k, k+1, \ldots, 2k, 2k+1, \ldots 3k+1 \\ k+1, \ldots 2k, 2k+1, \ldots 3k+1, 1, \ldots, k \\ 2k+1, \ldots 3k+1, 1, \ldots, k, k+1, \ldots 2k \end{array}$$

We easily see that every arc in our digraph is a forward arc in at least 2 of these orders, so the only thing we have to prove now is that the arcs which are backward once induce an acyclic digraph.
Assume that there exists a circuit C using only these arcs. We study its winding number. Remind that this number can be seen as the number of backward arcs in any enumeration but also as the sum of the lengths of the arcs in a circular representation. We are using these two definitions in order to get a contradiction. First, the winding number of this circuit in the three previous orders is the same, since these three linear orders belong to the same cyclic order. Moreover, since each arc is backward in exactly one of the orders, it means that the sum of the winding numbers (that is three times the winding number) is equal to $|C|$. But by assumption, each arc has a length at most $k/3k+1$, so the sum of the lengths in any of these three orders must be less than $k|C|/(3k+1)$. So we would have $k|C|/(3k+1) \geq |C|/3$, which gives the contradiction.

For the second family, we prove this by induction. Assume that for the k-th power of the C_4 there exist such 4 orders L_1, L_2, L_3, L_4 on the 4^k vertices. For the next digraph, we consider the sets V_1, V_2, V_3, V_4 of cardinality 4^k. We will denote by $L_i(V_j)$ the vertices of V_j in the order given by L_i. The 4 total orders we consider are then :

$$\begin{array}{c} L_1(V_1)\ L_2(V_2)\ L_3(V_3)\ L_4(V_4) \\ L_1(V_4)\ L_2(V_1)\ L_3(V_2)\ L_4(V_3) \\ L_1(V_3)\ L_2(V_4)\ L_3(V_1)\ L_4(V_2) \\ L_1(V_2)\ L_2(V_3)\ L_3(V_4)\ L_4(V_1). \end{array}$$

And we get 4 copies of the previous graph that are linked together to form a C_4. ■

Thus we see that the majority digraphs that can be represented by 4 orders contain all the extremal examples for the case $k = 3$ of the Caccetta-Häggkvist Conjecture 5.1. To conclude this section, we want therefore to insist on this particular interesting question on 4 orders, for which we were unable to give an answer.

Question : Given 4 orders on $\{1, \dots, n\}$, if D is the digraph obtained by keeping the arcs that are forward in at least 3 orders, does D contain a vertex with outdegree strictly less than $n/3$?

5.4 Linear Algebra approach

In this short section we want to show some ideas about the formulation of the conjecture with tools of linear algebra and basic spectral theory. We will in the end give a conjecture that implies the Caccetta-Häggkvist conjecture.

Let D be a digraph on n vertices. The adjacency matrix $\mathbf{A}$ of D is the $n \times n$ matrix defined by $A_{ij} = 1$ if (i, j) is an arc and 0 otherwise.

So, if $\mathbf{A}$ is the incidence matrix of D, the vector of outdegrees of D is given by $\mathbf{A}.\mathbf{1}$, where $\mathbf{1}$ is the column vector which entries are 1's.
The cycles of D can be interpreted with traces of the powers of $\mathbf{A}$. Indeed $tr(\mathbf{A}^n)$ represents the number of closed walk of length n. So the girth of a graph G is simply given by $g(D) = inf\{k, tr(\mathbf{A}^k) > 0\}$.
But this has another interpretation. Indeed, these traces are related with the coefficients of the characteristic polynomial of A, that is $\chi_{\mathbf{A}}(X) = det(\mathbf{A} - X.\mathbf{I}_n)$. This follows from the classical Newton relations on coefficients. From this we can derive the following proposition

Proposition 5.4 *Let $\chi_{\mathbf{A}}(X) = det(\mathbf{A} - X.\mathbf{I}_n) = (-1)^n X^n + \sum_{k=1}^{n} a_k X^{n-k}$ be the characteristic polynomial of a $n \times n$ matrix $\mathbf{A}$. $\chi_{\mathbf{A}}$ is entirely determined (by linear relations) by the set $\{tr(\mathbf{A}^i), i = 1 \dots n\}$. Thus, the following equivalence holds for all k.*

$$\forall 1 \leq i \leq k \; a_i = 0 \iff \forall 1 \leq i \leq p \; tr(\mathbf{A}^i) = 0$$

Finally one has the following corollary :

Corollary 5.1 *Let $D = (V, A)$ be a simple digraph on n vertices and $P(X) = (-1)^n X^n + \sum_{k=1}^{n} a_k X^{n-k}$ the characteristic polynomial of its incidence matrix.*
Then $g(D) = \inf\{k \geq 1,\ a_k \neq 0\}$

To conclude we are making a remark about the minimum out degree in order to give the conjecture.
Denote by ρ the spectral radius of the incidence matrix A of a digraph D. Then the following holds :

$$\min_{x \in V} d_+(x) \leq \rho \leq \max_{x \in V} d_+(x)$$

So a stronger conjecture could be the following

Conjecture 5.3 *Let $\mathbf{A}$ be a $n \times n$ matrix which entries belong to $\{0, 1\}$. Let $\chi_{\mathbf{A}}(X) = (-1)^n X^n + \sum_{k=1}^{n} a_k X^k$ denote its characteristic polynomial and ρ its spectral radius. Then :*

$$a_1 = a_2 = \ldots = a_k = 0 \qquad \Longrightarrow \qquad \rho < \frac{n}{k}.$$

5.5 Choice functions.

Let $k \leq n$ be two positive integers. A (k, n)-*choice function* is a mapping Φ from the set $\binom{n}{k}$ of k-element subsets of $\{1, \ldots, n\}$ into $\{1, \ldots, n\}$ such that $\Phi(X) \in X$ for all $X \in \binom{n}{k}$. We ask the following question.

Conjecture 5.4 *If Φ is a (k, n)-choice function, there exists i such that $|\bigcup \Phi^{-1}(i)| > (k-1)n/k$*

Proof of the equivalence of conjectures 5.1 and 5.4
Let us first prove that if D is a counterexample to the Caccetta-Häggkvist conjecture, then it is a counterexample to Conjecture 5.4. Assume that D is a digraph with girth greater than k and minimum outdegree at least n/k. We can define a (k, n)-choice function on the vertices of D. Indeed, since every induced subgraph on k elements is acyclic, it contains at least a vertex of outdegree 0, and this gives our choice function. Since by definition, for every vertex i, $\bigcup \Phi^{-1}(i)$ is disjoint from the outneighbourhood of i, it is clearly a counter-example to conjecture 5.4.

Now conversely, we would like to prove that a counterexample to Conjecture 5.4 would disprove Caccetta-Häggkvist . Consider a minimum counterexample Φ with respect to n and k. Construct a digraph D on $\{1,\ldots,n\}$ by letting an arc ij if $j \notin \bigcup \Phi^{-1}(i)$. By hypothesis, $|\bigcup \Phi^{-1}(i)| \leq (k-1)n/k$, so D has minimum outdegree at least n/k.
Let us prove that D has girth greater than k. If D has a circuit C of length k, observe that $\Phi(C)$ cannot exist. If D has a circuit C of length $l < k$, let us consider the initial section S of D generated by C (i.e. the set of vertices x for which there is a directed path from x to some vertex of C). Observe that $C \subseteq S$. If $l := |S| \geq k$, we have a contradiction since S contains a functional subdigraph F of size k (i.e. every vertex of F has outdegree 1 in F), and thus $\Phi(F)$ cannot be defined. Now we will discuss two cases:

- $|V(D) \setminus S| \geq k$
  By the definition of S, every vertex $i \notin S$ is such that $\bigcup \Phi^{-1}(i)$ contains S. Denote now by Φ' the restriction of Φ to $V(D)\setminus S$. By the minimality of the counterexample, there is $i \notin S$ such that $|\bigcup \Phi'^{-1}(i)| > (k-1)(n-l)/k$, and thus $|\bigcup \Phi^{-1}(i)| > (k-1)(n-l)/k + l \geq (k-1)n/k$, and we achieve a contradiction.

- $|V(D) \setminus S| < k$. In this case, we have $n \geq k > n/2$. If $k = n$, the problem is trivial. Thus we can assume that $k < n$. Since Φ is a counterexample, for every vertex i $|\bigcup \Phi^{-1}(i)| < (k-1)n/k \leq n-1$, and thus the minimum outdegree in D is at least 2. We use now the fact that the Caccetta-Häggkvist conjecture is true for minimum outdegree 2. We consider the set of terminal strongly connected components $T_1, T_2, \ldots, T_p$ of D. Since the minimum degree in each T_i is at least 2, there exists a circuit C_i of T_i which at most $T_i/2$ vertices. This means that there exists a functional spanning subgraph of D, which set of circuits is exactly $C_1, \ldots, C_p$. In particular, there exists functional subgraphs of D of all cardinal between $(|T_1| + \ldots + |T_p|)/2$ and n. So there is a functional subgraph of size k, which is again a contradiction.

■

5.6 Homogeneous structures.

age of graphs

An *age of graphs* $\mathcal{A}$ is a set of finite graphs having the following properties:

- If $G \in \mathcal{A}$ and H is an induced subgraph of G, $H \in \mathcal{A}$.
- If $G, G' \in \mathcal{A}$, there exists $H \in \mathcal{A}$ such that both G and G' are induced subgraphs of H.

It is easy to see that ages correspond to the class of finite induced subgraphs of a countable (finite or infinite) graph. We thus can speak of the *age* of a graph. An age $\mathcal{A}$ has the *amalgamation property* if given $G, H_1, H_2 \in \mathcal{A}$ and $f_1 : G \to H_1$, $f_2 : G \to H_2$, there exists $A \in \mathcal{A}$ and $g_1 : H_1 \to A$, $g_2 : H_2 \to A$ for which $g_1 \circ f_1 = g_2 \circ f_2$. Roughly speaking, $\mathcal{A}$ has the amalgamation property if for any pair H_1, H_2, and any of their common induced subgraph G, one can amalgamate H_1 and H_2 on G (mind that $H_1 \cap H_2$ is not necessarily G, it only contains G). Here is another characterization of amalgamation: Say that a countable graph G is *homogeneous* if every isomorphism f from a finite induced subgraph of G into a finite induced subgraph of G can be extended into an automorphism of G.

Theorem 5.1 Fraïssé *An age $\mathcal{A}$ has the amalgamation property if and only if there exists a countable homogeneous graph G with age $\mathcal{A}$.*

The countable homogeneous graphs were characterized by Lachlan and Woodrow [36]. The countable homogeneous directed graphs were characterized by Cherlin [24]. Observe that the age $\mathcal{D}_{\ni}$ of finite directed graphs with girth at least four have the amalgamation property: to amalgamate H_1, H_2 on G, just set $H_1 \cap H_2 = G$, and add no other arc.

Let $\mathcal{D}$ be the class of finite digraphs D such that for every induced subgraph S of D on s vertices, there exists a vertex with outdegree less than $s/3$. By definition, $\mathcal{D}$ is stable by taking induced subgraphs and whenever $D, D' \in \mathcal{D}$, the disjoint union of $D \cup D'$ is also in $\mathcal{D}$. Thus $\mathcal{D}$ is an age. Moreover, by applying the definition to subgraphs of size 3, we get immediately that $\mathcal{D} \subset \mathcal{D}_3$.

Our hope was that $\mathcal{D}$ could have the amalgamation property, and then make use of the characterization of countable homogeneous directed graphs. But it quickly turned out that, for minimum outdegree $n/3$, the following is indeed equivalent to the Caccetta-Häggkvist conjecture:

Conjecture 5.5 *The class $\mathcal{D}$ has the amalgamation property.*

Proof of the equivalence of conjectures 5.5 and 5.1
Observe first that the Caccetta-Häggkvist conjecture is equivalent to $\mathcal{D} = \mathcal{D}_3$, and thus it implies conjecture 5.5.
Conversely assume that $\mathcal{D}$ has the amalgamation property, then we prove $\mathcal{D} = \mathcal{D}_3$ by induction on the number n of vertices of the graphs of $\mathcal{D}$. Assume that every graph D of $\mathcal{D}_\ni$ with less than n vertices belongs to $\mathcal{D}$. Now consider $D \in \mathcal{D}_\ni$ with n vertices. If D is a tournament, it is certainly the transitive tournament, and thus belongs to $\mathcal{D}$. If not, there exists $x, y \in V(D)$ which are not linked by an arc. We can moreover assume that x and y do not have the same in and out neighbors, otherwise, D would be a transitive tournament, where vertices are blown-up with stable sets. By the induction hypothesis, both $D \setminus x$ and $D \setminus y$ are in $\mathcal{D}$. Using the amalgamation property of $\mathcal{D}$, one can amalgamate $D \setminus x$ and $D \setminus y$ on $D \setminus \{x, y\}$ to form a graph $D' \in \mathcal{D}$. If D' has n vertices, it is either D or D plus one arc linking x and y. In both cases, D' belongs to $\mathcal{D}$ ($\mathcal{D}$ is also stable by taking subgraphs). The last case is when D' has $n - 1$ vertices, but this case is not possible since x and y have different neighborhoods.

■

5.7 Linear Optimization

The object of this section is to prove some equivalent statement of the Seymour second neighbourhood conjecture 5.2. The objects of these statements is to reformulate the conjecture in terms of *positive weightings* of the vertices, that is $\omega : V(D) :\longrightarrow \mathbb{R}_+$.

Conjecture 5.6 *Let $D = (V, E)$ be a digraph on n vertices. Then:*
$\forall \omega \in V^{\mathbb{R}_+} \, \exists x \in V \, \omega(x^{++}) \geq \omega(x^+)$.

To prove that this is equivalent to 5.2, we can prove that this is true when the weightings are integer-valued. To do this, one has just to replace each vertex by a set of independent vertices of size equal to its weight. Then if it is true for integer valued weightings it is obviously true for rational valued ones just by linearity, and it follows by density that this is also true for real valued weightings.
Most surprising, this statement is also equivalent to the following one which results simply in a quantifier inversion.

Conjecture 5.7 *Let $D = (V, E)$ be a digraph on n vertices. Then:*
$\exists \omega \in V^{\mathbb{R}_+}, \omega \neq 0\,, \forall x \in V\, \omega(x^{++}) \geq \omega(x^{+})$.

Proof of the equivalence of Conjectures 5.6 and 5.7
Denote by $\{v_1, \ldots, v_n\}$ the vertices of D. Consider the $n \times n$ matrix $\mathbf{A}$, defined by :

$$A_{ij} = \begin{cases} 1 & \text{if} v_j \in v_i^{++} \\ -1 & \text{if} v_j \in v_i^{+} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{1}$ be the column vector which entries are all 1. and (P) be the linear problem defined by ::

$$\begin{array}{rrcl} \textit{find} & \mathbf{x} & & \\ \textit{subject to} & \mathbf{Ax} & \leq & \mathbf{-1} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

Then Conjecture 5.6 is equivalent to the non-feasibility of (P) for all digraphs. The dual problem (P^*) of (P) is simply :

$$\begin{array}{rrcl} \textit{maximize} & \mathbf{y.1} & & \\ \textit{subject to} & \mathbf{yA} & \geq & \mathbf{0} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

But this problem is feasible, it suffices to take $\mathbf{y} = \mathbf{0}$. So Conjecture 5.6 is equivalent to the fact that (P^*) is unbounded for all digraphs, which is precisely :

$$\forall D\ \exists \omega \in V(D)^{\mathbb{R}_+}\,, \omega \neq 0\, \forall x \in V \omega(x^{--}) \geq \omega(x^{-})\,,$$

which is of course Conjecture 5.7. ∎

5.8 Counting Subgraphs

We want in this short paragraph mention a approach to tackle this conjecture that has mainly been investigated by J.A. Bondy in [13].
The point is to obtain linear relations between the number of non-isomorphic

induced graphs of a given size. For example, if you study non isomorphic subgraphs of size 3 and forbid 3-directed cycles, there is 6 different possibilities. The number of subgraphs (not necessarily induced) of a given type is a linear combination of the number of induced subgraphs of different types. Moreover it is possible to calculate or bound these numbers of subgraphs in terms of the degrees. So it gives a system of linear inequalities which variables are the number of induced subgraphs of a given size. This study permits to obtain bounds on the degree of a graph that does not contained any induced C_3. Using these methods for induced subgraphs of size 4, J.A.Bondy has shown that a regular eulerian graph of degree cn where $c \geq (2\sqrt{6} - 3)/5$ contains a induced C_3.

Chapter 6

Embeddings of non-oriented graphs

6.1 Introduction

In the previous chapters, we have been studying different embeddings of digraphs in order to deduce combinatorial properties from the geometric aspects of these embeddings. In the present chapter we are interested in a problem concerning non-oriented graphs. The idea is the following : we want to embed graphs in some sphere in $\mathbb{R}^d$ (without any constraint on d) in such a way that two adjacent vertices are "far" on the sphere, that is the angle formed by the two vertices and the center of the sphere is strictly greater than a fixed angle θ (we will say that the spheric distance is at greater than θ). For example it is possible to realize the complete graph on n vertices in S^{n-1} with vectors that are pairwise orthogonal, just consider for this a orthonormal basis of $\mathbb{R}^n$ as representants of the vertices. If we fix $\theta = 2\pi/3$, we remark that a graph G that is embeddable that way is necessarily triangle-free. Indeed, if xyz is a triangle of G and if U_x, U_y, U_z are their images, we have:

$$0 \leq \|U_x + U_y + U_z\|^2 = 3 + 2(< U_x, U_y > + < U_x, U_z > + < U_y, U_z >)$$

So one of the two products has to be greater than $-1/2$, and thus two of the vectors U_x, U_y, U_z form an angle which is at most $2\pi/3$.

Thus, in 1972, M. Rosenfeld asked if every triangle-free graph could be embedded in the unit sphere S^d in such a way that two vertices joined by an edge have distance more than $\sqrt{3}$ (i.e. form an angle greater than $2\pi/3$). In 1978, D. Larman [37] disproved this conjecture, constructing a triangle-free graph for which the minimum length of an edge could not exceed $\sqrt{8/3}$. In addition, he conjectured that the bound is no better for the class of triangle-free graphs than for the class of all graphs, that is for every $\theta > \pi/2$, there exists a triangle free graph which is not embeddable in a sphere with edges longer than θ. Larman's conjecture was quickly popularized by P. Erdős (See Nešetřil and Rosenfeld [42] for a survey) and independently proved by M. Rosenfeld [42] and V. Rődl [41]. In this last paper it was shown that no bound better than $\sqrt{2}$ can be found for graphs with arbitrarily large odd girth. We prove in the next section that this is still true for arbitrarily large girth. We also discuss then the case of triangle-free graphs with linear minimum degree. Then in the last section we will explain why this is a problem of semidefinite optimization and how to express its dual.

6.2 Definitions and Theorem

Borsuk graph

Fix a real $0 < \alpha \leq 1$ and an integer $d \geq 1$. The *Borsuk graph* $Bor(d, \alpha)$ is the (infinite) graph defined on the d-dimensional unit sphere where two points are joined by an edge if and only if the distance on the sphere is at least $(1 + \alpha)\pi/2$. A graph is α *spherical* if it is a subgraph of $Bor(d, \alpha)$ for some d. Here is the key-observation:

α spherical

Lemma 6.1 *If G is α-spherical there exists a cut of G which has at least $(1 + \alpha)m/2$ edges, where m is the total number of edges.*

Proof
Embed G in some S^d in such a way that every edge has spherical length at least $(1 + \alpha)\pi/2$. Observe that a random hyperplane cuts an edge of G with probability $(1 + \alpha)/2$. By double counting, there is some hyperplane which cuts at least $(1 + \alpha)m/2$ edges. ■

Now we will prove that a graph G satisfying Lemma 6.1 is far from being random, by using Erdős' famous construction of graphs with large girth and large χ. This is the next result:

Lemma 6.2 *For every $\alpha > 0$ and every integer k, there exists a graph G, with girth at least k, in which every cut has less than $(1+\alpha)m/2$ edges, where m is the number of edges of G.*

Proof
We consider for this random graphs on n vertices with independently chosen edges with probability p. We first want to bound the size of a maximum cut. Let A be a subset of vertices. Let X denotes the number of edges between A and its complement. The worst case being when $|A| = n/2$, X is at most a binomial law $Bin(n^2/4, p)$. Its expectation is then at most $\frac{pn^2}{4}$. Thus:

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) \leq \Pr\left(Bin(n^2/4, p) - \frac{pn^2}{4} \geq \alpha\frac{pn^2}{4}\right)$$

We use the following form of the Chernoff Bound,
for any $0 \leq t \leq Np$:

$$\Pr(|Bin(N,p) - Np| > t) < 2e^{-t^2/3Np}$$

Thus, with $t = \alpha\frac{pn^2}{4}$, we get

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) < 2e^{-4\alpha^2p^2n^4/48pn^2} = 2e^{-\alpha^2n^2p/12}$$

Now, the probability that there exists a cut of size more than $(1+\alpha)\frac{pn^2}{4}$ is less than $2^n 2e^{-\alpha^2n^2p/12}$, the value 2^n being a bound to the number of cuts. Choosing $p = n^{-k/k+1}$, this probability goes to 0 as n tends to infinity.
. Let Y denotes the number of cycles of length at most k.

$$E(Y) = \sum_{i=3}^{k}\binom{n}{i}\frac{(i-1)}{2}p^i \leq \frac{1}{2}\sum_{i=3}^{k}n^ip^i \leq \frac{1}{2}(k-2)n^kp^k.$$

where the last inequality holds because $np = n^{\frac{1}{k+1}} \geq 1$.
Applying Markov's inequality,

$$\Pr(Y \geq \frac{n}{2}) \leq \frac{E(Y)}{n/2} \leq (k-2)n^{-\frac{1}{k+1}}.$$

Thus for n sufficiently large, we can find a graph on n vertices with $m = n^{\frac{k+2}{k+1}}$ edges such that the number of cycles of length at most k is less

than $n/2$ and the size of a maximum cut is at most $(1+\alpha)m/2$.
We can delete less than $n/2$ edges to make a graph of girth $\gamma > k$ and since $n/2$ is negligible compared to m, we still have that the size of a maximum cut is at most $(1+\alpha)m/2$. ■

Combining the two lemmas, we have proved the following :

Theorem 6.1 *For every $\alpha > 0$ and every integer k, there exists a graph G with girth at least k that is not α-spherical.*

Observe that this alone implies that there exists graphs with large girth and large chromatic number, since every k-chromatic graph has an homomorphism into K_k, which is in turn α_k-spherical for $\alpha_k = \arccos(\frac{-1}{k-1})$.

An interesting case arises when considering triangle-free graphs G with minimum degree δ such that $\delta \geq c.n$ for some fixed constant $c > 0$. Now, there exists a constant α_c, depending on c, such that every triangle-free graph G with minimum degree $\geq cn$ is α_c-spherical. To prove this, enumerate the vertices $v_1, \ldots, v_n$ of G. Fix $c := \delta/n$. For every v_i, we fix a set of neighbours N_i of v_i such that $|N_i| = \delta$. The unit vector associated with v_i is then $V_i = \frac{1}{\sqrt{c(1-c)n}}(x_1^i, \ldots, x_n^i)$ where $x_j^i = -c$ if $j \notin N_i$ and $x_j^i = 1-c$ if $j \in N_i$.

Since $N_i \cap N_j$ is empty whenever v_iv_j is an edge of G, we have $V_i.V_j = \frac{-c}{1-c}$. And thus the spherical length of an edge is exactly $\arccos(\frac{-c}{1-c})$. Let us raise the following:

Problem 6.1 *For every $c \in]0, 1/2]$, what is the largest α_c such that every triangle-free graph with minimum degree at least $c.n$ is α_c-spherical?*

One particular case of the previous construction is when $c = 1/3$, in which case we have $V_i.V_j = -1/2$. This means that if a triangle-free graph has minimum degree $\geq n/3$, its vertices can be positioned on some unit-sphere in such a way that edges have length on the sphere at least $2\pi/3$. Let us mention to conclude that the class of triangle-free graphs with minimum degree $> n/3$ has been recently entirely characterized [16] - they have chromatic number at most four. It would be interesting to try to obtain this bound from a geometric point of view. Moreover, a construction of Hajnál shows that minimum degree $> cn$, when $c < 1/3$ does not yield to any bound on the chromatic number. So the last question is to find what could be the largest chromatic number of a triangle-free graph with minimum degree $n/3$. The answer can be anything between four and infinity. Again, a geometric approach could be of use.

6.3 Semidefinite Programming and Expression of the Dual

Here we want to focus on the formulation of the problem, that is : Associate to each vertex of the graph a vector in $\mathbb{R}^d$ with some linear constraints on the scalar products of these vectors. In this section we are not going to prove new results but it is interesting to see that the problem studied in the previous section is an semidefinite optimization problem so that we can express its dual.

At this point, we have to insist on the fact that there is no constraint on the dimension d. In fact, imposing constraints on the dimension would make the problem much harder. To see this, consider for example the generalization of Rosenfeld's problem introduced by Karger, Motwani et Sudan [34]:

Problem of *vectorial colouring*: given a graph G, find an application ϕ from V into S^d that minimizes the greatest scalar product $< \phi(x), \phi(y) >$ among edges xy of G.

That is the problem we described in order to introduce the chapter. Intuitively, we want to place the vertices of G on a sphere while trying to have all edges longer than certain distance. If we force $d = 1$, the problem is equivalent to maximizing $p > 0$ such that we can embed the vertices of G on a circle of perimeter 1 with the constraints that every edge has length at least p. We recognize the notion of circular chromatic number $\chi_{circ}(G)$ we have encountered in section 4.3, and since for every graph

$$\chi_c(G) \leq \chi(G) \leq \lceil \chi_c(G) \rceil,$$

the determination of the circular chromatic number is at least as hard as finding the usual one. Thus, for $d = 1$ this problem is NP-Hard.

Let us mention that some instances of the case $d = 2$ contain famous problems. For example the very classical problem of the 13 spheres, asking whether it is possible to place 13 disjoints unity spheres that are tangent to the unit sphere centered at the origin, can simply be formulated as the question above for the graph K_{13} when we want each scalar product to be at most 1/2.

The main interest about these problems of semidefinite programming is that they are algorithmically easy to solve, in the sense that they are polynomially approximable at any precision rate (see [27] for a survey on this topic).

Here we are just going to give a few basic facts about semidefinite programming in order to express the dual of our problem.
Recall that a symmetric matrix M is *positive semidefinite* (psd) if for every vector x we have ${}^t xAx \geq 0$.
In an equivalent way, A psd if and only if there exists a matrix A such that $M = {}^t A.A$. In other words, $M = (m_{ij})$ is psd if and only if there is a family of n vectors U_i in $\mathbb{R}^d$ such that $m_{ij} = U_i.U_j$. Another characterization of psd matrices is the following : a symmetric matrix which has only non-negative eigenvalues.
We will note $A \succeq 0$ (resp. $A \succeq B$) if A psd (resp. $(A - B)$ psd).

Now we want to express with these notations the problem of vectorial colouring. Let G be a graph on vertices $\{1, \dots, n\}$. We want to :

$$\begin{array}{lll} \text{minimize } \; u \in \mathbb{R} & & \\ \textit{subject to the constraints} & a_{ii} = 1 & \text{for every } \; i = 1 \dots n \\ & a_{ij} \leq u & \text{for } ij \text{ edge of } \; G \\ & (a_{ij}) & \text{positive semidefinite} \end{array}$$

More generally, a problem of semidefinite optimization is a problem where we want to maximize a linear function of the variables a_{ij}, with linear constraints on the a_{ij} plus the constraint that the matrix (a_{ij}) is positive semidefinite.

In order to simplify notations, we use the following Frobenius product of two matrices on n rows and m columns A and B : $A \bullet B := \sum a_{ij} bij$. Remark that $A \bullet B = \text{Tr}({}^t AB) = \text{Tr}(AB)$. A psd program can thus be written :

$$\text{Maximize } C \bullet A \;\; \textit{with the constraints} \quad \begin{array}{l} B_i \bullet A \leq b_i \text{ for every } \; i = 1 \dots k \\ A \succeq 0 \end{array} \tag{6.1}$$

In order to see the similarity with linear programming, recall that such a problem can be written as :

$$\max\{c.x \;:\; M.x \leq b \text{ and } u \text{ non-negative } \}$$

and its dual is :

$$\min\{y.b \;: y.M \geq c \text{ eand } y \text{ non-negative}\}.$$

Hence, if we consider our matrices as vectors of size n^2, a psd program is a linear program where the non-negativity constraint is replaced by the constraint that the matrix is positive semidefinite..

In fact let us just mention that these two types of constraints : u non-negative and A psd are two particular cases of convex cones. And more generally, if K_1 and K_2 are two closed convex cones, we can study problems of the form

$$\max\{c.x \ : \ b - M.x \in K_2 \text{ and } x \in K_1\}.$$

The dual of such a program is:

$$\min\{y.b \ : \ y.M - c \in K_1^* \text{ and } y \in K_2^*\}.$$

where K_i^* denotes the polar cone of K_i, that is $\{s : s.x \geq 0 \ \text{ for every } \ x \in K_i\}$.
The case of linear programming corresponds to the case where K_1 and K_2 are the set of vectors with non-negative entries since in that case we have $K_i^* = K_i$ (the cone is said to be *autopolar*)

Moreover, this formulation implies the weak duality Theorem which asserts that the max of the primal is less than the min in the dual. Indeed, if x and y are respectively solutions of the primal and dual constraints, then :

$$yb \geq yMx + ys_1 = cx + s_2x + ys_1 \geq c.x$$

since $s_i \in K_i^*$.

In the case of semidefinite programming, K denotes the convex cone of positive semidefinite matrices, and things are similar. Indeed, if we are in the space of symmetric matrices we also have the equality $K^* = K$. Here is the proof of this fact :

Lemma 6.3 *A symmetric matrix A is positive semidefinite if and only if for every positive semidefinite matrix B, we have $A \bullet B \geq 0$.*

Proof :
First, if A and B are psd, we have $A =^t MM$, $B =^t NN$, and thus

$$A \bullet B = \operatorname{Tr}({}^t({}^tMM)^tNN) = \operatorname{Tr}({}^tMM^tNN) = \operatorname{Tr}({}^tN^tMMN) \geq 0$$

The converse is a theorem due to Fejer, that can be proved this way. If for every psd matrix B, we have $A \bullet B \geq 0$, then in particular for every vector x, we have

$$0 \leq A \bullet (x^t x) = tr(Ax^t x) = \mathrm{Tr}({}^t x A x) = {}^t x A x$$

■

The dual of the semidefinite program (6.1) eventually can be written as:

$$\text{Minimize } \sum_{i=1}^{n} y_i b_i \text{ under the constraints } \sum_{i=1}^{n} y_i A_i \succeq C, \quad y \geq 0 \qquad (6.2)$$

There is equality between this min and the max of the primal problem if these constraints are said to be strictly feasible, that is if there exists a positive definite matrix satisfying the constraints (these are called Slater conditions, see [48] for more details).

Let us come back to the initial problem. We wanted, for a graph $G = (V, E)$, to find unit vectors $v_i \in \mathbb{R}^d$, $i \in V$, such that $< v_i, v_j > \leq -c$ si $(i, j) \in E$. With the semidefinite programming notations, this becomes

$$\begin{array}{lll} Find\ A \succeq 0\ \ such\ that & & A \bullet E_{ii} = 1\ \ for\ i = 1 \dots n \\ & et & A \bullet E_{ij} < -c\ \ for\ every\ (i,j) \in E. \end{array}$$

where the matrices E_{ij} are the matrix which only non zero entry is the ij one which is equal to 1.

Thanks to the previous discussion on the dual of such a program, this is possible only if

$$\begin{array}{ll} For\ every\ matrix\ A \succeq 0\ \ such\ that & A_{ij} \geq 0\ \forall (i,j) \\ & A_{ij} = 0\ \ for\ every\ (i,j) \notin E \end{array}$$

$$on\ a\quad \sum_{i=1}^{n} A_{ii} - c \sum_{i \neq j} A_{ij} \geq 0.$$

Then in conclusion we can state the following proposition

Proposition 6.1 *A graph $G = (V, E)$ is 1/3-spherical unless :*

$$\begin{array}{ll} there\ exists\ vectors\ x_i\ i \in V,\ such\ that & x_i.x_j \geq 0\ \ \forall (i,j) \\ & x_i.x_j = 0\ \ for\ (i,j) \notin\ E \\ & \sum_{i=1}^{n} ||x_i||^2 < \sum_{ij \in E} x_i.x_j \end{array}$$

Chapter 7

Parity Matrices and Feedback Arc Sets

7.1 A Problem by Dimitri Grigoriev

The problem we are discussing in this section comes from a question that was initially studied by Dimitri Grigoriev as a lemma in his paper [30]. He considers a family of N vectors in the vector space $E = (\mathbf{F}_2)^d$ and is looking for a vector in E that is orthogonal to at least $N/3$ vectors and at most $2N/3$ vectors of the family. One can thus wonder if it is possible to find a vector that would be orthogonal to roughly half the number of vectors in our family. We are going to prove that this range $[N/3, 2N/3]$ can be replaced by the much better one of $[N/2 - \sqrt{N}/2, N/2 + \sqrt{N}/2]$ and we will give a polynomial algorithm to find this vector. We will also study the optimality of this bound.
The results shown in this chapter come from a joint work with Pascal Koiran, Sylvain Perifel et Stéphan Thomassé ([19]).

We can reformulate this problem as a set theory one. It suffices to see each vector as the incidence vector of a subset of a set of size d. Hence we have a family $\mathcal{F}$ a set of N non-empty distinct subsets of X. Since the value of the scalar product of two vectors in $\mathbf{F}_2^d$ corresponds in the set-theoretic point of view to the parity of the intersection, the goal is to find a subset F of X such that the number of elements of $\mathcal{F}$ which have an odd intersection with F, is as close as possible to $\frac{|\mathcal{F}|}{2}$.

We are giving two proof of the same result. The first one, that follows the intuition, proceeds by taking a random subset and showing that it satisfies the required condition. The second describes the question more precisely and allows us to give a simple deterministic polynomial algorithm to find the good subset.

7.1.1 A probabilistic proof

The first natural idea for this problem is to take for F a random subset of X.

Theorem 7.1 *Let X be a finite set and $\mathcal{F}$ be a set of N non-empty subsets of X. There is a subset $F \subseteq X$ such that*

$$-\frac{\sqrt{N}}{2} \leq |\{F_i \in \mathcal{F} : |F \cap F_i| \text{ even}\}| - \frac{N}{2} \leq \frac{\sqrt{N}}{2} \tag{7.1}$$

Proof
Let $\mathcal{F}$ be $\{F_1, \ldots, F_N\}$. We choose a random subset F of X obtained by selecting or not every element of X with probability $1/2$.
Let Y_i be the random variable defined by :

$$Y_i = 1 \text{ if } |F \cap F_i| \text{ is even and 0 otherwise.}$$

Therefore we are interested in the random variable $Y = \sum_{i=1}^{N} Y_i$.
First, let us prove that $P(Y_i = 1) = 1/2$. This follows immediately from the facts that every subset F occurs with same probability and that there are as many odd and even subsets in F_i. Thus, by linearity of the expectation we have: $E(Y) = \sum_{i=1}^{N} E(Y_i) = N/2$.

Then we prove that the events $\{Y_i = 1\}$ are *pairwise* [1] independent. For this let us consider two elements F_1 and F_2 of $\mathcal{F}$. We have to prove that

$$P(Y_1 = 1 \cap Y_2 = 1) = P(Y_1 = 1)P(Y_2 = 1) = 1/4 \tag{7.2}$$

There are three cases :

[1] It can be shown that these events are not 4-wise independent. **C'est bien 4 ?**

- F_1 and F_2 are disjoint. In this case, it is clear that the events are independent.
- $F_1 \subseteq F_2$. This case can be reduced to the previous one for F_1 and $F_2 \setminus F_1$ and we still have (7.2).
- The three sets $A = F_1 \setminus F_2$, $B = F_1 \cap F_2$ et $C = F_2 \setminus F_1$ are non empty. Then $X_1 = 1$ and $X_2 = 1$ imply that $|A \cap F| \equiv |B \cap F| \equiv |C \cap F| \mod 2$. But since these three sets are disjoints, we have a probability 1/8 to be in the case even-even-even and 1/8 to be in the case odd-odd-odd. Eventually we also have (7.2).

Since the events are pairwise independent we have the property that :

$$Var(Y) = \sum_{i=1}^{N} Var(Y_i) = N/4$$

To conclude we use Tchebycheff's inequality.

$$P(|Y - E(Y)| > t) < \frac{Var(Y)}{t^2} = \frac{N}{4t^2}$$

For $t = \sqrt{N}/2$ this becomes

$$P(|Y - \frac{N}{2}| > \frac{\sqrt{N}}{2}) < 1\,,$$

which implies the existence of the given set ■

Remark 7.1 *Tchebycheff's inequality also insures that at least* $3/4$ *of the subsets* F *fall within the range* $N/2 - \sqrt{N}$ *and* $N/2 + \sqrt{N}$, *since*

$$P(|Y - \frac{N}{2}| > \sqrt{N}) < 1/4.$$

This gives a trivial randomized algorithm for finding such a set. The deterministic algorithm of section 7.1.4 achieve the better range $[N/2-\sqrt{N}/2, N/2+\sqrt{N}/2]$ *obtained in the theorem.*

7.1.2 A deterministic way of getting this result

We want to find a subset F that minimizes the range between $|\{i, F \cap F_i \text{ even }\}|$ and $|\{i, F \cap F_i \text{ odd }\}|$. But this means exactly finding F that maximizes the number of pairs $\{F_i, F_j\}$ with $|F \cap F_i| \not\equiv |F \cap F_j| \bmod 2$. Indeed, if t denotes $|\{i : F \cap F_i \text{ odd }\}| - \frac{k}{2}$, the number of such pairs is exactly $(k/2 - t)(k/2 + t) = k^2/4 - t^2$.
The crucial fact is that if $F \subset X$ and F_i, F_j are two elements of $\mathcal{F}$:

$$|F \cap F_i| \not\equiv |F \cap F_j| \bmod 2 \iff |F \cap (F_i \bigtriangleup F_j)| \equiv 1 \bmod 2$$

Thus, finding F that minimizes the range between $|\{i, F \cap F_i \text{ even }\}|$ and $|\{i, F \cap F_i \text{ odd }\}|$, is exactly finding F that maximizes $|\{(i,j) : F \cap (F_i \bigtriangleup F_j) \text{ odd }\}|$.

We consider the following bipartite graph (V, E):

– $V = (V_1 \cup V_2)$ where $V_1 = \{(i,j) : 1 \le i < j \le k\}$, and $V_2 = \mathcal{P}(X)$

– $(F, (i,j)) \in E$ iff $|F \cap (F_i \bigtriangleup F_j)|$ odd.

What we are looking for is a vertex of V_2 of maximum degree. Let $N(x)$ denote the set of neighbours of x. We will only need to apply the following Lemma for $A = V_2$, as in Lemma 7.2. However it turns out that we can characterize in Lemma 7.1 all the subsets $A \subset V_2$ for which the proof still holds.

Lemma 7.1 *Let $A \subset V_2$ such that $\emptyset \in A$ and $\forall F, F' \in A$, $(F \bigtriangleup F') \in A$. Assume moreover that $\forall x \in V_1$, $N(x) \cap A \neq \emptyset$. Then*

$$\forall x \in V_1,\ |N(x) \cap A| = \frac{|A|}{2}.$$

Proof
Let $x \in V_1$. By hypothesis, there exists $F \in A$ such that (x, F) is an edge of the graph. And by the other hypothesis the following map is well-defined,

$$\begin{array}{rcl} \phi : & A & \longrightarrow A \\ & F' & \longmapsto (F \bigtriangleup F') \end{array}$$

and is a bijection of $N(x) \cap A$ onto $A \setminus N(x)$ which proves the result. ∎

Lemma 7.2 *There exists a subset $A \subset V_2$ satisfying the hypothesis of Lemma 7.1.*

Proof
It suffices to take $A = V_2$. ■

Corollary 7.1 *There exists $F \in V_2$ such that $|N(F)| \geq \frac{|V_1|}{2}$*

Proof Double counting. ■

Corollary 7.2 *There exists $F \subset X$ such that $||\{i, F \cap F_i \text{ even }\}| - \frac{k}{2}| \leq \frac{\sqrt{k}}{2}$*

Proof
Let F be given by Corollary 7.1. Define $t = |\{i, F \cap F_i \text{ even }\}| - \frac{k}{2}$. Then $|N(F)| = (\frac{k}{2} + t)(\frac{k}{2} - t) = \frac{k^2}{4} - t^2$ and by hypothesis on F :

$$\frac{k^2}{4} - t^2 \geq \frac{|V_1|}{2} = \frac{k(k-1)}{4} = \frac{k^2 - k}{4}$$

which implies $|t| \leq \frac{\sqrt{k}}{2}$. ■

7.1.3 Discussion of the bounds

With the help of Theorem 7.1, we know that it is possible to reach the expected value within a range of order $\sqrt{N}$. One can wonder whether it is possible to insure a constant range. The following examples prove that this cannot be achieved.

Let us consider a set X with $n = 4k^2 + 1$ elements and $\mathcal{F}$ be the set of all subsets of X of size 2. Let $N = |\mathcal{F}|$. In this context, the problem is to partition X into two parts and count the number of edges through the cut, which are precisely the sets of F with odd intersection. We want to find $0 \leq a \leq n/2$ such that $a(N - a)$ is as close as possible to $N/2 = k^2(4k^2 + 1)$. But:

$$(2k^2 - k + 1)(2k^2 + k) = 4k^4 + k^2 + k$$
$$(2k^2 - k)(2k^2 + k + 1) = 4k^4 + k^2 - k$$

The function $a \longmapsto a(n-a)$ being decreasing on $[0, n/2]$, this proves that these are the two best values and that the error is at least k, which is of the order of $N^{1/4}$.

It is possible to refine this argument further, and get the bound $N^{1/3}$. We still use the same idea but instead consider subsets with three elements among n (so that $N = |\mathcal{F}| = \binom{n}{3}$). Let F be a subset of a elements. The number of elements of $\mathcal{F}$ whose intersection with F is of odd cardinality is then $a\binom{n-a}{2} + \binom{a}{3}$. Therefore, let

$$f(a) = \frac{a(n-a)(n-a-1)}{2} + \frac{a(a-1)(a-2)}{6} - \frac{n(n-1)(n-2)}{12}$$

be the difference with $|\mathcal{F}|/2$. We aim at showing that f is far from zero on integer values, when n is well chosen.

The zeros of f are $n/2$ and $n/2 \pm \sqrt{3n-2}/2$. From the variations of f, we see that the integers i so that $|f(i)|$ is minimal are among the six integers around the zeros. Suppose $n = 4k^2/3 + 1$ where $k \equiv 0 \bmod 3$. Then n is odd, and we have

$$f(\lfloor n/2 \rfloor) = f(n/2 - 1/2) = n/4 - 1/4$$

$$f(\lceil n/2 \rceil) = f(n/2 + 1/2) = -n/4 + 1/4$$

Furthermore, if $k \geq 2$ then $\sqrt{3n-2} = \sqrt{4k^2+1}$ is at most $1/16$ away from $2k$, so that

$$f(\lfloor n/2 + \sqrt{3n-2}/2 \rfloor) \geq f(n/2 + \sqrt{3n-2}/2 - 1/2 + 1/16) = 7n/16 + O(\sqrt{n}).$$

Similarly, the other three integers around the zeros have $\Omega(n)$ as image. Since the total number N of subsets of three elements among n is $O(n^3)$, the error is at least $\Omega(N^{1/3})$.

The same kind of calculations for subsets with 5 elements yields $\Omega(N^{2/5})$. It may have some plausibility that subsets of $n/2$ elements could yield the expected bound $\Omega(\sqrt{N})$, which would show the sharpness of the bounds in the theorem.

7.1.4 A deterministic polynomial time algorithm

We now present a very simple polynomial algorithm which finds a subset F achieving inequality (7.1) from Theorem 7.1. We work from the point of view described in subsection 7.1.2: given the subsets F_i, we need to find a subset F that has an odd intersection with more than half of the $F_i \triangle F_j$ (considered as a multiset). Note that these symmetric differences are all nonempty since the F_i are distinct. The algorithm goes this way.

1. We construct all the sets $F_i \triangle F_j$ and denote by $\mathcal{G}$ the multiset obtained.
2. Let $x \in X$. Let $\mathcal{G}'$ be the multiset of all elements of $\mathcal{G}$ not containing x.
   Apply recursively the algorithm to $X \setminus \{x\}$ and $\mathcal{G}'$. Thus we get a subset F' of $X \setminus \{x\}$ that has an odd intersection with more than half of the elements of $\mathcal{G}'$. Now there are two cases:
   - F' has an odd intersection with more than half of the elements of $\mathcal{G} \setminus \mathcal{G}'$. In this case $F = F'$ is a solution to our problem.
   - Otherwise, since x belongs to all elements of $\mathcal{G} \setminus \mathcal{G}'$, taking $F = F' \cup \{x\}$ gives a solution.

7.2 The Problem of the Minimum Feedback Arc Set

In this last section we will answer a question of graph theory that was asked by Bang-Jensen and Thommassen ([4]) concerning feedback arc sets.
Recall that a feedback arc set in a digraph $D = (V, A)$ is a set of arcs F such that $D - F$ is acyclic. The minimum size of a feedback arc set of D is denoted by $mfas(D)$. A classical result of Karp [35] asserts that finding a minimum feedback arc set in a digraph is NP-hard. Bang-Jensen and Thomassen [4] conjectured that finding a minimum fas in a tournament is also NP-hard. A very close answer was given by Ailon, Charikar and Newman in [1] where they prove that the problem is NP-hard under randomized reductions. Our approach is similar but the reduction we use is simpler and therefore easily derandomized via parity-check matrices (see Alon and Spencer [3], p.255). Finally we prove that the minimum fas for tournaments is polynomially equivalent to the minimum fas for digraphs, and thus NP-hard.

Remark 7.2 *Here we need to point out that a different proof of this conjecture, but which was unknown to us when we proved it, was found by Noga*

Alon some months before and is available in preprint (voir [2]) on his web page.

We will come back on the concepts introduced in the previous section. Let us consider a finite set X with z elements and the $2^z \times 2^z$ square matrix $\mathbf{A}$ whose rows and columns are indexed by the subsets F_i of X (in any order) and whose entries are $a_{ij} = (-1)^{|(F_i \cap F_j)|}$. The results of the previous section 7.1.2 implies the following lemma :

Lemma 7.3 *With the definition of* $\mathbf{A}$ *given above, for every subset* J *of* r *columns we have:*

$$\sum_{i=1}^{2^z} |\sum_{j \in J} a_{ij}| \leq 2^z \sqrt{r}$$

Indeed, since on each row we count 1 for an even intersection and -1 for an odd one, summing the coefficients a_{ij} on a row and taking the absolute value is exactly counting the difference between the number of subsets that intersect in an odd way and the subsets intersecting in an even way the subset corresponding to the chosen line.

Lemma 7.4 *Let* z *be any positive integer divisible by three. Let* $k = 2^z$ *and let* A *be the* $k \times k$ *matrix introduced in Lemma 7.3. Let* $B = (b_{ij})$ *be the matrix obtained from* A *by an arbitrary permutation of the columns. Define* q_i *as follows.*

$$q_i = \max\{|\sum_{j=1}^{p} b_{ij}| \, : \, p = 1, 2, \ldots, k\}$$

We have $\sum_{i=1}^{k} q_i \leq 2k^{5/3}$.

Proof
Define the integers $l = k^{2/3}$ and $s = k^{1/3}$. For all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, s$, we let $c_i^j = |\sum_{j'=(j-1)l+1}^{jl} b_{ij'}|$. By Lemma 7.3 we have $\sum_{i=1}^{k} c_i^j \leq k\sqrt{l}$ for all $j = 1, 2, \ldots, s$. Therefore $\sum_{i=1}^{k} \sum_{j=1}^{s} c_i^j \leq ks\sqrt{l} = k^{5/3}$.

We now evaluate q_i. Assume that p is defined so that $q_i = |\sum_{j'=1}^{p} b_{ij'}|$. Let j be such that $(j-1)l \leq p < jl$. Note that $q_i \leq c_i^1 + c_i^2 + \ldots + c_i^{j-1} + l$, the term l being an upper bound on $|\sum_{i=(j-1)l+1}^{p} b_{ij}|$. Thus $\sum_{i=1}^{k} q_i \leq$

$(\sum_{i=1}^{k}\sum_{j=1}^{s} c_i^j) + kl \leq 2k^{5/3}$. ■

By probabilistic arguments we can show that there exist matrices A achieving the bound $\sum_{j=1}^{k} q_j = O(k^{3/2})$ for any permutation of the columns. But we need deterministic reductions, and the $(2k^{5/3})$-bound is sufficient for this purpose. A more careful proof would give a $(\sqrt{2}k^{5/3})$-bound for any z, but again, we do not need it.

Theorem 7.2 *Let z be any positive integer divisible by three and let $k = 2^z$. There exists a bipartite tournament G_k, whose partite sets both have k vertices ($|V(G_k)| = 2k$) and mfas$(G_k) \geq \frac{k^2}{2} - 2k^{5/3}$. Furthermore, we can construct G_k in polynomial time.*

Proof

Let $A = (a_{ij})$ be the $k \times k$ matrix given in Lemma 7.4. Observe that A has $k(k+1)/2$ positive entries since every column has $k/2$ positive entries, except the emptyset column which has k. Let the partite sets of G_k be $\{r_1, r_2, \ldots, r_k\}$ and $\{s_1, s_2, \ldots, s_k\}$ respectively. Now add an arc from r_i to s_j if $a_{ij} = -1$ in A, and add an arc from s_j to r_i if $a_{ij} = 1$ in A. This clearly defines a bipartite tournament, which can be constructed in polynomial time.

Let π be a minimum feedback arc set order of G_k, i.e. an enumeration of the vertices for which the number of backward arcs is *mfas*(G_k). Without loss of generality we may assume that the order of the s_j's in π is $s_1, s_2, \ldots, s_k$. Let $i \in \{1, 2, \ldots, k\}$ be arbitrary and define p such that $s_1, s_2, \ldots, s_p$ come before r_i in π and $s_{p+1}, s_{p+2}, \ldots, s_k$ come after r_i in π. Let m_i denote the number of "1" in row i and note that the number of backward arcs adjacent to r_i is the following:

$$\begin{aligned} & |\{a_{ij} : a_{ij} = -1,\, j \leq p\}| + |\{a_{ij} : a_{ij} = 1,\, j > p\}| \\ = \; & |\{a_{ij} : a_{ij} = -1,\, j \leq p\}| + (m_i - |\{a_{ij} : a_{ij} = 1,\, j \leq p\}|) \end{aligned}$$

Let $q_i = \min\{\sum_{j=1}^{p} a_{ij} : p = 1, 2, \ldots, k\}$ and note that the minimum feedback arc set of G_k is at least $\sum_{i=1}^{k}(m_i + q_i)$, which by Lemma 7.4 implies that $mfas(G_k) \geq \frac{k(k+1)}{2} - 2k^{5/3} > \frac{k^2}{2} - 2k^{5/3}$. ■

Theorem 7.3 *The minimum feedback arc set for tournaments is NP-hard.*

Proof
We will reduce from the minimum feedback arc set in general digraphs, so let D be any digraph of order n. We may assume that D has no cycles of length two, as deleting such a cycle decreases the minimum feedback arc set by exactly one. We may also assume that D has no loops. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$ and let $k = 2^{6\lceil 1+log_2(n)\rceil}$. Note that $k \in O(n^6)$ and $k \geq 64n^6$.

Let the vertices in the partite sets of G_k, which was defined in Theorem 7.2, be $\{r_1, r_2, \ldots, r_k\}$ and $\{s_1, s_2, \ldots, s_k\}$ respectively.

We now construct the tournament T with vertex set $\{w_i^j : i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k\}$ and the arc set described below. Let $a, b \in \{1, 2, \ldots, n\}$ and $i, j \in \{1, 2, \ldots, k\}$ be arbitrary. An arc between the vertices w_a^i and w_b^j is in T according to the following rules.

(a): $w_a^i w_b^j \in A(T)$ if $a = b$ and $i < j$.

(b): $w_a^i w_b^j \in A(T)$ if $v_a v_b \in A(D)$.

(c): If v_a and v_b have no arc between them in D and $a < b$ then $w_a^i w_b^j \in A(T)$ if $r_i s_j \in A(G_k)$ and $w_b^j w_a^i \in A(T)$ if $s_j r_i \in A(G_k)$.

Roughly speaking, we blow-up every vertex of D by a transitive tournament of size k, and we fill-in the bipartite gaps resulting from non-edges of D by copies of G_k.

We will now bound $mfas(T)$ from both above and below. Without loss of generality assume that $|\{v_a v_b : v_a v_b \in A(D),\ a > b\}| = mfas(D)$. Note that Theorem 7.2 implies that the arcs generated by (c) above will always contribute at least $(\binom{n}{2} - |A(D)|)(\frac{k^2}{2} - 2k^{5/3})$ to $mfas(T)$ and at most $(\binom{n}{2} - |A(D)|)(\frac{k^2}{2} + 2k^{5/3})$. Now consider the following order of the vertices in T.

$$w_1^1, w_1^2, \ldots, w_1^k, w_2^1, w_2^2, \ldots, w_2^k, w_3^1, w_3^2, \ldots \ldots, w_n^k$$

This order implies the following bound on $mfas(T)$.

$$mfas(T) \leq k^2 mfas(D) + \left(\binom{n}{2} - |A(D)|\right)\left(\frac{k^2}{2} + 2k^{5/3}\right)$$

In order to bound $mfas(T)$ from below let π be an ordering of the vertices in T, such that exactly $mfas(T)$ arcs are backward in the ordering. Let $i_1, i_2, \ldots, i_n$ all be integers from $\{1, 2, \ldots, k\}$ and note that there are at least $mfas(D)$ arcs between vertices in $\{w_1^{i_1}, w_2^{i_2}, w_3^{i_3}, \ldots, w_n^{i_n}\}$ which are backward arcs in π, as this set of vertices induce a digraph isomorphic to D. By summing over all possible values of $i_1, i_2, \ldots i_n$ we get $k^n mfas(D)$ backward arcs, where each arc can be counted at most k^{n-2} times. This implies the following bound.

$$mfas(T) \geq \frac{k^n mfas(D)}{k^{n-2}} + \left(\binom{n}{2} - |A(D)|\right)\left(\frac{k^2}{2} - 2k^{5/3}\right)$$

Note that as $k^{1/3} \geq 64^{1/3} n^2 = 4n^2$ we get that $(\binom{n}{2} - |A(D)|) \times 2k^{5/3} < k^2 \frac{2n^2}{k^{1/3}} \leq \frac{k^2}{2}$. The above two bounds now imply the following.

$$mfas(D) - \frac{1}{2} < \frac{mfas(T)}{k^2} - \frac{1}{2}\left(\binom{n}{2} - |A(D)|\right) < mfas(D) + \frac{1}{2}$$

So if we could compute $mfas(T)$ in polynomial time, we would also have computed $mfas(D)$. As our reduction is polynomial, this implies the result. ■

Bibliography

[1] N. Ailon, M. Charikar and A. Newman, Aggregating Inconsistent Information: Ranking and Clustering. preprint.

[2] N. Alon, Ranking tournaments. *SIAM Journal of Discrete Mathematics*, to appear

[3] N. Alon and J. Spencer, The probabilistic method. Second edition. *Wiley-Interscience Series in Discrete Mathematics and Optimization*, 2000.

[4] J. Bang-Jensen and C. Thomassen, A polynomial algorithm for the 2-path problem for semicomplete digraphs. *SIAM J. Discrete Math.*, 5: 366-376, 1992..

[5] J. Bang-Jensen and G. Gutin, Digraphs :Theory, Algorithms and Apllications. *Springer Monographs in Mathematics* , 2001.

[6] M. Behzad, Minimal 2-regular digraphs with given girth. *J. Math. Soc. Japan* 25:1–6, 1973.

[7] M. Behzad, G. Chartrand and C.E. Wall, On minimal regular digraphs with given girth. *Fund. Math* 69:227-231, 1970.

[8] C. Berge, Graphs. *North-Holland*, 1985.

[9] C. Berge, k-optimal partitions of a directed graph. *Europ. J. Combonatorics* 3:97-101, 1982.

[10] J.-C. Bermond, 1-graphes re'guliers minimaux de girth donne. *Colloque sur la The'orie des Graphes (Paris, 1974). Cahiers Centre E'tudes Recherche Ope'r. 17* 2-4:125–135, 1975.

[11] J. A. Bondy and P. Charbit, Cyclic Orders and Circtuis Covering of Gigraphs. *preprint*, 2004

[12] J. A. Bondy, Large Cycles in Digraphs. *Discrete Math.* 1(2):121-132, 1971.

[13] J. A. Bondy, Counting subgraphs: a new approach to the Caccetta-Häggkvist conjecture. *Discrete Math.*, 15:165-166, 1997.

[14] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications. *North-Holland*, 1976.

[15] S. Bessy and S. Thomassé, Spanning a strong digraph by α circuits: A proof of Gallai's conjecture. *Combinatorica*, to appear.

[16] S. Brandt and S. Thomassé, Dense Triangle-Free Graphs are Four-Colourable : A Solution to the Erdős-Simonovits Problem. preprint.

[17] L. Caccetta, R. Ha"ggkvist, On minimal digraphs with given girth. *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing* 181–18, 1978.

[18] V Chvátal, E. Szemere'di, Short cycles in directed graphs. *J. Combin. Theory Ser. B* 35:323–327, 1983.

[19] P. Charbit, P. Koiran, S. Perifel and S. Thomassé, Finding a Vector Orthogonal to Roughly Half a Collection of Vectors. preprint.

[20] P. Charbit and András Sebő, Cyclic Orders : Equivalence and Duality. preprint.

[21] P. Charbit and S. Thomassé, Graphs with Large Girth not Embeddable in the Sphere. *Combinatorics Probability and Computing*, to appear.

[22] P. Charbit, S. Thomassé, Anders Yeo, The Minimum Feedback Arc Set Problem is NP-Hard for Tournaments. *Combinatorics Probability and Computing*, to appear.

[23] V. Chvátal, Linear Programming. *Freeman, A Series of Books in the Mathematical Sciences*, 1983.

[24] G. L. Cherlin, Homogeneous directed graphs. The imprimitive case. *Logic Colloquium 1985* 67-88, 1985.

[25] R. Diestel. Graph Theory. *Springer-Verlag, Heidelberg Graduate Texts in Mathematics, Volume 173* , 2000.

[26] T. Gallai, Problem 15, *Theory of Graphs and its Applications* 161, 1964.

[27] M. Goemans, Semidefinite Programming in Combinatorial Optimization. *Mathematical Programming*, 79:143-161, 1997.

[28] M. X. Goemans and D. P. Williamson, Improved Approxiamtion Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming, *J. ACM* 42:1115-1145, 1995.

[29] R. L. Graham and M. Grő;tschel and L. Lovász, Handbook of combinatorics, *MIT Press*, 1995.

[30] D. Grigoriev, Topological complexity of the range searching.*Journal of Complexity*, 16:50–53, 2000.

[31] Y.O. Hamidoune, Quelques proble‘mes de connexite’ dans les graphes oriente’s. *J. Combin. Theory Ser. B* 30:1–10, 1981.

[32] Y.O. Hamidoune, A note on minimal directed graphs with given girth. *J. Combin. Theory Ser. B* 43:343–348, 1987.

[33] C.T. Ho‘ang, B. Reed, A note on short cycles in digraphs. *Discrete Math* 66:103–107, 1987.

[34] D. Karger, R.Motwani and M.Sudan, Approximate graph colouring by semidefinite programming, *J. ACM* 45:162-169, 1978.

[35] R. Karp, Reducibility among combinatorial problems, *Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.*, 85-103, 1972.

[36] A. H. Lachlan and R. Woodrow, Countable ultrahomogeneous undirected graphs. *Transactions of the American Mathematical Society,*. 262 (1): 51-94, 1980.

[37] D.G. Larman, A Triangle Free Graph Which Cannot be square root of 3 Imbedded in any Euclidean Unit Sphere. *J. Comb. Theory, Ser. A* 24(2):162–169, 1978.

[38] N. Linial, Extending the Greene-Kleitman Theorem to Directed Graphs, *J. Comb. Theory, Ser. A* 30:331-334, 1981.

[39] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.*, 10:96–115, 1927.

[40] J. Nešetřil and M. Rosenfeld, Embedding graphs in Euclidean spaces, an exploration guided by Paul Erdős. *Geombinatorics* 6:143–155, 1997.

[41] V. Rődl, On Combinatorial Properties of Spheres in Euclidiean Spaces. *Combinatorica* 4:345-349: 1984.

[42] M. Rosenfeld, Triangle free graphs that are not $\sqrt{3}$-embeddable in S^d. *J. Comb. Theory Series B* 33:191–195, 1982.

[43] A. Schrijver, Combinatorial Optimization, Springer, 2003.

[44] A. Sebő, Minmax Relations for Cyclically Ordered Graphs. *Cahiers du Laboratoire Leibniz*, 2004.

[45] J. Shen, Directed triangles in digraphs. *J. Combin. Theory Ser. B* 74:405–407, 1998.

[46] J. Shen, On the girth of digraphs. *Discrete Math.* 211:167–181, 2000.

[47] J. Shen, On the Caccetta-Häggkvist conjecture. *Graphs and Combinatorics* 18(3):645–654, 2002.

[48] M. Todd, Semidefinite Optimization, *Acta Numerica* 10:515–560, 2001.

[49] X. Zhu, Circular chromatic number: a survey, *Discrete Mathematics*, 229 (1-3): 371-410, 2001.

UNIVERSITE CLAUDE BERNARD - LYON I

DIPLOME DE DOCTORAT
Auteur: Pierre Charbit
Titre: Circuits in Graphs via Embeddings

Résumé de la thèse:

When one studies properties or invariants even purely combinatorial ones, of graphs, it is often useful to consider their embedings in various surfaces. This thesis presents several results in this direction, concerning the circuits of non-oriented or directed graphs.

In the first chapters, we consider the circular embeddings by giving a detailed study of the notion of cyclic order, that was introduced by S. Bessy and S. Thomassé in their recent proof of a forty year old conjecture of T. Gallai. New proofs of their theorems as well as new results on this notion are exposed.

Then we are interested in embeddings of triangle free graphs, in particular when there are conditions on the minimum degree. In the oriented case, we consider toric embeddings and will be related to a question that is now open since thirty years : the conjecture of Caccetta-Häggkvist . This chapter will contain results concerning this question, we in particular state several equivalent formulations. Then in the non-oriented case, we prove a new result related to a question of M. Rosenfeld concerning spherical embeddings.

Eventually, the last chapter of this thesis contains two results. The first one concerns some properties of orthogonality of vectors in $\mathbb{F}_2^n$ and permits to prove the second one, which is the answer to a question raised by J Bang-Jensen and C. Thomassen : prove that the determination of the size of minimum feedback arc set is a NP-Hard problem for the class of tournaments.