Taylor expansion, finiteness and strategies

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Programs as transformations

$P : A \to B$ transforms elements of A into elements of B.

A and B can be ordinary types, or more complex objects as power series, probability distributions,. . .

But P is still a transformation, P *does* something.

Resource consumption

A program $P : A \to B$, as every natural process, needs energy to run. It *consumes* the elements of A.

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Quantitative semantics

Interpret precisely this mecanisms of consuption in a model.

Denotational semantics of Call-By-Name λ-calculus

$M, N ::= x \mid \lambda x M \mid MN$
$(\lambda x M)N \to_\beta M[N/x]$
$[[M]]_{\mathcal{M}} \rightsquigarrow$ Something in a structure $\mathcal{M}$ invariant under $\to_\beta$

For a program $P : A \to B$, $[[P]]$ can be *e.g* a function or a relation between $[[A]]$ and $[[B]]$.

Quantitative approach

Think about resources

- Quantitative meaning of $[[M]]$ in $(\lambda x(xx))M$? *wrt* $(\lambda xx)M$?
- Interpret probabilistic reduction ?
- ...

Girard (Normal functors, 1988)

Uses of arguments $\rightsquigarrow$ degree of a monomial in a power series.

Types: $[[A]] \subseteq \mathbb{S}^{|A|}$ where $\mathbb{S}$ is a semiring
Programs : power series

$\mathbb{S}$	$\rightsquigarrow$	Boolean semiring
Types	$\rightsquigarrow$	$[[A \to B]] = \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Programs	$\rightsquigarrow$	$P : A \to B \Rightarrow [[P]] \subseteq \mathcal{M}_{\text{fin}}(\mid A \mid) \times \mid B \mid$
Invariance	$\rightsquigarrow$	Composition of multirelations.

Key idea

let $M : A \to B,\ N : A$.
$([a_1, \ldots, a_k], b) \in [[M]]$ will match with k uses of the argument N in the application (MN).

Quantitative semantics leads to fertile studies when applied to probabilistic programming languages.

Key result

Probabilistic coherence spaces (Danos & Ehrhard) are fully abstract for probabilistic PCF (Ehrhard, Pagani, Tasson 2015)

The proof uses Taylor expansion.

From quantitative semantics, a syntax with multisets and sums is derived.

The *resource terms* can be seen as points of the relational semantics, with a dynamics.

$$m, n ::= x \mid \lambda x m \mid \langle m \rangle [n_1, \ldots, n_k] \text{ (k-linear application)}$$

$$\langle \lambda x m \rangle [n_1, \ldots, n_k] \to_\partial \sum_{\sigma \in \mathfrak{S}_n} m \left[n_{\sigma(1)}/x_1, \ldots, n_{\sigma(k)}/x_k \right]$$

$$\langle \lambda x \langle x \rangle [x] \rangle [z] \to_\partial 0 \ {}_\partial\!\leftarrow \langle \lambda x x \rangle [z, z]$$

λ-calculus		resource calculus
MN	$\leadsto$	$\langle M \rangle [N_1, \ldots, N_k]$
$(\lambda x (x) x) z$	$\leadsto$	$\langle \lambda x \langle x \rangle [x] \rangle [z, z]$
$\downarrow_\beta$		$\downarrow_\partial$
zz	$\leadsto$	$\langle z \rangle [z] + \langle z \rangle [z]$

Multilinear Approximations

We define the approximation $m \lhd M$:

- $x \lhd x$
- $\lambda x n \lhd \lambda x N$ if $n \lhd N$.
- $\langle m \rangle [n_1, \ldots, n_k] \lhd MN$ for all $k \in \mathbb{N}$ if $m \lhd M$ and $n_i \lhd N$.

(Resource terms can be seen as polynomials that approximate power series.)

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Property : simulation of $\to_\beta$ with approximants

If $m \lhd M$ and $M \to M'$, $m \to 0$ or $\exists m'$ s.t. $m \rightrightarrows_\partial m'$ and $m' \lhd M'$.

Parallel reduction

Definition

We extend $\to_\partial$ to a parallel reduction $\rightrightarrows_\partial$.

Example

- $MN \to MN'$
- Let $\langle m\rangle[n_1, \ldots, n_k] \lhd MN$.
- $\langle m\rangle[n_1, \ldots, n_k] \rightrightarrows_\partial \langle m\rangle[n'_1, \ldots, n'_k] \lhd MN'$ if $n_i \rightrightarrows_\partial n'_i$ for **all** i.

In the resource setting, Taylor expansion consists in taking infinite sums of resource terms.

The idea is that taken together, the combination of all $m \lhd M$, behave exactly as M.

Semantic approach : Interpret a term/function as an infinite series of approximants.

Syntactic Taylor expansion :

$$\mathcal{T}(MN) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \langle \mathcal{T}(M) \rangle [\mathcal{T}(N), \ldots, \mathcal{T}(N)]_k$$

$\mathcal{T}(\lambda x M) = \lambda x \mathcal{T}(M) \qquad \mathcal{T}(x) = x.$

Remark

$\mathcal{T}(M)$ is a weighted sum of all resource nets m s.t. $m \lhd M$

Simulation

Wanted result

Extend $\rightrightarrows_\partial$ to infinite sums of terms ($\Rightarrow_\partial$), in order to have $M \to_\beta N \Rightarrow \mathcal{T}(M) \Rightarrow_\partial \mathcal{T}(N)$.

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Can $\Rightarrow_\partial$ be always well-defined ?

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Counterexample

$$\sum_{k \in \mathbb{N}} \langle \lambda x x \rangle [\langle \lambda x x \rangle \dots [y]]] \dots] \Rightarrow_\partial \infty \cdot y$$

If $\mathbb{S}$ is not a complete semiring, the reduction is not defined for all series.

Some convergence results

$\Rightarrow_\partial$ and normalization are well defined and commute with Taylor expansion:

- Classical Λ: Ehrhard Regnier 2007.
- Non deterministic Λ with finite sums : Pagani, Tasson, Vaux-Auclair 2016.
- Algebraic calculus: Vaux-Auclair 2017.

Different proof methods

- Uniform calculi: Coherence relation on resource terms
- Non uniform calculi (and Linear Logic proof nets): Bound the applicative depth of resource terms

Contribution

Are these results and proof methods
valid for other operational semantics?

Call-by-Value, Call-by-Need, PCF (with explicit fixpoint),
Call-by-Push-Value, . . . ?

It depends on the definition of approximants.

Contribution

Infinite coefficients never appear
when computing Taylor expansion

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Call-By-Value Taylor expansion

Resource terms

$v ::= x \mid \lambda x m$
$m, n ::= [v_1, \dots, v_k] \mid mn$

Approximation

- $[x, \dots, x] \lhd x$
- $[\lambda x m_1, \dots, \lambda x m_k] \lhd \lambda x M$ if $m_i \lhd M$.
- $mn \lhd MN$ if $m \lhd M$ and $n \lhd N$.

Linear reduction

$[\lambda x m][n_1, \dots, n_k] \to_v \sum_{\sigma \in \mathfrak{S}_k} m[n_{\sigma(1)}/x_1, \dots, n_{\sigma(k)/x_k}]$
$[\lambda x m_1, \dots, \lambda x m_k]m \to 0$ if $k \neq 1$.

Property : If $[\lambda x m]n \lhd (\lambda x M)N$, it is a redex if and only if N is a variable or an abstraction.

Fixpoint operator

$\mathbf{Fix}(M) \to M(\mathbf{Fix}(M))$

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We have to consider an infinite set of resource terms $F = \{f_n \mid n \in \mathbf{N}\}$ such that $\langle f_n \rangle [m, \ldots, m] \to^n \langle m \rangle [\langle m \rangle [\ldots \langle m \rangle [\,]]]$.

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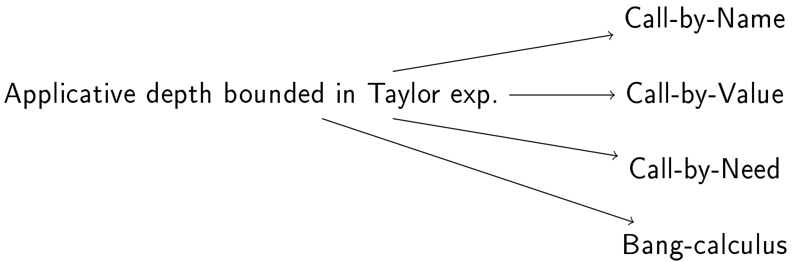
- F has an unbounded applicative depth
- F is uniform

Overview



Provided there is no explicit fixpoint, we can prove finiteness of coefficients without relying on uniformity. And then consider algebraic extensions of the calculus.

- Designing the syntax of linear approximants is the key to establish semantical properties of various calculi
- Algebraic extensions of the calculi are not always possible, depending on the structure of resource terms.
- Untyped fixpoint $<$ Typed fixpoint.
- Ongoing work (with Christine Tasson) : design a convenient resource calculus for Call-By-Push-Value.

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- Thank you