Some dynamical systems associated with continued fractions and $S$-adic systems

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Relying on joint work with:

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Generalized continued fractions:

Natural extensions, Suspensions, Symbolic dynamics, tilings, toral translations and the Weyl chamber flow
Generalized unimodular continued fraction

- Generalized unimodular continued fraction:
- A piecewise projective map $T$ on a cone $\Lambda \subset \mathbb{R}^d$
- It is associated with a map $M : \Lambda \rightarrow GL(d, \mathbb{Z})$.
- $T(x) = M(x)^{-1}x$.
- This is the Gauss map
- Being a projective map, it admits many presentations,
- depending on the presentation
The map $T$ is defined on the positive quarter plane

$M(x, y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ if $x < y$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $x > y$

If we normalize by $x + y = 1$, we obtain $T_1 : [0, 1) \to [0, 1)$,

$x \mapsto \frac{x}{1-x}$ if $x < \frac{1}{2}$, $x \mapsto 2 - \frac{1}{x}$ if $x > \frac{1}{2}$.

$T(x) = M(x)^{-1}x$.

If we normalize by $y = 1$, we obtain $T_2 : [0, +\infty) \to [0, +\infty)$,

$x \mapsto \frac{x}{1-x}$ if $x < 1$, $x \mapsto x - 1$ if $x > 1$.

It is the same map.
The Farey map $T_1$
Natural extensions and Gauss measures

- There is a natural formula for a natural extension
- $\tilde{T}(x, y) = (M(x)^{-1}x, tM(x)y)$.
- $\tilde{T}$ preserves Lebesgue measure
- It preserves the symplectic form $\sum x_i y_i$
- If $T$ is strictly expanding, $\tilde{T}$ has a unique compact invariant set $K$
- If $K$ has nonempty interior, it is a model for the natural extension of $T$. 

The natural extension of the Farey map

- The map $\tilde{T}_1$ is defined on $\{(x, y) | 0 < x < 1, \frac{1}{1-x} < y < \frac{1}{x}\}$.
- $(x, y) \mapsto \left( \frac{x}{1-x}, (1-x)^2 y + 1 - x \right)$ if $x < \frac{1}{2}$.
- $(x, y) \mapsto (2 - \frac{1}{x}, x^2 y - x)$ if $x < \frac{1}{2}$.
- This map preserves Lebesgue measure.
- Hence $T_1$ has invariant (infinite) density $\frac{1}{x(1-x)}$. 
The natural extension of the Farey map

![Graph showing the natural extension of the Farey map.](image-url)
A Boole type map

- If we conjugate by a primitive of the invariant density,
- we obtain a Boole type map $T_3 : \mathbb{R} \to \mathbb{R}$
- $x \mapsto \log(e^x - 1)$ if $x > 0$
- $x \mapsto -\log(e^{-x} - 1)$ if $x < 0$
- This is an ergodic map on $\mathbb{R}$ which preserves Lebesgue measure.
- Can we classify all such maps? The first was the original Boole map
- $x \mapsto x - \frac{1}{x}$. 
Generalised continued fractions
S-adic models associated with continued fractions
Gauss maps and their natural extensions
Suspensions

Some dynamical systems associated with continued fractions
A simple example: Brun Continued Fractions

- Given 3 positive real numbers x, y, z
- Subtract the second biggest from the biggest, and iterate
- It is associated with the 6 elementary matrices on the 6 sorted subcones of the positive cone.
- It satisfies a Markov condition.
Natural extension of Brun Continued Fractions
Natural extension of Brun Continued Fractions

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Some dynamical systems associated with continued fractions
A 2-adic example: Saito’s algorithm

- This GCD algorithm is defined on pairs of integers \((p, q)\) with \(q\) odd.
- If \(p\) is even, we replace the pair by \((\frac{p}{2}, q)\).
- If \(p\) is odd, we replace the pair by \((\frac{p+q}{2}, p)\).
- This algorithm converges to \((d, d)\), where \(d\) is the GCD of \(p\) and \(q\).
A 2-adic example: Saito’s algorithm

- It is associated to a Gauss map on $\mathbb{Z}_2$ defined by
- $T(x) = x/2$ if $x$ is not invertible
- $T(x) = \frac{1}{x} + \frac{1}{2}$ if $x$ is invertible.
- $T$ preserves the Haar measure on $\mathbb{Z}_2$
- This map should allow to perform a probabilistic analysis of this algorithm.
Suspensions

- It is natural to look for a suspension flow of the natural extension
- and there is an obvious candidate
- the flow $\phi_t(x, y) = (e^t x, e^{-t} y)$ commutes
- and preserves the symplectic form $x.y$
- It gives a suspension of the natural extension, by a suitable quotient space.
The question is to give a meaning to the underlying space

This is sometimes possible:

- Modular spaces
A dynamic model for continued fractions

- We want to find a family of systems $R_\alpha$.
- where $\alpha$ is in the domain of the Gauss map $T$.
- and a family of subsets $A_\alpha$.
- such that the induced map $R_\alpha|A_\alpha$ is conjugate to $R_{T\alpha}$.
- Example 1: continued fraction, rotations and the geodesic flow on the modular surface.
- Example 2: interval exchange maps, Rauzy induction and the Teichmüller flow.
- this has been hugely successful (but highly technical)
A dynamic model for continued fractions

- Extend this to a higher dimensional continued fraction map?
- Problem: find the family of dynamical systems (translations on compact groups).
- Difficult in dimension $> 1$.
- The set $A_\alpha$ has special properties:
  - It should be a bounded remainder set (not easy to find)
  - It should give symbolic dynamics with linear complexity
  - In the periodic case:
  - It needs to have fractal boundaries (Markov partition).
Symbolic dynamics

- Geometry is difficult: we are going to use symbolic dynamics.
- The basic idea:
  - With a given point, we associate a sequence $M_n$ of matrices.
  - With each matrix $M_n$, we associate a substitution $\sigma_n$ having this matrix.
  - This infinite sequence of substitution defines an infinite limit sequence of letters ($S$-adic sequence).
Symbolic dynamics

- The $S$-adic sequence defines a symbolic dynamical system which is renormalizable by construction.
- Think of a generalized Sturmian sequence, with its infinite sequence of renormalizations.
- This associates to the initial point $\alpha$ a symbolic system $(\Omega_\alpha, S)$.
- We could take this as an answer, but we want a geometric model.
- We consider the stepped line obtained by abelianization of $u \in \Omega_\alpha$ : we want it to be a model set.
Some properties of symbolic systems

- We want \((\Omega_\alpha, S)\) to be minimal.
- We want \((\Omega_\alpha, S)\) to be uniquely ergodic: asymptotic direction for the stepped line.
- We want this asymptotic direction to be totally irrational.
- We want the stepped line to be *balanced*, that is, remain within bounded distance of the asymptotic direction.
Some properties of symbolic systems

- If the stepped line is \textit{balanced}, it is (almost) a model set.
- We can project it on the diagonal plane to obtain a compact set with nonempty interior which gives a locally finite covering by action of the diagonal subgroup of $\mathbb{Z}^3$.
- If this covering is a tiling, the stepped line is a true model set.
- All these properties can be ensured by a generalized Pisot property on the sequence of matrices, plus some combinatorics.
- For some classical algorithms, the Pisot property was proved by Avila-Delecroix.
A plan of attack

1. Study some properties of sequence of positive matrices.
2. Define the corresponding $S$-adic system.
3. Prove that the asymptotic properties of sequence of matrices imply properties of the $S$-adic system.
4. Define the dynamical geometric model for this sequence of matrices: Rauzy fractals and Rauzy boxes.
5. Prove that these properties are almost everywhere defined for some algorithms, using Oseledets and computations by Avila-Delecroix.
6. This plan has been realised in several cases (Brun’s algorithm, for example)
The classical example

- The basic algorithm is the geodesic flow on the modular surface
- We will show a few movies (due to Edmund Harriss) showing this flow
- and various arithmetic properties
- A first movie: *the shape of lattices* can be found at
- https://www.youtube.com/watch?v=vLrlIp4Uc0
A last curious example

- The positive rationals can be generated by the Calkin Wilf tree
- Where \( \frac{p}{q} \) has descendants \( \frac{p}{p+q} \) and \( \frac{p+q}{q} \).
- Moshe Newman (see A. Malter, D. Schleicher, and D. Zagier) proved that the map
  \[ x \mapsto \frac{1}{2\lfloor x \rfloor + 1 - x}, \] starting from 0, gives a breadth-first enumeration of the tree.
- This is a simple and interesting dynamical system
Generalised continued fractions
S-adic models associated with continued fractions

Continued fraction as renormalisation
A curious dynamical system

Some dynamical systems associated with continued fractions
A last curious example

- This map preserves the tail of the continued fraction expansion
- Here is the orbit of $e - 1$
- given by the continued fraction
Some dynamical systems associated with continued fractions

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[[1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[0, 1, 3, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[4, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 4, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[1, 3, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 1, 1, 2, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[2, 2, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 2, 1, 1, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[1, 1, 1, 1, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[0, 1, 2, 1, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[3, 1, 2, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 3, 3, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[1, 2, 3, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 1, 1, 1, 3, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[2, 1, 3, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 2, 4, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[1, 1, 4, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 1, 5, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[6, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1],
[0, 6, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[1, 5, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 1, 1, 4, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[2, 4, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 2, 1, 3, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[1, 1, 1, 3, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[0, 1, 2, 3, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[3, 3, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
[0, 3, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[1, 2, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
[0, 1, 1, 1, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1]]
A last curious example

- The sum of the coefficients is preserved!
- This map exchanges $[0, 1]$ and $[1, +\infty]$
- Its square is a map from $[0, 1]$ to itself.
Generalised continued fractions
S-adic models associated with continued fractions

Continued fraction as renormalisation
A curious dynamical system

Some dynamical systems associated with continued fractions
A last curious example

- If we conjugate by the Question Mark function of Minkowski,
- we obtain an interesting result
Generalised continued fractions

$S$-adic models associated with continued fractions

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Some dynamical systems associated with continued fractions
A last curious example

- This map is conjugate to the 2-adic odometer
- Hence it is uniquely ergodic
- Its unique ergodic invariant measure
- is a singular measure
- Image of Lebesgue measure by the Minkowski function
- This map could be related to the horocycle flow.