Some dynamical systems associated with continued fractions and S-adic systems

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Relying on joint work with :

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Generalized continued fractions :

Natural extensions, Suspensions, Symbolic dynamics, tilings, toral translations and the Weyl chamber flow

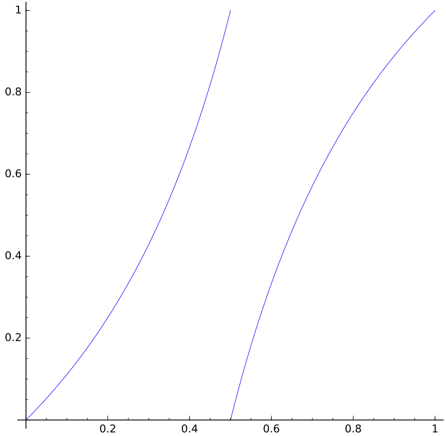
Generalized unimodular continued fraction

- Generalized unimodular continued fraction :
- A piecewise projective map T on a cone $\Lambda \subset \mathbb{R}^d$
- It is associated with a map $M : \Lambda \to GL(d, \mathbb{Z})$.
- $T(x) = M(x)^{-1}x$.
- This is the Gauss map
- Being a projective map, it admits many presentations,
- depending on the presentation

One example : the Farey map

- The map T is defined on the positive quarter plane
- $M(x,y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ if $x < y$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $x > y$
- If we normalize by $x + y = 1$, we obtain $T_1 : [0,1) \to [0,1)$,
- $x \mapsto \frac{x}{1-x}$ if $x < \frac{1}{2}$, $x \mapsto 2 - \frac{1}{x}$ if $x > \frac{1}{2}$.
- $T(x) = M(x)^{-1}x$.
- If we normalize by $y = 1$, we obtain $T_2 : [0,+\infty) \to [0,+\infty)$,
- $x \mapsto \frac{x}{1-x}$ if $x < 1$, $x \mapsto x - 1$ if $x > 1$.
- It is the same map.

The Farey map T_1



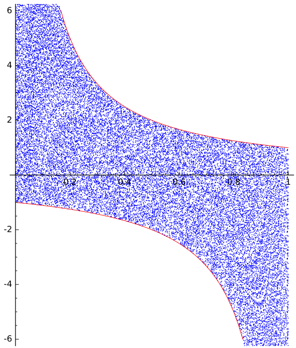
Natural extensions and Gauss measures

- There is a natural formula for a natural extension
- $\tilde{T}(x, y) = (M(x)^{-1}x, {}^{t}M(x)y)$.
- $\tilde{T}$ preserves Lebesgue measure
- it preserves the symplectic form $\sum x_i y_i$
- If T is strictly expanding, $\tilde{T}$ has a unique compact invariant set K
- if K has nonempty interior, it is a model for the natural extension of T.

The natural extension of the Farey map

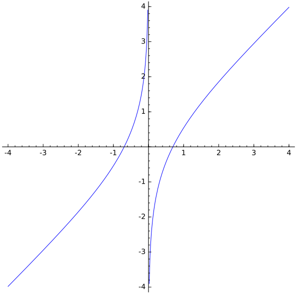
- The map $\tilde{T}_1$ is defined on $\{(x,y)|0<x<1, \frac{1}{x-1}<y<\frac{1}{x}$
- $(x,y)\mapsto \left(\frac{x}{1-x},(1-x)^2y+1-x\right)$ if $x<\frac{1}{2}$.
- $(x,y)\mapsto \left(2-\frac{1}{x},x^2y-x\right)$ if $x<\frac{1}{2}$.
- This map preserves Lebesgue measure
- Hence T_1 has invariant (infinite) density $\frac{1}{x(1-x)}$.

The natural extension of the Farey map



A Boole type map

- If we conjugate by a primitive of the invariant density,
- we obtain a Boole type map $T_3 : \mathbb{R} \to \mathbb{R}$
- $x \mapsto \log(e^x - 1)$ if $x > 0$
- $x \mapsto -\log(e^{-x} - 1)$ if $x < 0$
- This is an ergodic map on $\mathbb{R}$ which preserves Lebesgue measure.
- Can we classify all such maps ? The first was the original Boole map
- $x \mapsto x - \frac{1}{x}$.


4
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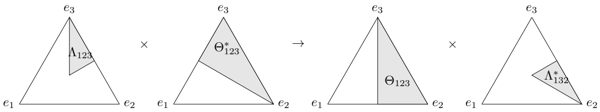
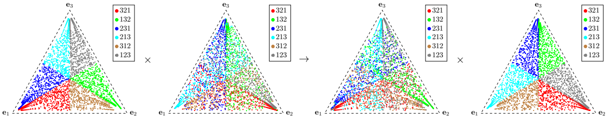
A simple example : Brun Continued Fractions

- Given 3 positive real numbers x,y,z
- Subtract the second biggest from the biggest, and iterate
- It is associated with the 6 elementary matrices on the 6 sorted subcones of the positive cone.
- It satisfies a Markov condition.

Natural extension of Brun Continued Fractions



Natural extension of Brun Continued Fractions



A 2-adic example : Saito's algorithm

- This GCD algorithm is defined on pairs of integers (p, q) with q odd.
- If p is even, we replace the pair by $(\frac{p}{2}, q)$
- If p is odd, we replace the pair by $(\frac{p+q}{2}, p)$
- This algorithm converges to (d, d), where d is the GCD of p and q

A 2-adic example : Saito's algorithm

- It is associated to a Gauss map on $\mathbb{Z}_2$ defined by
- $T(x) = x/2$ if x is not invertible
- $T(x) = \frac{\frac{1}{x}+1}{2}$ if x is invertible.
- T preserves the Haar measure on $\mathbb{Z}_2$
- This map should allow to perform a probabilistic analysis of this algorithm.

Suspensions

- It is natural to look for a suspension flow of the natural extension
- and there is an obvious candidate
- the flow $\phi_t(x, y) = (e^t x, e^{-t} y)$ commutes
- and preserves the symplectic form x.y
- It gives a suspension of the natural extension, by a suitable quotient space.

Suspensions

- The question is to give a meaning to the underlying space
- This is sometimes possible :
- Modular spaces

A dynamic model for continued fractions

- We want to find a family of systems R_α.
- where α is in the domain of the Gauss map T.
- and a family of subsets A_α.
- such that the induced map $R_{\alpha|A_\alpha}$ is conjugate to $R_{T\alpha}$.
- Example 1 : continued fraction, rotations and the geodesic flow on the modular surface.
- Example 2 : interval exchange maps, Rauzy induction and the Teichmüller flow.
- this has been hugely successful (but highly technical)

A dynamic model for continued fractions

- Extend this to a higher dimensional continued fraction map ?
- Problem : find the family of dynamical systems (translations on compact groups).
- Difficult in dimension > 1.
- the set A_α has special properties :
- It should be a bounded remainder set (not easy to find)
- It should give symbolic dynamics with linear complexity
- In the periodic case :
- It needs to have fractal boundaries (Markov partition).

Symbolic dynamics

- Geometry is difficult : we are going to use symbolic dynamics.
- The basic idea :
- With a given point, we associate a sequence M_n of matrices.
- With each matrix M_n, we associate a substitution σ_n having this matrix.
- This infinite sequence of substitution defines an infinite limit sequence of letters (S-adic sequence).

Symbolic dynamics

- The S-adic sequence defines a symbolic dynamical system which is renormalizable by construction.
- Think of a generalized Sturmian sequence, with its infinite sequence of renormalizations.
- This associates to the initial point α a symbolic system (Ω_α, S).
- We could take this as an answer, but we want a geometric model.
- We consider the stepped line obtained by abelianization of $u \in \Omega_\alpha$: we want it to be a model set.

Some properties of symbolic systems

- We want (Ω_α, S) to be minimal.
- We want (Ω_α, S) to be uniquely ergodic : asymptotic direction for the stepped line.
- We want this asymptotic direction to be totally irrational.
- We want the stepped line to be *balanced*, that is, remain within bounded distance of the asymptotic direction.

Some properties of symbolic systems

- If the stepped line is *balanced*, it is (almost) a model set.
- We can project it on the diagonal plane to obtain a compact set with nonempty interior which gives a locally finite covering by action of the diagonal subgroup of $\mathbb{Z}^3$.
- If this covering is a tiling, the stepped line is a true model set.
- All these properties can be ensured by a generalized Pisot property on the sequence of matrices, plus some combinatorics.
- For some classical algorithms, the Pisot property was proved by Avila-Delecroix.

A plan of attack

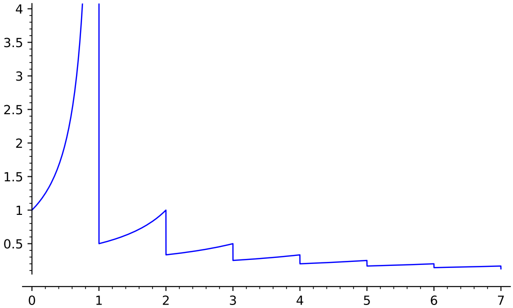
1. Study some properties of sequence of positive matrices.
2. Define the corresponding S-adic system.
3. Prove that the asymptotic properties of sequence of matrices imply properties of the S-adic system.
4. Define the dynamical geometric model for this sequence of matrices : Rauzy fractals and Rauzy boxes.
5. Prove that these properties are almost everywhere defined for some algorithms, using Oseledets and computations by Avila-Delecroix.
6. This plan has been realised in several cases (Brun's algorithm, for example)

The classical example

- The basic algorithm is the geodesic flow on the modular surface
- We will show a few movies (due to Edmund Harriss) showing this flow
- and various arithmetic properties
- A first movie : *the shape of lattiices* can be found at
- https ://www.youtube.com/watch ?v=vLrliPt4Uc0

A last curious example

- The positive rationals can be generated by the Calkin Wilf tree
- where $\frac{p}{q}$ has descendants $\frac{p}{p+q}$ and $\frac{p+q}{q}$.
- Moshe Newman (see A. Malter, D. Schleicher, and D. Zagier) proved that the map
- $x \mapsto \frac{1}{2\lfloor x \rfloor + 1 - x}$, starting from 0, gives a breadth-first enumeration of the tree.
- This is a simple and interesting dynamical system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7

A last curious example

- This map preserves the tail of the continued fraction expansion
- Here is the orbit of $e-1$
- given by the continued fraction

```
[[1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
 [0, 1, 3, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10],
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 [3, 3, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1],
 [0, 3, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
 [1, 2, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1, 1, 10],
 [0, 1, 1, 1, 1, 2, 1, 3, 1, 1, 6, 1, 1, 8, 1]]
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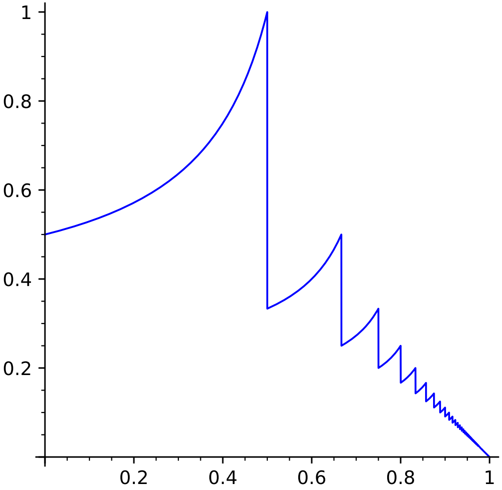
A last curious example

- The sum of the coefficients is preserved !
- This map exchanges $[0, 1]$ and $[1, +\infty]$
- Its square is a map from $[0, 1]$ to itself.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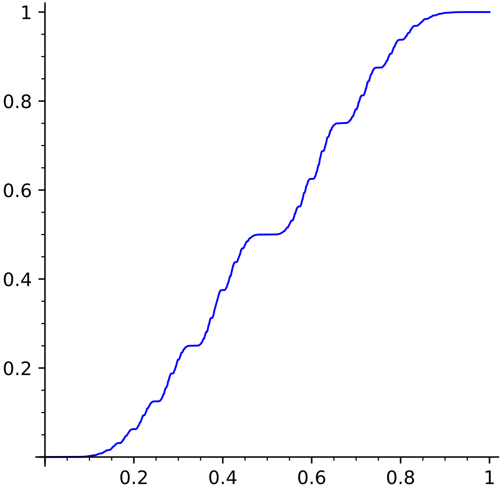
A last curious example

- If we conjugate by the Question Mark function of Minkowski,
- we obtain an interesting result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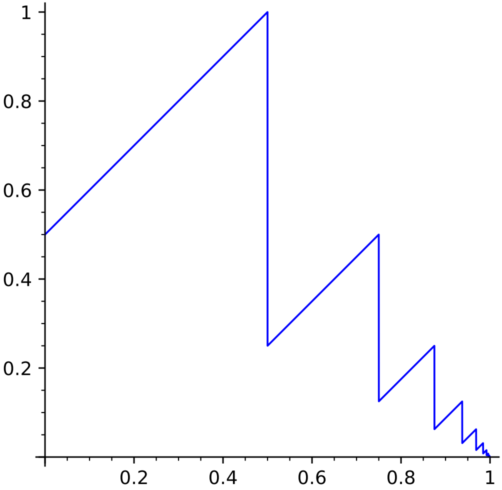
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A last curious example

- This map is conjugate to the 2-adic odometer
- Hence it is uniquely ergodic
- Its unique ergodic invariant measure
- is a singular measure
- Image of Lebesgue measure by the Minkowski function
- This map could be related to the horocycle flow.