A. - PARRISH - TSENG.

$$\boxed{\left|\alpha - p/q\right| < C/q^2}$$

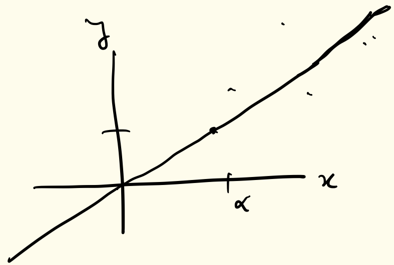
∞'ly many $p/q \in \mathbb{Q}$

$\gcd(p,q) = 1.$

$C = 1 \longrightarrow$ Dirichlet's theorem.

$C = 1/\sqrt{5} \longrightarrow$ Hurwitz. [sharp, $\alpha = \frac{1+\sqrt{5}}{2} = [1;1,1,1,1 \ldots]$]

$\begin{pmatrix} p \\ q \end{pmatrix}$ to have inverse slope $\sim \alpha$



$$|\alpha - p/q| < c/q^2$$

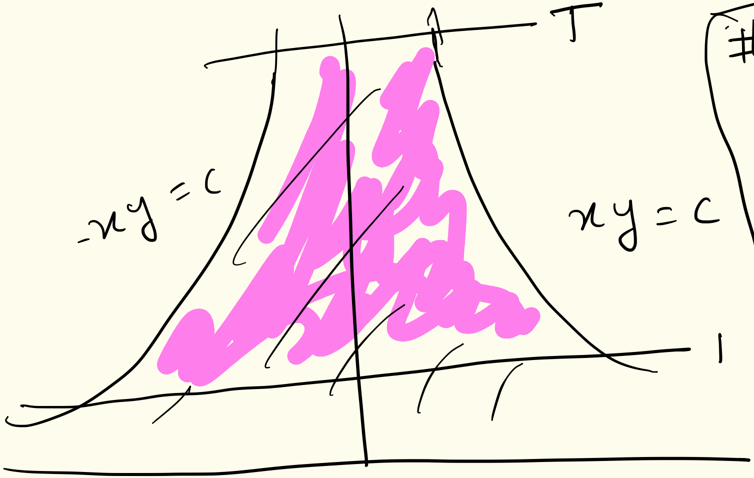
$$q^2 |\alpha - p/q| < C$$

$$q\,|q\alpha - p| < C. \qquad (*) \qquad q > 0$$

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p - q\alpha \\ q \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$|x|\,y < C$$

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 = h_{-\alpha}\mathbb{Z}^2 = \Lambda_\alpha \qquad h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$



$$\#\left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2_{\text{prim}} : \begin{matrix} |p - q\alpha|\, q < c \\ 0 < q \leq T \end{matrix} \right\}$$

$$= \#\Lambda_{\alpha,\text{prim}} \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{matrix} |x|\, y < c \\ 1 \leq y < T \end{matrix} \right\}.$$

$$\boxed{= N(\alpha, c, T)}$$

region has $\underline{\infty \text{ area}}$

What can we say about growth of $N(\alpha, c, T)$ w/ T?

$X_2 = SL_2\mathbb{R}/SL_2\mathbb{Z}$ = unimodular lattices in $\mathbb{R}^2$

$$g\, SL_2\mathbb{Z} \longleftrightarrow g\mathbb{Z}^2$$

$$\Lambda_\alpha = h_\alpha \mathbb{Z}^2 \longleftrightarrow \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} SL_2\mathbb{Z}.$$

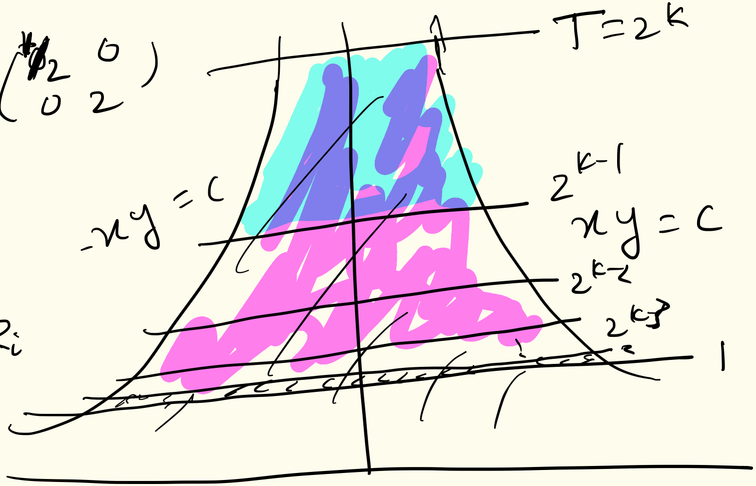
$T = 2^k$

$$g_{\log 2} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad g_{-\log 2} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{array}{l} |x|y < c \\ 1 < y \le 2^k \end{array} \right\}$$

$$= \bigsqcup_{i=0}^{k-1} \underbrace{\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{array}{l} |x|y < c \\ 2^i < y \le 2^{i+1} \end{array} \right\}}_{:= R_i}$$

$$R_i = g_{-\log 2}^{i} R_0$$



$$X_2 = SL_2\mathbb{R}/SL_2\mathbb{Z}$$

$$\bigsqcup_{i=0}^{k-1} R_i = \bigsqcup_{i=0}^{k-1} g_{-\log 2}^{i} R_0$$

$$\# \Lambda_{\alpha,\text{prim.}} \cap \bigsqcup_{i=0}^{k-1} R_i = \#\left(\Lambda_{\alpha,\text{prim.}} \cap \bigsqcup_{i=0}^{k-1} g_{-\log 2}^{i} R_0\right)$$

$$= \sum_{i=0}^{k-1} \# \Lambda_{\alpha,\text{prim}} \cap g_{-\log 2}^{i} R_0$$

$$= \sum_{i=0}^{k-1} \#\left(g_{\log 2}^{i} \Lambda_{\alpha,\text{prim.}} \cap R_0\right)$$

$$= \sum_{i=0}^{k-1} f_0\left(g_{\log 2}^{i} \Lambda_\alpha\right)$$

$$f_0(\Lambda) = \# \Lambda_{\text{prim.}} \cap R_0$$

$$\frac{N(\alpha, c, 2^k)}{k} = \frac{1}{k}\sum_{i=0}^{k-1} f_0(g_{1,0,2}^{\,i}\,\Lambda_\alpha)$$

Hopefully, by Birkhoff ergodic thm, RHS

$$\longrightarrow \int_{X_2} f_0 \, d\mu_2.$$

μ_2 Haar probability meas. on X_2.

a.e. α $\longrightarrow$ $g_{1,0,2} \curvearrowright X_2$ is ergodic

& $\{\Lambda_\alpha\}$ is a unstable manifold for this action.

[A. - Parrish - Tsay $\longrightarrow$ Nonlinearity]

$$\int_{X_2} f_0 \, d\mu_2 = \frac{6}{\pi^2} |R_0|$$

$$= \frac{6}{\pi^2} 2c \log 2.$$

$$= \frac{12}{\pi^2} c \log 2.$$

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h in $C_c(\mathbb{R}^2)$, $\hat{h} \in L^1(X_2)$

$$\hat{h}(\Lambda) = \sum_{v \in \Lambda_{prim}} h(v)$$

Siegel: $\int_{X_2} \hat{h} \, d\mu_2 = \frac{6}{\pi^2} \int_{\mathbb{R}^2} h \, d\mathrm{leb}.$

$f_0 = \hat{\chi}_{R_0}$

$h \longmapsto \int_{X_2} \hat{h} \, d\mu_2$ $SL_2\mathbb{R}$-inv. linear functional on $C_c(\mathbb{R}^2)$