Ergodic theorems and particular trajectories

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Plan of the talk

(I) Main features of the Ergodic Theorem

(II) Another point of view
with focus in the case of a dynamical system of the interval

(III) Probabilistic studies of the truncated generic trajectories
Using the weighted density transformer

(IV) Particular trajectories for the Euclidean system:
- Finite trajectories (rational inputs)
- Periodic trajectories (irrational quadratic inputs)
- Statement of the main result

(V) Study of particular trajectories with three tools
- Dirichlet generating functions
- Singularity analysis
- the weighted transfer operator.

(VI) Return to the notion of size for a periodic trajectory

(I) the Ergodic Theorem and its main features

General ergodic framework.

Definitions. A probability space $(X, \mathcal{X}, \mu)$, and a measurable mapping $T : X \to X$.

– The subset $E \in \mathcal{X}$ is T-invariant iff $T^{-1}E = E$.

– The measure μ is T-invariant [or the map T preserves the measure μ]
iff for every subset $E \in \mathcal{X}$, one has $\mu(T^{-1}E) = \mu(E)$.

– Let $T : X \to X$ be a measure-preserving transformation on $(X, \mathcal{X}, \mu)$.
The map T is ergodic iff
for every T–invariant subset $E \in \mathcal{X}$, one has either $\mu(E) = 0$ or $\mu(E) = 1$.

Particular case of a dynamical system.

– A dynamical system is related to the particular case when:
X (compact) topological space, its Borel set $\mathcal{X}$, $T : X \to X$ continuous.

– A dynamical system (X, T) is said to be uniquely ergodic if
there exists a unique T-invariant Borel probability measure on X.
Such a measure is necessary ergodic for T.

Ergodic Theorem

Definition. Let $T : X \to X$ be a measure-preserving transformation on $(X, \mathcal{X}, \mu)$.
Consider a μ-integrable function c [i.e., $c \in \mathcal{L}^1(\mu)$] (a "cost" or a "weight")
Then, there are the following averages:

– space–average of c

$$\mu[c] = \int_X c(x) d\mu(x) .$$

– time–average of c along a n-th truncated trajectory $(x, Tx, \dots, T^{n-1}x)$

$$\frac{1}{n} C_n(x), \qquad C_n(x) := \sum_{\ell=0}^{n-1} c(T^\ell x) .$$

Ergodic Theorem. If T is μ-ergodic, then one has

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{\ell=0}^{n-1} c(T^\ell x)\right] = \mu[c] \qquad \text{for almost } \mu\text{-every } x \in X$$

In the case of the unique ergodicity, when c is moreover continuous,
the previous holds for every $x \in X$.

Main features of the Ergodic Theorem

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{\ell=0}^{n-1} c(T^\ell x)\right] = \mu[c] \qquad \text{for almost } \mu\text{-every } x \in X$$

– Almost everywhere?

No information about the exceptional subset $\mathcal{E}$ (of measure 0)
where the Ergodic Theorem "fails"

– Speed of convergence? No information

$\Longrightarrow$ A possible complementary point of view,
notably for dynamical systems of the interval

– Replace almost everywhere by on average

– First on the total set X

– Why not on smaller sets $Y \subset X$?

– Why not on discrete subsets Y .... (of zero measure !)

– Obtain information about the speed of convergence

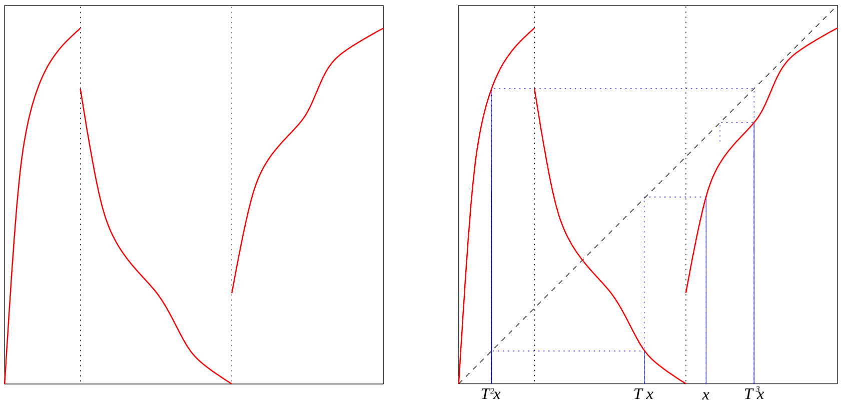
(II) Dynamical system of the interval : generalities

A dynamical system $(\mathcal{I}, T)$ of the unit interval $\mathcal{I}$ is defined by

- a finite or infinite denumerable alphabet Σ,
- a topological partition of $\mathcal{I} :=]0, 1[$ with open intervals $\mathcal{I}_{m, m\in\Sigma}$,
- a shift mapping T
  s.t. $T|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ from $\mathcal{I}_m$ to $\mathcal{J}_m := T(\mathcal{I}_m)$.

Given an input x of $\mathcal{I}$, this gives rise to the trajectory

$$\mathcal{T}(x) := (x, Tx, T^2x, \dots)$$



A dynamical system, with $\Sigma = \{a, b, c\}$ and a word $M(x) = (c, b, a, c \ldots)$.

Correlations between symbols due to

– the geometry of the branches
– the shape of the branches

The geometry of the branches [position of $T(\mathcal{I}_m)$ wrt $\mathcal{I}_\ell$] ;
it describes the set $s(m)$ of possible successors of the symbol m.

Particular cases:

- Complete systems $T(\mathcal{I}_m) = \mathcal{I}$
- Markovian systems $T(\mathcal{I}_m) =$ union of some $\mathcal{I}_\ell$

give rise to a finite characterization of $s(m)$.

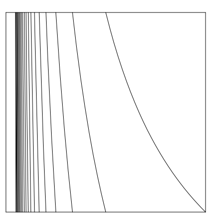
The shape of the branches [derivatives of the branches] also explains how the distribution evolves.

General case of interest.

A complete – or a Markovian – system
– with a possible infinite denumerable alphabet
– topologically mixing – and expansive.

Main instances: – (in this talk) the system defined with the Gauss map

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad T(0) = 0$$



– (in Frédéric's talk), the system defined with the β-shift (for $\beta > 1$) with
$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor$$

(III) A probabilistic point of view on truncated trajectories.

Focus on dynamical systems (of the interval)

A probabilistic point of view on truncated trajectories.

– Describe the behaviour of the n-th weighted truncated trajectories
– and their limit for $n \to \infty$

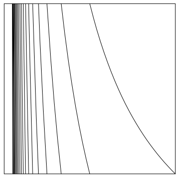
Interest in the random variables $C_n : X \to \mathbb{R}$ associated with a cost c

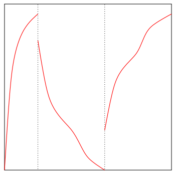
$$x \mapsto C_n(x), \quad \text{with} \quad C_n(x) := \sum_{\ell=0}^{n-1} c(T^\ell x)$$

– What about the sequence of expectations $\mathbb{E}[C_n]$?

– Are there limit distributional results for the random variables C_n?
for instance a limit gaussian law?

The weighted density transformer associated with a complete system.





$\mathcal{H} :=$ {inverse branches}

Density Transformer: for an initial density f on $[0,1]$,
$\mathbf{H}[f]$ is the density on $[0,1]$ after one iteration of the shift

$$\mathbf{H}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|\, f \circ h(x)$$

Weighted density transformer : extension of $\mathbf{H}$ into $\mathbf{H}_{1,w}$.

$$\mathbf{H}_{1,w}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|\, e^{wc(h(x))} f \circ h(x), \qquad \mathbf{H}_{1,0} = \mathbf{H}$$

The n-th iterate generates the weighted n-th truncated trajectories

$$\mathbf{H}^n_{1,w}[f](x) = \sum_{h \in \mathcal{H}^n} |h'(x)|\, e^{wC_n(h(x))}\, f \circ h(x)$$

$$\frac{d}{dw}\Big|_{w=0} \left[w \mapsto \mathbf{H}^n_{1,w}[f](x)\right] = \sum_{h \in \mathcal{H}^n} |h'(x)|\, C_n(h(x))\, f \circ h(x)$$

Remark: If the system is not complete, we must add indicator functions.
For a Markovian sytem, we can replace them by an operator matrix.

Weighted density transformer and probabilistic study of generic trajectories

$$\mathbf{H}^n_{1,w}[f](x) = \sum_{h \in \mathcal{H}^n} |h'(x)|\, e^{wC_n(h(x))}\, f \circ h(x)$$

$$\frac{d}{dw}\Big|_{w=0} \left[w \mapsto \mathbf{H}^n_{1,w}[f](x)\right] = \sum_{h \in \mathcal{H}^n} |h'(x)|\, C_n(h(x))\, f \circ h(x)$$

Take the integral over the unit interval + perform a change of variables

$$\mathbb{E}_f[e^{wC_n}] = \int_{\mathcal{I}} \mathbf{H}^n_{1,w}[f](t)dt, \qquad \mathbb{E}_f[C_n] = \frac{d}{dw}\Big|_{w=0} \left[w \mapsto \int_{\mathcal{I}} \mathbf{H}^n_{1,w}[f](t)dt\right]$$

Good dynamical system and cost of moderate growth
$\Longrightarrow$ $\mathbf{H}^n_{1,w}$ behaves as the n-th power of its dominant eigenvalue $\lambda(1,w)$,
uniformy when w is near 0.

$$\mathbb{E}_f[C_n] = n \cdot \mu[c]\,(1 + O(\rho^n)), \qquad \mu[c] = \lambda'_w(1,0)$$

$\mathbb{E}_f[e^{wC_n}]$ behaves as a (uniform) n-th power + Quasi-Powers Theorem
$\Longrightarrow$ Asymptotic Gaussian law for C_n

(IV) Study of particular trajectories of the Euclidean system

Discrete sets $\mathcal{X} \subset \mathcal{I}$ and trajectories $\mathcal{T}(x) := (x, Tx, T^2x, \ldots, T^\ell x, \ldots)$ for $x \in \mathcal{X}$.

Natural discrete sets $\mathcal{X}$:

– The set $\mathcal{Q}$ of x for which the trajectory $\mathcal{T}(x)$ is finite

– The set $\mathcal{P}$ of x for which the trajectory $\mathcal{T}(x)$ is (purely) periodic.

In the particular case of the Euclidean system (Gauss map):

$\mathcal{T}(x)$ finite $\iff x$ rational,

$\mathcal{T}(x)$ (purely) periodic $\iff x$ (reduced) quadratic irrational

In each case, a bijection between $\mathcal{X}$ and a (subset) of $\mathcal{H}^\star = \cup_{p\geq 0}\mathcal{H}^p$

$\mathcal{Q} := \mathbb{Q} \cap [0,1] = \{h(0) \mid h \in \mathcal{H}^\star \times \mathcal{F}\}$

$\mathcal{P} := \mathbb{I} \cap [0,1] = \{h(x_h) \mid h \in \mathcal{H}^\star,\ h \text{ primitive}, x_h \text{ fixed point of } h\}$

There is also a natural notion of size ϵ and a natural definition of cost C

$(\mathcal{Q})$: $\epsilon(x) :=$ denominator of $x = |h'(0)|^{-1/2}$

$(\mathcal{P})$: $\epsilon(x) := |\widehat{h}'(x_{\widehat{h}})|^{-1/2}$, $\widehat{h}$: = the primitive h of even period

$$C(x) := \sum_{\ell=0}^{P(x)-1} c(T^\ell(x)),\quad P(x) := \text{length or period of the trajectory } \mathcal{T}(x)$$

(V) Proof of the result with three tools

– Dirichlet generating functions

– Singularity analysis

– the weighted transfer operator.

Study of the mean cost along particular trajectories: Main result

For $\mathcal{X} \in \{\mathcal{Q}, \mathcal{P}\}$, the set of interest is $\mathcal{X}_N := \{x \in \mathcal{X} \mid \epsilon(x) \leq N\}$

There are two important costs on $\mathcal{X}_N$, the length P, and the cost C.

The mean values of the cost C or the cost P on $\mathcal{X}_N$ are the ratios

$$\mathbb{E}_N[P] := \frac{1}{|\mathcal{X}_N|}\sum_{x\in\mathcal{X}_N} P(x), \qquad \mathbb{E}_N[C] := \frac{1}{|\mathcal{X}_N|}\sum_{x\in\mathcal{X}_N} C(x)$$

Main Theorem. The asymptotic estimates hold on $\mathcal{Q}_N$ or $\mathcal{P}_N$

$$\mathbb{E}_N[P] \sim \frac{2}{h(T)}\log N, \quad \mathbb{E}_N[C] \sim \frac{2}{h(T)}\mu[c]\log N, \qquad (N \to \infty)$$

and involve the entropy $h(T)$ of the Euclidean system

Corollary. When $N \to \infty$, the asymptotic estimate holds: $\dfrac{\mathbb{E}_N[C]}{\mathbb{E}_N[P]} \to \mu[c]$

The particular trajectories (finite or periodic) behave on average in the same way as the generic trajectories behave almost everywhere.

Generating functions and analytic combinatorics

A general tool: A (bivariate) generating function, of Dirichlet type

$$F(s,w) := \sum_{x\in\mathcal{X}} \epsilon(x)^{-s} \cdot e^{wC(x)}, \qquad G(s) = \sum_{x\in\mathcal{X}} C(x)\cdot \epsilon(x)^{-s}$$

Main principles of analytic combinatorics: A good knowledge of

– the dominant singularity of G,

– the behaviour of G at this singularity

leads to the asymptotics of its coefficients, namely the behaviour of

$$\sum_{\substack{x\in\mathcal{X}\\ \epsilon(x)\leq N}} C(x), \quad |\mathcal{X}_N| := \sum_{\substack{x\in\mathcal{X}\\ \epsilon(x)\leq N}} 1, \quad (\text{for } N \to \infty)$$

This solves our question.

But, how to study the function $s \mapsto G(s)$ and "discover" its dominant singularity?

We express $F(s,w)$ – and thus $G(s)$ – in terms of the transfer operator.

Generating functions and transfer operators (I)

The bivariate generating function $F(s, w) := \sum_{x \in \mathcal{X}} \epsilon(x)^{-s} \, e^{wC(x)}$

is first expressed as a sum over the (convenient subset) of $\mathcal{H}^\star$

$$(\mathcal{Q}) \quad F(2s, w) \approx \sum_{h \in \mathcal{H}^\star} |h'(0)|^s \, e^{wC(h(0))} = \sum_{n \geq 0} \sum_{h \in \mathcal{H}^n} |h'(0)|^s \, e^{wC(h(0))}$$

$$(\mathcal{P}) \quad F(2s, w) \approx \sum_{h \in \mathcal{H}^\star} |h'(x_h)|^s \, e^{wC(x_h)} = \sum_{n \geq 0} \sum_{h \in \mathcal{H}^n} |h'(x_h)|^s \, e^{wC(x_h)}$$

Three differences with the previous study (truncated n-th trajectories)

- A sum over all possible "depths"
- The parameter s leads to the weighted transfer operator

$$\mathbf{H}_{s,w}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s \; e^{wc(h(x))} f \circ h(x)$$

- For $(\mathcal{P})$, a variable point x_h for each h and the replacement of $h \mapsto \widehat{h}$.

Some principles for Singularity Analysis

We are interested by the singularities of the operator (as a function of s)

$$\frac{d}{dw}\Big|_{w=0} \left[w \mapsto (I - \mathbf{H}_{s,w})^{-1} \right] = (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}$$

Two main operators:

- The quasi inverse $(I - \mathbf{H}_s)^{-1}$ of the (unweighted) transfer operator

$$\mathbf{H}_s[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s f \circ h(x)$$

It has a dominant singularity at $s = 1$ (in fact a simple pole)

- In the Euclidean case, no other singularities on the line $\Re s = 1$.
  Good news for applying the Tauberian Theorem !!
- Not the case for the β-shift ....! There are an infinite set of poles
  on $\Re s = 1$ located at s for which $1 - \beta^{1-s} = 0$

- The weighted operator $\mathbf{H}_s^{[c]}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s \, c(h(x)) \, f \circ h(x)$

For a cost of moderate growth, it is regular at $s = 1$. It "brings" the factor $\mu[c]$.

Generating functions and transfer operators (II)

In case $(\mathcal{Q})$ (finite trajectories)

$$F(2s, w) \approx \sum_{h \in \mathcal{H}^\star} |h'(0)|^s e^{wC(h(0))} = \sum_{n \geq 0} \sum_{h \in \mathcal{H}^n} |h'(0)|^s e^{wC(h(0))}$$

$$F(2s, w) \approx (I - \mathbf{H}_{s,w})^{-1}[1](0)$$

In case $(\mathcal{P})$ (periodic trajectories)

$$F(2s, w) \approx \sum_{n \geq 0} \sum_{h \in \mathcal{H}^n} |h'(x_h)|^s e^{wC(x_h)}$$

the presence of the fixed point x_h of h leads to the trace of the operator.
Each component of the transfer operator is a composition operator,

$$f \mapsto |h'|^s e^{wC \circ h} f \circ h$$

On a good functional space, the set of its eigenvalues is

$$\{ |h'(x_h)|^s e^{wC(x_h)} \cdot (-1)^n h'(x_h)^n \mid n \in \mathbb{N} \}$$

Finally:

$$F(2s, w) \approx \mathrm{Tr} \left[\mathbf{H}_{s,w}^2 (I - \mathbf{H}_{s,w}^2)^{-1} \right]$$

(VI) Return to the notion of size for a periodic trajectory
Case of the Euclidean system

A natural notion of the size of a (reduced) quadratic irrational (rqi) ?
The arithmetical point of view.

A real x for which the trajectory $\mathcal{T}(x)$ is purely periodic of period $P(x)$.
Then x is defined by the relation $h(x) = x$,
with a (primitive) inverse branch $h \in \mathcal{H}^{P(x)}$

$$h(x) = x \Longrightarrow ax^2 + bx + c = 0, \qquad \text{with } \gcd(a, b, c) = 1$$

$$\Longrightarrow x \in \mathbb{Q}(\sqrt{\Delta}), \qquad \text{with } \Delta = b^2 - 4ac > 0$$

There is a fundamental unit in the quadratic (real) number field $\mathbb{Q}(\sqrt{\Delta})$
When chosen > 1 it is denoted as $\epsilon(x)$ and is chosen as the size of x.

Its computation involves the smallest inverse branch $\widehat{h}$ of even length.

$$\epsilon(x) = |\widehat{h}'(x_{\widehat{h}})|^{-1/2}$$

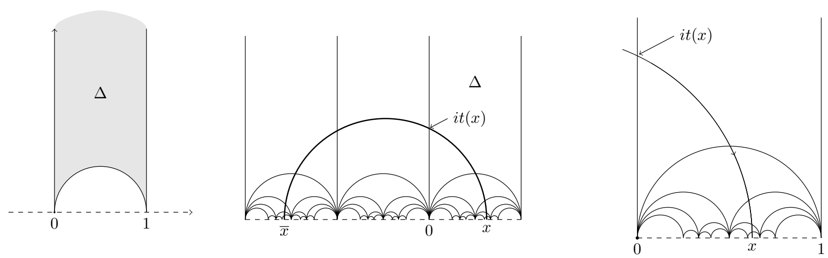
A natural notion of the size of a (reduced) quadratic irrational (rqi) ?
The dynamical point of view.

The hyperbolic plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$.
When endowed with the metric $ds = |dz|/y$, its geodesics are
the semi-circles centered on the real axis and the vertical lines.

The set $\{h(\Delta) \mid h \in SL_2(\mathbb{Z})\}$ where Δ is the triangle with cusps $i\infty$, 0 and 1
defines the Farey tessellation of $\mathbb{H}$.

With a rqi number, $x \in [0, 1]$ one associates
- its conjugate $\bar{x}$ that satisfies $\bar{x} < -1$.
- its minimal even period q
- the sum of the digits along the even period $M := \sum_{i \le q} m_i$.

A natural notion of the size of a (reduced) quadratic irrational (rqi) ?
The dynamical point of view : Some geodesic pictures



The oriented geodesic $\gamma(x)$ that links $\bar{x}$ to x
- intersects the imaginary axis at $it(x)$.
- defines the oriented curve $\gamma_+(x)$ that links $it(x)$ to x
- this curve "crosses" domains $h(\Delta)$,

The primitive part of the geodesic is defined by the portion of $\gamma_+(x)$
that corresponds to the first M domains it crosses.
Its length $\rho(x)$ is related to the size $\epsilon(x)$ via $\rho(x) = \log \epsilon(x)$