The domino problem for word-hyperbolic groups

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The domino problem for surface groups

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Consider a finite set $\tau$ of Wang tiles

![Wang tiles](image.png)
Consider a finite set $\tau$ of Wang tiles

Question:
is there a function $x : \mathbb{Z}^2 \rightarrow \tau$ such that adjacent tiles share the same color?
Classical domino problem

Question:
is there an algorithm which given a finite set of Wang tiles decides whether they tile the plane or not?

Theorem (Berger 66’)
No.
The domino problem is Undecidable.
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Domino problem

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Theorem (Kari 08')
The domino problem is undecidable in the binary hyperbolic tiling.
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Theorem (Kari 08’)

*The domino problem is undecidable in the binary hyperbolic tiling.*
Let us consider the following ingredients:

- A directed, labeled (infinite) graph $\Gamma = (V, E, L)$.
- A finite set of colors $\mathcal{A}$.
- A finite list of forbidden colored labeled connected finite graphs $\mathcal{F}$. 
General setting

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- A directed, labeled (infinite) graph $\Gamma = (V, E, L)$.
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**Domino problem for $\Gamma$:**

Is there an algorithm which decides, given $(\mathcal{A}, \mathcal{F})$, whether there exists a coloring $x : V \rightarrow \mathcal{A}$ such that no graph from $\mathcal{F}$ embeds?
The original domino problem:
Binary hyperbolic tiling:
Binary hyperbolic tiling:
A particularly interesting case is when $\Gamma = (V, E, L)$ is the Cayley graph of a finitely generated group $G$ given by the set of generators $S$.

- $V = G$.
- $E = \{(g, gs) \mid g \in G, s \in S\}$.
- $L(g, gs) = s$. 

Remark: the domino problem does not depend upon the set of generators $S$. These problems are all computationally (many-one) equivalent. $DP(G)$ is the domino problem of the group $G$. 

General setting: Cayley graphs
A particularly interesting case is when $\Gamma = (V, E, L)$ is the **Cayley graph** of a finitely generated group $G$ given by the set of generators $S$.

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$\text{DP}(G)$ is the domino problem of the group $G$.  

Cayley graph of free group.
Domino problem on groups.

List of facts:

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Domino problem on groups.

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- $\text{DP}(\mathbb{Z}^2)$ is undecidable.
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- $\text{DP}(G)$ is decidable whenever $G$ is virtually free (Cayley graph looks like the previous tree [finite tree-width]).

Domino conjecture
A finitely generated group has decidable domino problem if and only if it is virtually free.

Verified for polycyclic groups, Baumslag-Solitar groups, Branch groups.
Domino problem on groups.

**Domino conjecture**

A finitely generated group has decidable domino problem if and only if it is virtually free.

Why should one care about this?

**Theorem (Muller & Schupp '85)**

A graph has decidable monadic second order logic (MSO) if and only if it has finite tree-width.

**Fact 1**

A group is virtually free if and only if its Cayley graphs have finite tree-width.

**Fact 2**

The domino problem can be expressed in MSO. If $DC$ holds, then the domino problem contains all the complexity of MSO for finitely generated groups.
Domino problem on groups.

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If DC holds, then the domino problem contains all the complexity of MSO for finitely generated groups.
Consider the fundamental group of a closed orientable surface.
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Surface groups

Consider the fundamental group of a closed orientable surface.

\[ \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle \]
Surface groups

\[1\text{https://math.stackexchange.com/questions/1834108/cayley-graph-of-the-fundamental-group-of-the-2-torus}\]
Theorem (Aubrun, B. Moutot)

The domino problem of the fundamental group of any closed orientable surface of positive genus is undecidable.
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The domino problem of the fundamental group of any closed orientable surface of positive genus is undecidable.

Remark: we just need to show that the domino problem of

\[ \pi_1 \left( \begin{array} \end{array} \right) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle \]

is undecidable.
Proof idea: use hyperbolicity.

- Step 1: show undecidability of DP for a class of graphs which embed nicely in the hyperbolic plane.
How to prove it

Proof idea: use hyperbolicity.

- Step 1: show undecidability of DP for a class of graphs which embed nicely in the hyperbolic plane.
- Step 2: show that one of these graphs $\Gamma$ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of $\pi_1(\mathbb{H}^2)$.
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- Step 1: show undecidability of DP for a class of graphs which embed nicely in the hyperbolic plane.
- Step 2: show that one of these graphs $\Gamma$ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of $\pi_1\begin{picture}(200,0)
  \drawline(100,0)(100,20)
  \drawline(90,20)(90,0)
  \drawline(110,20)(110,0)
\end{picture}$.  
- Step 3: reduce $\text{DP} \left( \pi_1\begin{picture}(200,0)
  \drawline(100,0)(100,20)
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  \drawline(110,20)(110,0)
\end{picture} \right)$ to $\text{DP}(\Gamma)$.
Proof idea: use hyperbolicity.

- Step 1: show undecidability of $\text{DP}$ for a class of graphs which embed nicely in the hyperbolic plane.
- Step 2: show that one of these graphs $\Gamma$ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of $\pi_1(\bigcirc)$.\[\begin{array}{c}
\end{array}\]
- Step 3: reduce $\text{DP} \left( \pi_1(\bigcirc) \right)$ to $\text{DP}(\Gamma)$.
- Step 4: profit.
A (non-deterministic) substitution is a pair \((\mathcal{A}, R)\) where \(\mathcal{A}\) is a finite alphabet and \(R\) is a set of pairs \((a \mapsto w) \in \mathcal{A} \times \mathcal{A}^*\).
A (non-deterministic) **substitution** is a pair \((\mathcal{A}, R)\) where \(\mathcal{A}\) is a finite alphabet and \(R\) is a set of pairs \((a \mapsto w) \in \mathcal{A} \times \mathcal{A}^*\).

**Example**

- \(\mathcal{A} = \{0\}, R = \{(0 \mapsto 00)\}\).
- \(\mathcal{A} = \{0, 1\}, R = \{(1 \mapsto 0), (0 \mapsto 01)\}\).
A (non-deterministic) **substitution** is a pair \((A, R)\) where \(A\) is a finite alphabet and \(R\) is a set of pairs \((a \mapsto w) \in A \times A^*\).

**Example**

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An infinite word \(u = \ldots u_{-1} u_0 u_1 u_2 \ldots \in A^\mathbb{Z}\) **produces** a word \(v = \ldots v_{-1} v_0 v_1 v_2 \ldots \in A^\mathbb{Z}\) if \(v\) can be obtained from \(u\) by applying a rule of \(R\) on each symbol.

That is, there exists a function \(\Delta : \mathbb{Z} \rightarrow \mathbb{Z}\) such that :

\[
(u_i \mapsto v_{\Delta(i)} \cdots v_{\Delta(i+1)-1}) \in R \text{ for every } i \in \mathbb{Z}
\]
Let $\{u_i\}_{i \in \mathbb{Z}}$ be a sequence of bi-infinite words such that $u_i$ produces $u_{i+1}$ (with $\Delta_i$). We can associate an orbit graph.
Let \( \{u_i\}_{i \in \mathbb{Z}} \) be a sequence of bi-infinite words such that \( u_i \) produces \( u_{i+1} \) (with \( \Delta_i \)). We can associate an orbit graph.

- Join all consecutive symbols of \( u_i \) by edges from left to right.
- Join each symbol of \( u_i \) with the corresponding sequence of symbols it produces in \( u_{i+1} \) assigning labels from left to right.
Example 1: trivial substitution gives $\mathbb{Z}^2$.

$A = \{0\} \quad R = \{(0 \mapsto 0)\}$. 
Example 2: Doubling substitution gives bin hyp tiling.

\[ A = \{0\} \quad R = \{(0 \mapsto 00)\} \]
Undecidability: reduce to example 2.

Idea: take an orbit graph $\Gamma$. 

![Graph Diagram]
Undecidability: reduce to example 2.

In each vertex code a finite subgraph of the binary orbit graph + information on how to locally paste them together.
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In each vertex code a finite subgraph of the binary orbit graph + information on how to locally paste them together.

Impose local consistency rules.
Suppose $\mathsf{DP}(\Gamma)$ is decidable.

Use the previous tiling to encode the binary orbit graph.

Let $(\mathcal{A}, \mathcal{F})$ be an alphabet and a set of forbidden patterns for the binary orbit graph. Use the encoding to simulate tilings in $\Gamma$.

As $\mathsf{DP}(\Gamma)$ is decidable, we may use the associated algorithm to decide whether $(\mathcal{A}, \mathcal{F})$ admits a tiling of the binary orbit graph.

contradiction $\checkmark$. 

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Warning

- We must check that the language of coded subgraphs is finite.
Undecidability: reduce to example 2.

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- As $\text{DP}(\Gamma)$ is decidable, we may use the associated algorithm to decide whether $(\mathcal{A}, \mathcal{F})$ admits a tiling of the binary orbit graph.
- contradiction $\checkmark$.

**Warning**

- We must check that the language of coded subgraphs is **finite**.
- We must check that the set of encodings is **non-empty**.
A substitution \((A, R)\) has an **expanding eigenvalue** if there exists \(\lambda > 1\) and \(\nu: A \rightarrow \mathbb{R}^+\) such that for every \((a \mapsto u_1 \ldots u_k) \in R:\)

\[
\lambda \nu(a) = (\nu(u_1) + \nu(u_2) + \cdots + \nu(u_k))
\]

**Example**

\[A = \{0\}, \quad R = \{(0 \mapsto 00)\}\] admits the expanding eigenvalue \(\lambda = 2\).

\[
2\lambda \nu(0) = (\nu(0) + \nu(0))
\]
To every orbit of a substitution with an expanding eigenvalue we can associate canonically a tiling of $\mathbb{H}^2$. 

\[ v(a) \cdot e^y \]

\[ (a, w_1 \ldots w_k) \text{-tile} \]
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Hyperbolic geometry to the rescue!

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We superpose a tiling of \((A, R)\) and a binary tiling.

\((a, w_1 w_2 w_3)\)

Remark: Tiling superpositions were introduced by D.B. Cohen and C. Goodman-Strauss to produce aperiodic tilings of surface groups.
Hyperbolic geometry to the rescue!

We superpose a tiling of \((\mathcal{A}, R)\) and a binary tiling.

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Finitely many (coded) ways to intersect \(\iff\) finite alphabet
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Finitely many (coded) ways to intersect \(\Rightarrow\) finite alphabet
There is an encoding \(\Rightarrow\) non-emptiness
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Theorem (Aubrun, B., Moutot)

For every orbit graph $\Gamma$ of a substitution with an expanding eigenvalue $DP(\Gamma)$ is undecidable.
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Question

How does this relate to the fundamental group of $\infty$?
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Question

How does this relate to the fundamental group of $\infty$?

There is a "hidden" substitution in that group, namely $A = \{a, b\}$ and

$$\{\left( a \mapsto ab^5ab^5ab^5ab^4 \right), \left( b \mapsto ab^5ab^5ab^5ab^5ab^5ab^4 \right) \}$$

with $\lambda = 17 + 12\sqrt{2}$ and $v(b)/v(a) = \frac{1+\sqrt{2}}{2}$.
A way to look at this Cayley graph is as a translation surface obtained by pasting together octagons.
If a vertex is **connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:
If a vertex is **not connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:
Encode substitution structure using a finite alphabet and local rules. ✓
Surface group: proof of undecidability

- Encode substitution structure using a finite alphabet and local rules. ✓
- Assume the domino problem of the surface group is decidable.
Encode substitution structure using a finite alphabet and local rules. ✓

Assume the domino problem of the surface group is decidable.

Use the previous construction to reduce the domino problem in the orbit graph of the substitution to the one of the surface group.
Encode substitution structure using a finite alphabet and local rules. ✓

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Contradiction.
Encode substitution structure using a finite alphabet and local rules. ✓
Assume the domino problem of the surface group is decidable.
Use the previous construction to reduce the domino problem in the orbit graph of the substitution to the one of the surface group.
Contradiction.

Theorem (Aubrun, B., Moutot)

The domino problem is undecidable on the fundamental group of the closed orientable surface of genus 2.
Word-hyperbolic group

A finitely generated group is **word-hyperbolic** if the geodesic triangles of one of its Cayley graphs are $\delta$-slim for some $\delta > 0$.

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2

Facts about word-hyperbolic groups:

- Virtually free groups ✓.
- Surface groups (genus $g \geq 2$) ✓.
- Nice computability properties: Finitely presented, decidable word problem, Dehn’s algorithm works, language of shortlex geodesics is regular, etc.
- A random group is almost surely word-hyperbolic.
Word-hyperbolic groups

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- Virtually free groups ✓.
- Surface groups (genus $g \geq 2$) ✓.
- Nice computability properties: Finitely presented, decidable word problem, Dehn’s algorithm works, language of shortlex geodesics is regular, etc.
- A random group is almost surely word-hyperbolic.

Bottom line: testing ground for

**Domino conjecture**
A finitely generated group has decidable domino problem if and only if it is virtually free.
Gromov’s conjecture

The fundamental group of \( \infty \) embeds into any one-ended word-hyperbolic group.
Gromov’s conjecture

The fundamental group of \(\infty\) embeds into any one-ended word-hyperbolic group.

Facts:

- If a group \(H\) embeds into a group \(G\), then the domino problem of \(G\) is computationally harder than the domino problem of \(H\).
- If a word-hyperbolic group is not virtually free, it contains an embedded one-ended word-hyperbolic group.
- If \(GC\) holds, then every word-hyperbolic group which is not virtually free contains an embedded copy of the fundamental group of \(\infty\).
Gromov’s conjecture

The fundamental group of \( \infty \) embeds into any one-ended word-hyperbolic group.

Theorem

If \( GC \) holds, then the domino problem conjecture holds for every word-hyperbolic group.
The fundamental group of $\infty$ embeds into any one-ended word-hyperbolic group.

**Theorem**

If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

- **Fun fact**: find a (non virt free) word-hyperbolic group with decidable domino problem and you shall attain fame and glory disprove Gromov’s conjecture!
Gromov’s conjecture

The fundamental group of $\infty$ embeds into any one-ended word-hyperbolic group.

**Theorem**

*If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.*

- **Fun fact**: find a (non virt free) word-hyperbolic group with decidable domino problem and you shall attain fame and glory disprove Gromov’s conjecture!

- **Fun fact**: Same can be shown with weaker versions of GC.
The domino problem is undecidable on surface groups.
https://arxiv.org/abs/1811.08420