Computations for the spectrum of the Euclid transfer operator

Beginning around 1994,

Underlying a series of papers

Stating a set of conjectures

which seems to be recently proven in 2013 by Alkauskas

The transfer operator which underlies the Euclid algorithm is

$$\mathbf{G}_s[f](x) := \sum_{m=1}^{\infty} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right)$$

Philippe was interested to compute the spectrum

for $s = 1$ [Euclid algorithm] or $s = 2$ [Gauss reduction algorithm]

Truncation matrices (I)

On the polynomial $(x-a)^j$

$$h_j(x) := \mathbf{G}_s[(x-a)^j](x) = \sum_{m=1}^{\infty} \frac{1}{(m+x)^{2s}} \left(\frac{1}{m+x} - a\right)^j .$$

With the binomial theorem, and using the Hurwitz zeta function $\zeta(s,w)$

$$h_j(x) = \sum_{\ell=0}^{j} \binom{j}{\ell} (-a)^{j-\ell}\ \zeta(2s+\ell, x+1), \qquad \zeta(s,w) = \sum_{m=0}^{\infty} \frac{1}{(m+w)^s}$$

With the expansion of $s \mapsto \zeta(s, 1+a)$ at any point a:

$$\zeta(s, x+1) = \sum_{i=0}^{\infty} (-1)^i \binom{s+i-1}{i} \zeta(s+i, 1+a)\,(x-a)^i.$$

Truncation matrices (II)

On series expansions at $x = a$, the operator $\mathbf{G}_s$ is an infinite matrix

$$(M_{i,j}) := \text{ the coefficient of } (x-a)^i \text{ in } \mathbf{G}_s[(x-a)^j],$$

$$M_{i,j} = (-1)^i \sum_{\ell=0}^{j} \binom{j}{\ell} \binom{i+\ell+2s-1}{i} (-a)^{j-\ell} \zeta(2s+\ell+i, a+1).$$

We choose $a = 1/2$ for faster convergence. Then,

$$\zeta\left(s, \frac{3}{2}\right) = 2^s \left[\frac{1}{3^s} + \frac{1}{5^s} + \dots\right] = 2^s \left[-1 + \zeta(s)(1+2^{-s})\right].$$

The finite matrix $\mathbf{T}_s^{[m]}$ is the submatrix with indices $0 \le i, j < m$.
$\mathbf{T}_s^{[m]}$ is the truncated operator of $\mathbf{G}_s$ on polynomials with degree $< m$.

Matrices $\mathbf{T}_s^{[m]}$ provide a sequence of approximations to operator $\mathbf{G}_s$
$\Longrightarrow$ their spectrum provide good approximations to the spectrum of $\mathbf{G}_s$.

Computing the spectrum of matrices $\mathbf{T}_s^{[m]}$ (I)

In 1994, using Maple, Philippe computed the eigenvalues of the $\mathbf{T}_2^{[m]}$ for many values of $m \leq 100$ and numerical accuracy up to 150 digits.

He obtained a proven numerical value dominant eigenvalue $\lambda(2)$ of $\mathbf{G}_2$

$$\lambda(2) = 0.19945\,88183\,43767 \pm 10^{-15}$$

As m increases, the set of eigenvalues of $\mathbf{T}_2^{[m]}$ stabilize....
This stable set yields precise information on the complete spectrum of $\mathbf{G}_2$.

This provides convincing (but not proven) values for the first eigenvalues:

$$\begin{aligned}
\lambda^{(1)}(2) &\doteq +0.19945\,88183\,43767\,26019\,18456 \\
\lambda^{(2)}(2) &\doteq -0.07573\,95140\,84360\,60892\,78089 \\
\lambda^{(3)}(2) &\doteq +0.02856\,64037\,69818\,52783\,00174 \\
\lambda^{(4)}(2) &\doteq -0.01077\,74165\,76612\,69829\,31408 \\
\lambda^{(5)}(2) &\doteq +0.00407\,09406\,93426\,42144\,86407.
\end{aligned}$$

Computing the spectrum of matrices $\mathbf{T}_s^{[m]}$ (II)

The 37 eigenvalues found are all simple and they alternate in sign.

The ratios $r_j = \lambda^{(j}(2)/\lambda^{(j+1)}(2)$ show a remarkable stability,

$$r_1 = -2.633, \;\; r_2 = -2.651, \;\; r_3 = -2.650, \;\; r_4 = -2.647, \;\; r_5 = -2.644, \ldots$$

The spectrum of $\mathcal{G}$ is very nearly a geometric progression of ratio -2.64.

Introduction of a simplified model.

For large s, the operator $\mathbf{G}_s$ is dominated by its first term $\mathbf{C}_s$,

$$\mathbf{C}_s[f](x) = \frac{1}{(1+x)^{2s}} f(\frac{1}{1+x}),$$

The spectrum of $\mathbf{C}_s$ provides an approximation for the spectrum of $\mathbf{G}_s$.

The spectrum of $\mathbf{C}_s$ is an exact geometric progression,

which involves the fixed point $1/\phi$ of the map $x \mapsto 1/(1+x)$.

$$\mathrm{Sp}\,\mathbf{C}_s = \left\{ \lambda^{(j)}(s) = \frac{1}{\phi^{2s}} \cdot \frac{(-1)^{j-1}}{\phi^{2j}}, \;\; j \geq 0 \right\}, \qquad \phi = \frac{1+\sqrt{5}}{2}$$

The ratio between successive eigenvalues is $-\phi^2 = -2.61803$.

Conjectures on the spectrum of $\mathbf{G}_s$.

In theory of numbers, interest for the Gauss-Kusmin operator $\mathbf{G} := \mathbf{G}_1$

$$\mathbf{G}[f](x) := \sum_{m=1}^{\infty} \frac{1}{(m+x)^2} f\left(\frac{1}{m+x}\right)$$

Conjecture. The following statements about $\operatorname{Sp}\mathbf{G} := \{\lambda^{(n)}\}$ are true :

(i) The eigenvalues are simple, $|\lambda^{(n)}|$ strictly decreases.

(ii) They have alternating sign: $(-1)^n \lambda^{(n)} > 0$

(iii) The ratios have a limit .... [statement due to our works with Philippe]

$$\lim_{n\to\infty} \frac{\lambda^{(n)}}{\lambda^{(n+1)}} = -\phi^2, \qquad \text{and even} \quad \lambda^{(n)} \sim (-1)^{n+1}\phi^{-2n}$$

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Alkauskas announced in 2013 a proof of the conjecture.

He contacted me and asked if we had performed other experiments.

With Julien, we found other computations that were made by Philippe

(for $m = 100$)

```
#############################################################
# Compute Eigenvalues of G_4 by truncation method at z=a
#############################################################

###################################################
######### ACCURACY
###################################################
# Digits=10 is adequate till m=6  |
# Digits=15 is adequate till m=10 | => use Digits:=trunc(1.6*m).
# Digits=20 is adequate till m=14         |
# Digits=40 is adequate till truncation order of 30 (s=4)
# The accuracy is about 3.5^(-m) for the dominant eigenvalue
# that is to say:
#        [s=4] 0.5*10^(-10) for m=16 and 0.2*10^(-17) for m=30.
#        [s=2] 1.0*10^(-9) for m=16.
# This suggests that the eigenvalue for m=64 must be
# correct with an error less than 10^(-34)!
# Expect to fish about 3m/5 correct eigenvalues
# and about 1/3 spurious ones. (37 out of 64 for m=64)
###################################################
```

```
# From file eigenG4
# This file contains the first 37 eigenvalues of G_4
# probably correct to 10^(-16) each. This checks up
# to 0.5*10^(-17) with values computed for the trace.
# This sequence was "digested" from an eigenvalue computation
# involving:
#       m=64 (truncation order)
#       Digits:=100 (initial computation)
#       z0=a=1/2 (gives better convergence, it seems)
#       4 hours of CPU time on 31/12/94
#        -- cf procedure clean2 for cleaning the raw file "eigen64"
#
```

```
T64 := [.19945881834376726019184568598O, -.
757395140843606089278089485787e-1, .
285664037698185278300174698951e-1, -.
107774165766126982931408206228e-1, .
407094069342642144864070572698e-2, -.
153952180018239528860782668416e-2, .
582805733584502053498627548492e-3, -.
220823348268191236243289963002e-3, .
837325358853106433640466159955e-4, -.
317705979374238814577743179357e-4, .
120614646075357503402006049755e-4, -.
458127475393509489219335659010e-5, .
174083531597627046233302946233e-5, -.
661746767463171651407014547539e-6, .
251634386362587644553320401109e-6, -.
957141719446664013559114496044Oe-7, .
364164004463695812982581969824e-7, -.
138586405397045330950656708015e-7, .
527517673571311193500250769867e-8, -.
200834084247128475405019242296e-8, .
764740714110974322822834769879e-9, -.
291246485284948234718675736930e-9, .
110935606406100723633065236933e-9, -.
422610025809908283847794233095e-10, .
161013598339084136090662374610e-10, -.
613529002820587871724682219387e-11, .
233804914499454834510321712616e-11, -.
891076555583903764520535962109e-12, .
339637447359374500377396316216e-12, -.
129466840179437412125760751310e-12, .
493535166831712593143620280927e-13, -.
188141809398799938649294595742e-13, .
717382564780043982306975292912e-14, -.
273327928397144529492414050601e-14, .
103931726919482977829126764200e-14, -.
400572318378184997265264257954e-15, .
153441062476148303982245982378e-15]:
```

```
# Digits:=150; sp(100,4); lprint(%); quit;
# bytes used=6976975640, alloc=10876984, time=8490.55

# eigenvalues computed with 150 Digits and truncation order=100
# bytes used=6976975640, alloc=10876984, time=8490.55 <tricastin>
T100_raw:=
array(1 .. 100,
[(63)=.675184198154171255498757236262736735928849952192539596636\
61537347556481878102981125181336753687846419875102315860692151868546
3786927304\
3088544604386399e-19,
(64)=-.345088510843781299668803591242173244947117129281 62\
44670110714915244385989771623442446049595110616520471408667496093002
3156483825\
867065101287715537839 1e-19,
(1)=.19945881834376726019184568597987908069287418 75\
7141070624835226462238356595924491845490905020840126691628747337173 9
2767872462\
9445703328697908619653549 2,
(2)=-.757395140843606089278089485787484596279339492\
2771626520711263802728095687764310343797093473600128819368056340683 9
9946902396\
1617722918964587544343464 87e-1,
(3)=.28566403769818527830017469895078362554027 7\
6343184766949170829166483613588120514045758982925856822072343620436 7
9404289887\
5309105551294292127044505073 27e-1,
(4)=-.196607062352356154981795233420850004 87\
2803747199515457560747449277095692471776387225831484679823933132442 6
1111616202\
229912468019678989297995891172290 1e-1,
(5)=-.107774165766126982931408206228723 4\
1382891980143377860139411930590136356328493944544168130616016887880 4
6910970801\
0689721713390084792435081738381716250 8e-1,
(6)=.407094069342642144864070572703 6\
0162027717825103450001533504214209338594223941909850489368171343782 3
6160415516\
452934978970365973114061284442393467880 29e-2,
(7)=-.15395218001823952886078266 8\
4615075169465477937429460791348595393332989412717393224757940601805 9
1354206885\
2371171081214735412423659584691552943167009 16e-2,
(8)=.582805733584502053498627\
5482353104312705782322953267527596379115970106520220988276952521389 7
3221856613\
226937126054161482786593155705216472303787581313e-3,
(9)=-.220823348268191236 24\
3289974247912903355730627248964030792648919535454936460137544562119 4
7830902876\
514364524403127189478125684614083724121506348292599 1e-3,
(10)=.837325358853106 4\
3364047336051143224077845038579960078769303510307330870420457346215 9
9603304361\
204696538862681316605998011034966405993041577398444355 51e-4,
(11)=.491578617826\
9630649664587728155485107430655059007027085157437560888177182588425 9
3840404764\
049350959339064138946048146638395742749184229922811057964539e-4,
(12)=-.3177059\
7937423881457774536231142436579300018775302830296268783448390195975 1
0937004937\
3263251013408437480832915466930380772324893500423153126666305220 2e-4
,
(13)=.120\
6146460753575034020203015517890051684342384147708170545783421129266 2
3562962249\
4060860514772489098909019766644584687978210722651865818126154582549 1
3e-4,
(14)=
-.
4581274753935094892193651478010932488709411538485211832517127381174 2
63440613\
8328749504821117036835023028691966928147797900842130345937109610111 9
970571e-5,
(
15)=.
1740835315976270462356083667593469512498414362368046992378793873077 7
32673\
148889431108931023371037505132423321564831243816965775697109132\
90601145119281e-5,
(16)=-.
6617467674631716524868596509245135830920909969340128544019222120095 4
2\
8151175242649536644810588810766160234691287486778018443671471468138 9
2750432245\
041e-6,
```