Super-normality

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Outline

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Super-normality

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Expansion of real numbers (in some base $b$)

Fix an integer base $b \geq 2$. The alphabet is $A = \{0, 1, \ldots, b - 1\}$.

- if $b = 2$, $A = \mathbb{B} = \{0, 1\}$,
- if $b = 10$, $A = \{0, 1, 2, \ldots, 9\}$.

Each real number $\xi \in [0, 1)$ has an expansion in base $b$:

$x = a_1 a_2 a_3 \cdots$ where $a_i \in A$ and

$$\xi = \sum_{k \geq 1} \frac{a_k}{b^k}.$$

In the rest of this talk:
real number $\xi \in [0, 1)$ $\longleftrightarrow$ sequence $x \in A^\mathbb{N}$

$1/3 \longleftrightarrow 010101\cdots = (01)^\mathbb{N}$

$\pi/4 \longleftrightarrow 1100100100001111\cdots$
Normal sequences

A sequence is normal if all finite words (aka blocks) of the same length occur in it with the same (limit) frequency.

If $x \in A^\mathbb{N}$ and $w \in A^*$, the frequency of $w$ in $x$ is defined as

$$\text{freq}(x, w) = \lim_{N \to \infty} \frac{|x[1 : N]|_w}{N}.$$ 

where $x[1 : N]$ is the prefix of length $N$ of $x$ and $|x[1 : N]|_w$ is the number of occurrences of $w$ in it.

A sequence $x \in A^\mathbb{N}$ is normal if for each finite word $w \in A^*$:

$$\text{freq}(x, w) = \frac{1}{(\#A)|w|}$$

where

- $\#A$ is the cardinality of the alphabet $A$
- $|w|$ is the length of the finite word $w$. 
Normal sequences (continued)

Theorem (Borel, 1909)

The decimal expansion of almost every real number in $[0, 1)$ is a normal sequence over the alphabet $\{0, 1, \ldots, 9\}$.

Nevertheless, not so many examples have been proved normal. Some of them are:

- Champernowne 1933 (natural numbers):

  
  \[12345678910111213141516171819202122232425\ldots\]

- Besicovitch 1935 (squares):

  
  \[149162536496481100121144169196225256289324\ldots\]

- Copeland and Erdős 1946 (primes):

  
  \[235711131719232931374143475359616771737983\ldots\]
Super-normality

- Introduced in 2010 by Zeev Rudnick
- Talk in 2010 by Benjamin Weiss (to be found on YouTube)

Let \( \lambda > 0 \) be a positive real number. A binary (that is over the alphabet \( \mathbb{B} = \{0, 1\} \)) sequence \( x \in \mathbb{B}^\mathbb{N} \) is \( \lambda \)-super-normal if for each fixed integer \( k \geq 0 \)

\[
\lim_{n \to \infty} \frac{\# \{ w \in \mathbb{B}^n : |x[1 : \lfloor \lambda 2^n \rfloor]|_w = k \}}{2^n} = e^{-\lambda} \frac{\lambda^k}{k!}.
\]

A binary sequence \( x \) is super-normal if it is \( \lambda \)-super-normal for each real number \( \lambda > 0 \).
Binomial law

Suppose that the probability of an event $X$ is $0 \leq p \leq 1$. The probability of having $k \geq 0$ occurrences of $X$ in $N$ independent draws is

$$\binom{N}{k} p^k (1 - p)^{N-k}$$

Of course

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1 - p)^{N-k} = 1$$

and $k$ and $p$ being fixed

$$\lim_{N \to \infty} \binom{N}{k} p^k (1 - p)^{N-k} = 0$$
Convergence to the Poisson law

Let $\lambda > 0$ fixed and $N \geq 1$ and $0 \leq p \leq 1$ such that $Np = \lambda$.

$$ \lim_{N \to \infty, Np=\lambda} \binom{N}{k} p^k (1-p)^{N-k} = e^{-\lambda} \frac{\lambda^k}{k!} $$

Of course

$$ \sum_{k \geq 0} e^{-\lambda} \frac{\lambda^k}{k!} = 1 $$

This is the Poisson law.
Convergence to the Poisson law (continued)

\[
\lim_\substack{N \to \infty \\ Np = \lambda} \binom{N}{k} p^k (1 - p)^{N-k} \\
= \lim_\substack{N \to \infty} \binom{N}{k} \left( \frac{\lambda}{N} \right)^k \left( 1 - \frac{\lambda}{N} \right)^{N-k} \\
= \lim_\substack{N \to \infty} \frac{N(N-1)\cdots(N-k+1)}{N^k} \left( \frac{\lambda}{N} \right)^k \left( 1 - \frac{\lambda}{N} \right)^N \\
= e^{-\lambda} \frac{\lambda^k}{k!}
\]
Binary words of length $n$ as events

Suppose that symbols 0 and 1 in $\mathbb{B}$ have probability $1/2$. The probability of a word $w \in \mathbb{B}^n$ of length $n$ is $p = 2^{-n}$.

The prefix of length $N = \lfloor \lambda 2^n \rfloor$ of $x$ is seen as $\lfloor \lambda 2^n \rfloor$ independent draws of words of length $n$. For each $k \geq 0$,

$$\lim_{n \to \infty} \frac{\#\{w \in A^n : |x[1:\lfloor \lambda 2^n \rfloor]|_w = k\}}{2^n} = e^{-\lambda} \frac{\lambda^k}{k!}$$
What is known

Theorem (Weiss)

*If* $x$ *is* $\lambda$*-super-normal* *for some* $\lambda > 0$, *then* $x$ *is normal.*

Key ingredient:

Lemma (Hot spot/Pyatetskiǐ–Shapiro)

*If there is a constant* $K$ *such for each word* $w \in A^*$,

$$
\limsup_{N \to \infty} \frac{|x[1 : N]|_w}{N} \leq \frac{K}{(#A)|w|}
$$

*then* $x$ *is normal.*

Proposition (Weiss)

*The* Champernowne sequence *is not* 1*-super-normal.*
What is not known (yet)

Theorem (Weiss)

The set of super-normal sequences has measure 1.

- An explicit example of a super-normal sequence?
Open questions

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Thank you