Super-normality

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Expansion of real numbers (in some base b)

Fix an integer base $b \geqslant 2$. The alphabet is $A = \{0, 1, \ldots, b-1\}$.

- if $b = 2$, $A = \mathbb{B} = \{0, 1\}$,
- if $b = 10$, $A = \{0, 1, 2, \ldots, 9\}$.

Each real number $\xi \in [0, 1)$ has an expansion in base b:
$x = a_1 a_2 a_3 \cdots$ where $a_i \in A$ and

$$\xi = \sum_{k \geqslant 1} \frac{a_k}{b^k}.$$

In the rest of this talk:

$$\begin{array}{rcl} \text{real number } \xi \in [0,1) & \longleftrightarrow & \text{sequence } x \in A^{\mathbb{N}} \\ 1/3 & \longleftrightarrow & 010101 \cdots = (01)^{\mathbb{N}} \\ \pi/4 & \longleftrightarrow & 110010010000111 1\cdots \end{array}$$

Normal sequences

A sequence is normal if all finite words (aka blocks) of the same length occur in it with the same (limit) frequency.

If $x \in A^{\mathbb{N}}$ and $w \in A^*$, the frequency of w in x is defined as

$$\mathrm{freq}(x, w) = \lim_{N \to \infty} \frac{|x[1:N]|_w}{N}.$$

where $x[1:N]$ is the prefix of length N of x and $|x[1:N]|_w$ is the number of occurrences of w in it.

A sequence $x \in A^{\mathbb{N}}$ is normal if for each finite word $w \in A^*$:

$$\mathrm{freq}(x, w) = \frac{1}{(\#A)^{|w|}}$$

where
- $\#A$ is the cardinality of the alphabet A
- $|w|$ is the length of the finite word w.

Normal sequences (continued)

Theorem (Borel, 1909)

The decimal expansion of almost every real number in $[0,1)$ *is a normal sequence over the alphabet* $\{0,1,...,9\}$.

Nevertheless, not so many examples have been proved normal. Some of them are:

- Champernowne 1933 (natural numbers):

$$12345678910111213141516171819202122232425\cdots$$

- Besicovitch 1935 (squares):

$$149162536496481100121144169196225256289324\cdots$$

- Copeland and Erdős 1946 (primes):

$$235711131719232931374143475359616771737983\cdots$$

Super-normality

- Introduced in 2010 by Zeev Rudnick
- Talk in 2010 by Benjamin Weiss (to be found on YouTube)

Let $\lambda > 0$ be a positive real number. A binary (that is over the alphabet $\mathbb{B} = \{0, 1\}$) sequence $x \in \mathbb{B}^{\mathbb{N}}$ is λ-super-normal if for each fixed integer $k \geqslant 0$

$$\lim_{n\to\infty} \frac{\#\{w \in \mathbb{B}^n : |x[1 : \lfloor \lambda 2^n \rfloor]|_w = k\}}{2^n} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

A binary sequence x is super-normal if it is λ-super-normal for each real number $\lambda > 0$.

Binomial law

Suppose that the probability of an event X is $0 \leqslant p \leqslant 1$. The probability of having $k \geqslant 0$ occurrences of X in N independent draws is

$$\binom{N}{k} p^k (1-p)^{N-k}$$

Of course

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = 1$$

and k and p being fixed

$$\lim_{N \to \infty} \binom{N}{k} p^k (1-p)^{N-k} = 0$$

Convergence to the Poisson law

Let $\lambda > 0$ fixed and $N \geqslant 1$ and $0 \leqslant p \leqslant 1$ such that $Np = \lambda$.

$$\lim_{\substack{N\to\infty \\ Np=\lambda}} \binom{N}{k} p^k (1-p)^{N-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

Of course

$$\sum_{k\geqslant 0} e^{-\lambda} \frac{\lambda^k}{k!} = 1$$

This is the Poisson law.

Convergence to the Poisson law (continued)

$$\lim_{\substack{N\to\infty \\ Np=\lambda}} \binom{N}{k} p^k (1-p)^{N-k}$$

$$= \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1-\frac{\lambda}{N}\right)^{N-k}$$

$$= \lim_{N\to\infty} \frac{N(N-1)\cdots(N-k+1)}{N^k} \left(1-\frac{\lambda}{N}\right)^N \frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\frac{\lambda^k}{k!}$$

Binary words of length n as events

Suppose that symbols 0 and 1 in $\mathbb{B}$ have probability 1/2.
The probability of a word $w \in \mathbb{B}^n$ of length n is $p = 2^{-n}$.

The prefix of length $N = \lfloor \lambda 2^n \rfloor$ of x is seen as $\lfloor \lambda 2^n \rfloor$ independent draws of words of length n. For each $k \geqslant 0$,

$$\lim_{n \to \infty} \frac{\#\{w \in A^n : |x[1 : \lfloor \lambda 2^n \rfloor]|_w = k\}}{2^n} = e^{-\lambda} \frac{\lambda^k}{k!}$$

What is known

Theorem (Weiss)

If x is λ-super-normal for some $\lambda > 0$, then x is normal.

Key ingredient:

Lemma (Hot spot/Pyatetskiĭ–Shapiro)

If there is a constant K such for each word $w \in A^*$,

$$\limsup_{N\to\infty} \frac{|x[1:N]|_w}{N} \leqslant \frac{K}{(\#A)^{|w|}}$$

then x is normal.

Proposition (Weiss)

The Champernowne sequence is not 1-super-normal.

What is not known (yet)

Theorem (Weiss)

The set of super-normal sequences has measure 1.

- An explicit example of a super-normal sequence ?

Open questions

- An explicit example of a super-normal sequence ?

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- An explicit example of a super-normal sequence ?

Thank you