Multi-dimensional Continued fractions and Euclidean dynamics

March 23-27, 2020

Geometry & Dynamics of best approximation

GOAL: Motivate a class of generalizations (of CFRAC) that
- retain most of the theory
- relevant to Diophantine approximation.

- Simple continued fraction - think of as giving coordinates on $\mathbb{R}$, turning it into a tree
- Any question about a real number reduces to a question about a sequence of integers

$$\alpha \in \mathbb{Q}/\mathbb{Z} \rightsquigarrow (a_k)_{k\geq 1} \overset{(*)}{\rightsquigarrow} (q_k)_{k\geq 1} \quad \text{increasing}$$

$$q_{k+1} = a_{k+1} q_k + q_{k-1} \qquad \text{"recurrence relations"}$$

- The map (*) has a one-sided inverse: Euclidean Algorithm.

BOTH SEQUENCES CONTAIN ESSENTIALLY THE SAME INFORMATION, SO WE ARE WILLING TO SACRIFICE ONE FOR THE OTHER, IF NECESSARY.

Essential feature: terms of either come from a countable set.

Lagarias (1980s-90s)

- "Simplex cutting" algorithm don't work
- initiated systematic study of "best approximants"
- Guiding Principle: Renormalization dynamics

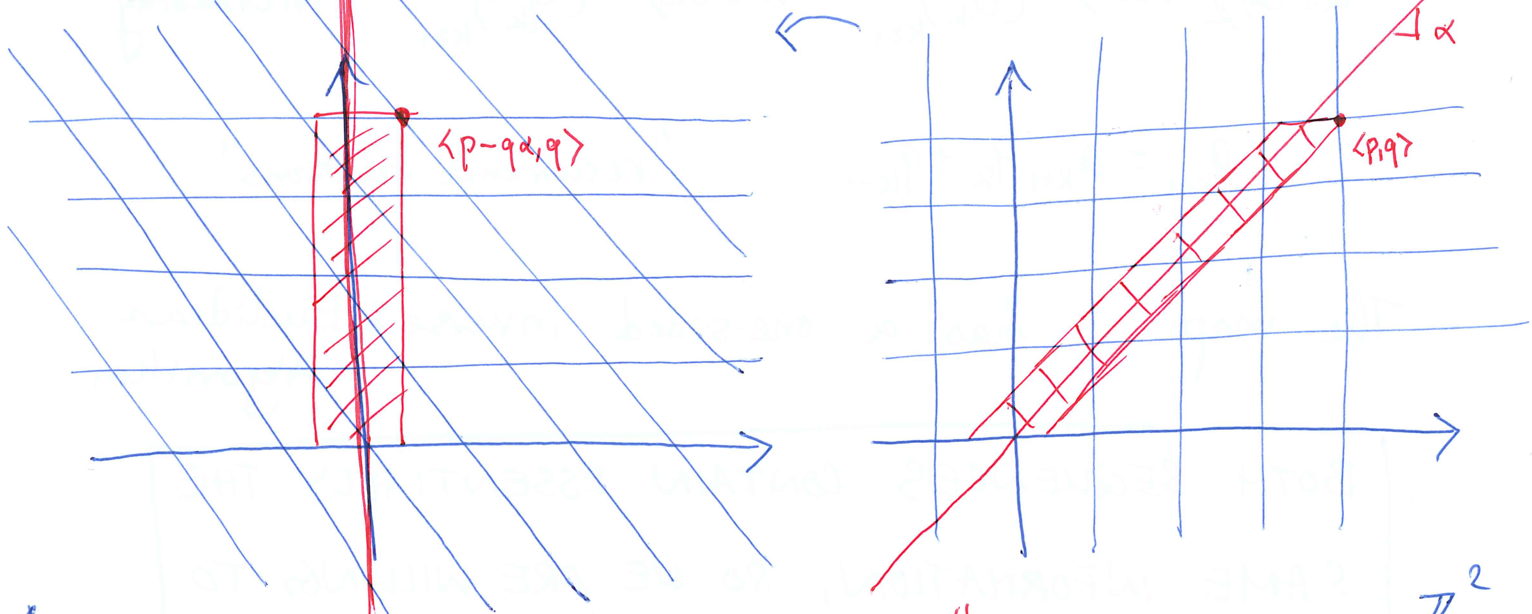
Fancy type of symmetry (that is sometimes non-invertible)

Gauss map $\alpha \rightsquigarrow \{1/\alpha\}$

Lagrange: p/q is a best approximant* to α (of 2^{nd} kind) if and only if it is a convergent of CFRAC.

* $|q\alpha - p| \leq |n\alpha - m|$ $\forall$ $\frac{m}{n} \in \mathbb{Q}$ s.t. $1 \leq n \leq q$ & strict ineq. if $n < q$.

Geometric interpretation $h_\alpha = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$ "inverse slope"

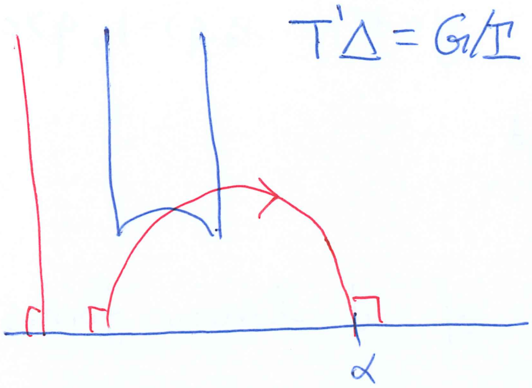


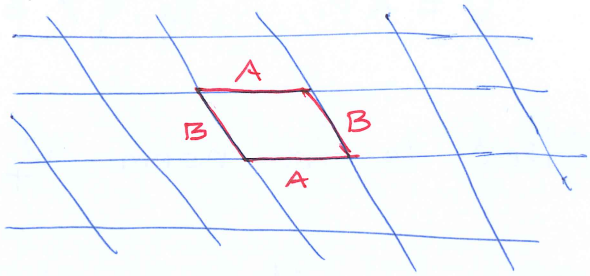
$\Lambda_\alpha = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$ "sheared integer lattice"

"EMPTY PARALLELOGRAM CONDITION" $\mathbb{Z}^2$

<u>Magnifying</u> <u>Glass</u> $\quad \alpha \rightsquigarrow \Lambda_\alpha \quad \in SL_2\mathbb{R}/SL_2\mathbb{Z}$.

$T^1\Delta = G/\Gamma$





Number Theory $\qquad$ linear action on $\mathbb{R}^2$

2D lattice $\Leftrightarrow$ flat genus one surface $\rightsquigarrow$ n-tori

$\rightsquigarrow$ higher genus

"Euclidean dynamics"

<u>Translation</u> <u>surface</u> $\quad X = (\sqcup P_i)/\sim \quad P_i \subset \mathbb{C}$ (polygons)

ω = holomorphic 1-form on X induced by dz

$SL_2\mathbb{R}$ acts on $\Omega M_g = \{(X,\omega) : \ldots\}/\sim$ induced by linear action on $\mathbb{R}^2$

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* Teichmüller geodesic flow = restriction to $g_t = \mathrm{diag}(e^t, e^{-t})$.

i.e. linear action on discrete subgroups of $\mathbb{R}^2$.

Q: What does this have to do with continued fractions?

A: Gauss map $G_1(x) = \{1/x\}$ can be recovered

from a Poincaré section $G_2(x,y) = (G_1(x), \frac{1}{k+y})$

where $k = [1/x]$.

<u>Simultaneous</u> <u>approximation</u> $\theta \in \mathbb{R}^d$, $\frac{\vec{p}}{q} \in \mathbb{Q}^d$

$\vec{v} = \langle p_1, \dots, p_d, q \rangle \in \mathbb{Z}^{d+1}$ s.t. $\gcd(p_1, \dots, p_d, q) = 1$, $q > 0$.

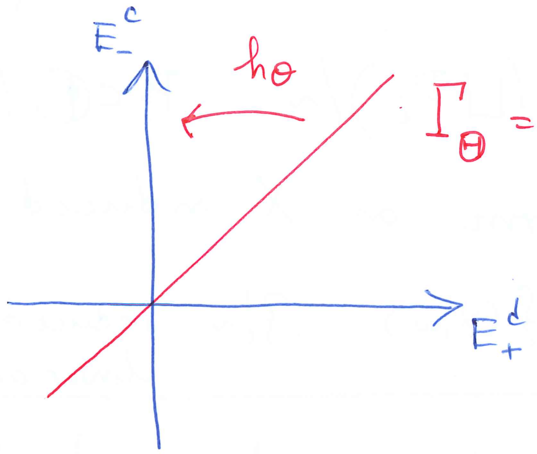
Need to fix norm ν_+ on $\mathbb{R}^d$.

More generally, consider system of d linear forms in c variables.

$d \times c$ matrix $\Theta : E_-^c \to E_+^d$

"eigenspaces"

$$\mathbb{R}^n = \mathbb{R}^{d+c} = E_+^d \oplus E_-^c \curvearrowleft g_t = \left[\begin{array}{c|c} e^{ct}\mathbb{1}_d & \\ \hline & e^{-dt}\mathbb{1}_c \end{array}\right]$$



Γ_Θ = graph of Θ

$$\boxed{h_\Theta(\Gamma_\Theta) = E_-^c}$$

"horizontal shear" (def)

<u>Number</u> <u>Theory</u>: Position of Γ_Θ rel. to $\mathbb{Z}^{d+c}$

<u>Dynamics</u>: Evolution of $\underbrace{\Lambda_\Theta = h_\Theta \mathbb{Z}^{d+1}}_{\text{sheared integer lattice}}$ under g_t

<u>DEF</u> $(\vec{p}, \vec{q}) \in \mathbb{Z}^{d+c}$ <u>best</u> <u>approx</u>. <u>vector</u>* rel. $\nu_\pm$ on $E_\pm$

if $\|\vec{p} - \Theta\vec{q}\|_+ \leq \|\vec{m} - \Theta\vec{n}\|_+$ $\forall (\vec{m}, \vec{n}) \in \mathbb{Z}^{d+c}$ s.t. $0 < \|\vec{n}\|_- \leq \|\vec{q}\|_-$

(with strict ineq. if $\|\vec{n}\|_- < \|\vec{q}\|_-$)

* to Θ

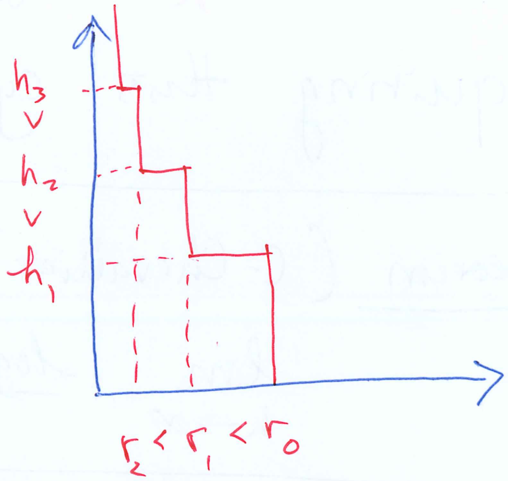
(1D) Staircase of Λ (rel. $\nu_\pm$)

Λ-cylinder $\quad C(r,h) = \overline{B}_+(r) \times \overline{B}_-(h)$

- $\Lambda \cap \mathrm{Int}(C) = \{0\}$
- $\Lambda \cap \mathrm{Bd}(C) \neq \emptyset$

The collection of Λ-cylinders with partial order induced by inclusion we shall call the staircase of Λ.

(Remark: They happen to be linearly ordered by "slope" $\frac{h}{r}$.)



Theorem The local minima of the staircase of Λ_Θ ~~correspond to~~ are realized by best approximation vectors.

$\langle \vec{p}, \vec{q} \rangle$ best. approx to Θ $\rightsquigarrow$ $\langle \vec{p} - \Theta\vec{q}, \vec{q} \rangle \rightsquigarrow (\|\vec{p}-\Theta\vec{q}\|_+, \|\vec{q}\|_-)$

This generalizes the geometric interpretation of Lagrange.

Remark: Staircase can also be seen as graph of systole.

$$\Sigma(\Lambda_\theta) = \{ (r,h) \in \mathbb{R}^2_+ : C(r,h) \text{ is a } \Lambda_\theta\text{-cylinder} \}$$

$$\left\{ \begin{array}{c} \text{minimal} \\ \Lambda\text{-cylinders} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{best approx.} \\ \text{vectors} \end{array} \right\} / \sim$$

Balanced cylinder : $C(r,h)$ $r = h$.

Largest balanced cylinder without nonzero lattice points in its interior belongs to $\Sigma(\Lambda_\theta)$

A transversal S to g_t-flow is defined by requiring this cylinder be maximal.

Theorem (C-Chevallier) For a.e. θ

$$\lim_{k \to \infty} \frac{\log h_k(\Lambda_\theta)}{k} = d \cdot \frac{\mu(\mathcal{L}_{d+c})}{\nu(S)}$$

- $d = c = 1$ Khintchine-Lévy 1936 $\frac{\pi^2}{12 \ln 2}$
- Here, $\mu = \text{Haar}(SL_{d+c}\mathbb{R}/SL_{d+c}\mathbb{Z})$ (Note: ratio indep. of normalization)
  ν = induced measure on S

$$\boxed{\frac{\mu(\mathcal{L}_{d+c})}{\nu(S)} = \text{average return time to } S}$$

- Entropy (of g_{-t}) along E^d_- is d (instead of 1)
  $\parallel$
  $\sum_{|\lambda|>1} \log|\lambda|$

Higher dim'l staircases $\Lambda \subset \mathbb{R}^n$ $(n \geq 3)$

Λ-box $B[r] = \prod_{1}^{n} [-r_i, r_i]$ $(r_i \geq 0)$

$\Updownarrow$

- $\Lambda \cap \operatorname{Int} B = \{0\}$
- $\Lambda \cap (\partial B) \neq \phi$

Notation: $u \in \mathbb{R}^n$ write $B(u) = B[r_u]$

where $(r_u)_i = |u_i|$.

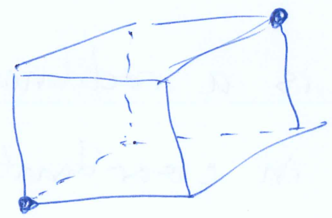
$B(u)$ = smallest box containing u.

DEF $\Sigma(\Lambda) = \{ u \in \mathbb{R}^n_+ : B(u) \text{ is a } \Lambda\text{-box} \}$

We say u is a corner of B if $B = B(u)$.

Lemma $B \in \Sigma(\Lambda)$ is a local min $\Leftrightarrow$ $\Lambda \cap (\partial B) \subset$ "corners" B.

Pf see picture:



Definition $\vec{\theta} \in \mathbb{R}^d$, $\vec{p}/q \in \mathbb{Q}^d$ $(d \geq 2, n = d+1)$. $c=1$

$\vec{p}/q$ is a convergent of $\vec{\theta}$ if $B(h_\theta \langle \vec{p}, q \rangle)$

is a local minimum of $\Sigma(\Lambda_\theta)$.

Lemma $\inf_{n>0} n \|n\theta_1\| \cdots \|n\theta_d\| = \inf_{n \in \pi(\theta)} - \text{"} -$

$\pi(\theta)$ = convergent denominators

Minkowski(-Voronoi) continued fraction algorithm

$Q(\Lambda) :=$ largest cube w/o nonzero lattice points in its interior

"systole cube"

$= [-r, r]^n$ where $r = \min\{\|u\|_\infty : u \in \Lambda, u \neq 0\}$

$$A = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : \prod a_i = 1,\ a_i > 0 \right\} \curvearrowright SL_n\mathbb{R}/SL_n\mathbb{Z}$$

Since $n \geq 3$, this is a higher rank (abelian) action.

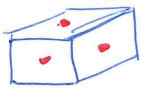
Problem: How well can we control A-orbits?

Define a transversal to A-action by

$$\Xi_n = \{ \Lambda \in G/\Gamma : Q(\Lambda) \text{ is a } \underline{\text{loc. max}} \text{ of } \Sigma(\Lambda) \}$$

(**Remark**: Ξ_n is a rectilinear domain in coordinates)

Lemma $B \in \Sigma(\Lambda)$ loc. max $\iff \Lambda \cap \partial_i B \neq \emptyset \ \forall i$

(Pf) 

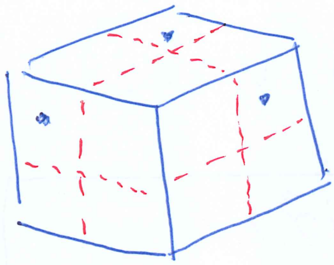
Ξ_n has, essentially, 3 connected components (if $n=3$)

This was known to Minkowski, but mostly forgotten.

We shall explain this using a combinatorial invariant. (signature)

<u>Signature</u> $\Lambda \in \Xi_n \setminus \underbrace{\{\ldots\}}_{\text{positive codimension}}$ $n=3$

$Q(\Lambda)$ is maximal (as a Λ-box). Assume



$[-1,1]^n \cong$

(1) unique elmt on each of Λ open face.

(2) ~~the~~ none of these lie on coord. hyperplanes

Each face is divided into $\cancel{2^{n-1}}$ 4 unit ~~cubes~~ squares exactly one of which contains a lattice pt.

Define $\nu(\text{edge})$ = total # occupied adjacent squares $\in \{0, 1, 2\}$

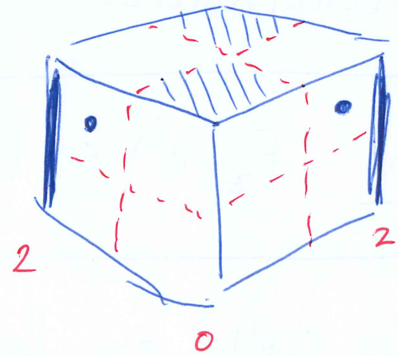
Note $\nu(-\text{edge}) = \nu(\text{edge})$.

Signature = multi-set of order 6 recording $\nu(\text{edge})$

<u>Theorem</u> Signature $= \{0, 1^4, 2\}$, $\{0^3, 2^3\}$ OR $\{1^6\}$.

<u>Proof</u> $\exists 2 \Leftrightarrow \exists 0$

"Corner condition"



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Let $u, v, w \in \Lambda \cap \partial B$ denote the unique elmts on the "positive" x-, y-, z-faces, respectively.

(W.l.o.g. may assume $u_3 > 0$ & $v_3 > 0$.)

Theorem (Minkowski) Either $w = u + v$ $\{0^3, 2^3\}$

OR $\Lambda = \mathbb{Z}u + \mathbb{Z}v + \mathbb{Z}w$ "index 1"

Q: Analog in higher dimensions?

Define $F_z : \Xi_3^* \circlearrowleft$ by 3 steps

(1) Shrink the Λ-box $Q(\Lambda)$ in z-direction until first time it is extendible in x- or y-direction.

(2) Expand it until a maximal box is obtained.

(3) Renormalize to make new box a cube.

Theorem (Minkowski) $F_z(\Lambda) \in \{0^3, 2^3\} \Rightarrow F_z^2(\Lambda) \notin \{0^3, 2^3\}$.

Minkowski CFRAC := first return to $\underbrace{}_{\text{union of}}$ "index 1" components