Reachability for multidimensional continued fractions and minimality of interval exchange transformations

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Reachability / orbit problem

Let $\mathcal{I}$ be a set of triples (f, X, Y) where $f : D \to D$ is a function $X \subset D$ and $Y \subset D$.

$\texttt{REACH}(\mathcal{I})$: **input:** a triple (f, X, Y) in $\mathcal{I}$
output: whether there exists $x \in X, y \in Y, n \geq 0$ such that $f^n(x) = y$

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Our functions $f : D \to D$ will be piecewise affine maps on K^d where $K \subset \mathbb{R}$ and X, Y will be polyhedron.

Example 1: piecewise translations

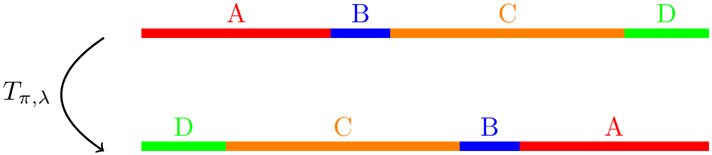
$d \geq 1$

$\Lambda \subset \mathbb{R}$ stable under addition

$f : [0,1]^d \to [0,1]^d$ bijective piecewise translations with discontinuities contained in hyperplanes $\{x_i = \alpha\}$ with $i \in \{1, \ldots, d\}$ and $\alpha \in \Lambda$ and translations contained in Λ^d.

Example 1: piecewise translations $d = 1$

interval exchange transformations with permutation $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$



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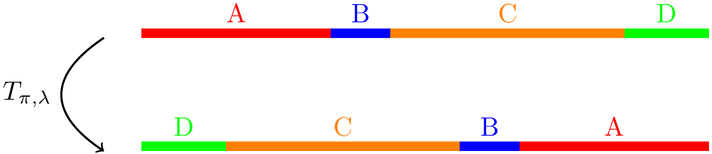

Theorem (Keane condition)

$X = \{\lambda_D, \lambda_D + \lambda_C, \lambda_D + \lambda_C + \lambda_B\}$
$Y = \{\lambda_A, \lambda_A + \lambda_B, \lambda_A + \lambda_B + \lambda_C\}$
If there is no $n \geq 0$, $x \in X$ and $y \in Y$ such that $T^n_{\pi,\lambda}(x) = y$ then all infinite orbits of $T_{\pi,\lambda}$ are dense.

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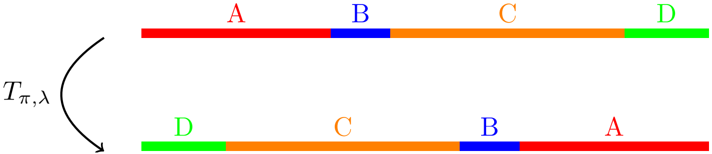

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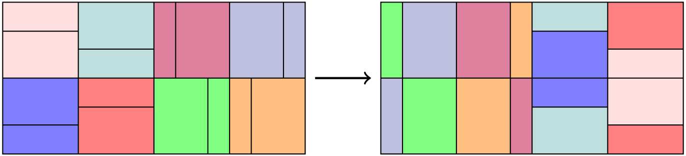
Remark:

- $\lambda \in \mathbb{Q}^4 \Rightarrow T_{\pi,\lambda}$ completely periodic
- λ totally irrational $\Rightarrow T_{\pi,\lambda}$ satisfies Keane condition

Example 1: piecewise translations $d = 2$

P. Hooper rectangle exchange system ($d = 2$)



Example 2: simplicial systems

Piecewise affine functions with Markovian topological dynamics.

Why do we care about reachability?

1. Dynamical systems: periodic orbits, minimality of interval exchange transformations, $\mathrm{GL}(2,\mathbb{R})$-orbit closures of translation surfaces,
2. Theoretical computer science
3. Control theory

Decidability

For any $d \geq 1$, $\texttt{REACH}(\mathrm{Mat}_{d \times d}(K))$ is decidable (Kannan-Lipton 1980 for $K = \mathbb{Q}$).

Undecidability result

We define a restricted class $\mathcal{F}$ of 2-dimensional rational piecewise affine functions on $D = \left([0,1) \cap \mathbb{Z}\left[\frac{1}{2}\right]\right)^2$ with discontinuities contained in hyperplanes $x = p/2^n$ or $y = p/2^n$
Piecewise $(x, y) \mapsto (ax + by + c, dx + ey + f)$ with $a, b, d, e \in \{1, 2, \frac{1}{2}\}$ and $c, f \in \mathbb{Z}\left[\frac{1}{2}\right]$.

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Theorem (folklore bis)

There exists an explicit function $f \in \mathcal{F}$ *such that the problem* $\mathtt{REACH}(\{(f, x, (0,0))\}_{x \in D})$ *is undecidable.*

Open problems

Whether the reachability problem is decidable or undecidable is an open question for the following classes of maps

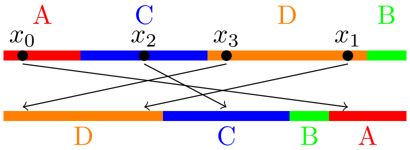
- interval exchange transformations over number fields,
- 1-dimensional piecewise rational affine maps,
- d-dimensional piecewise translations over number fields,
- simplicial systems,
- ...

This talk

1. Partial result for reachability of interval exchange transformations.
2. Link with a specific reachability problem for simplicial systems.

Periodic orbits and relations

(π, λ): interval exchange transformation.
A periodic orbit gives rise to a non-negative relation on translations



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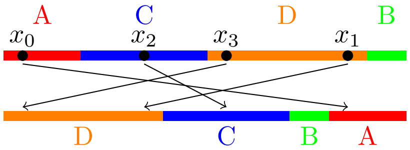

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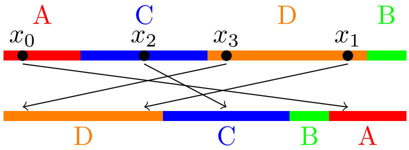

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Lemma

Let π be a permutation and $\lambda \in \mathbb{R}^d$. If $\mathbb{Z}^d_{\geq 0} \cap \mathrm{R}(\tau) = \{0\}$ then the iet $T_{\pi,\lambda}$ has no periodic orbit.

Rauzy induction

Idea: dynamics on the space of interval exchanges $\mathcal{R} : (\pi, \lambda) \mapsto (\pi', \lambda')$

integral non-negative matrix cocycle $A_n(\pi, \lambda)$: $\lambda = A_n(\pi, \lambda) \cdot \lambda^{(n)}$.

We have $\tau^{(n)} = A_n(\pi, \lambda)^t \tau^{(0)}$.

Improved lemma with induction

Replace $C_0(\pi, \lambda) := \mathbb{Z}^d_{\geq 0}$ by the subcone $C_n(\pi, \lambda) := A_n(\pi, \lambda)\mathbb{Z}^d$.

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Let π be a permutation and $\lambda \in \mathbb{R}^d$. If the Rauzy induction is well defined up to step n and $C_n(\pi, \lambda) \cap \mathrm{R}(\tau) = \{0\}$ then the iet $T_{\pi,\lambda}$ has no periodic orbit.

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Questions:

1. under which condition Rauzy induction is well defined?
2. what does look like $C_n(\pi,\lambda)$?

Infinite orbits of the Rauzy induction

Theorem (Rauzy, Veech)

Let π be an irreducible permutation and $\lambda \in \mathbb{R}^A$. Then the following conditions are equivalent

- (π, λ) *satisfies the Keane condition,*
- *Rauzy induction is defined for all times and* $\lambda^{(n)} \to 0$,
- *for all* $n \geq 0$, *there exists* m *such that* $A_m(\pi^{(n)}, \lambda^{(n)}) > 0$.

Short digression on Multidimensional continued fractions

The shape of the cones $C_n(\pi, \lambda)$

Theorem (Rauzy, Veech)

Let π be an irreducible permutation and $\lambda \in \mathbb{R}^A$. Then

- *if (π, λ) does not satisfy Keane condition, then there exists n such that the rightmost intervals of $(\pi^{(n)}, \lambda^{(n)})$ have equal lengths ("trivial saddle connection"),*
- *if (π, λ) satisfies the Keane condition, then $C_n(\pi, \lambda)$ converges to the cone of invariant measures of $T_{\pi,\lambda}$.*

The semi-algorithm

input: (π, λ)

- n = 0
- `repeat`
  - `if` $C_n(\pi, \lambda) \cap R(\pi, \lambda) = \{0\}$: `return` "no periodic orbit"
  - `if` Keane condition is violated at the *n*-th step: `return` "found saddle connection"
  - n = n + 1

bad news: SAF=0

Starting from 6 letters, there exists some (π, λ) with

- Keane condition
- $C_\infty(\pi, \lambda) \cap \mathrm{R}(\tau) \neq \{0\}$.

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The most famous one is the so-called Arnoux-Rauzy example

$$\pi = \begin{pmatrix} A_{1,\ell} & A_{1,r} & A_2 & B_1 & B_2 & C_1 & C2 \\ A_{1,r} & B_2 & B_1 & C_2 & C_1 & A_2 & A_{1,\ell} \end{pmatrix} \quad \lambda = (\theta+1, \theta^2-\theta-1, \theta^2, \theta, \theta, 1, 1)$$

where $\theta^3 - \theta^2 - \theta - 1 = 0$.

Lemma

The semi-algorithm does not terminate on minimal SAF=0 instances.

Hope (out of reach)

Conjecture

Let π be an irreducible permutation on d letters and $\lambda \in (\mathbb{R}_{\geq 0} \cap \overline{\mathbb{Q}})^d$. Then if $T_{\pi,\lambda}$ is minimal, it is uniquely ergodic.

If the conjecture was true, then the semi-algorithm would always terminate on algebraic λ with $\mathrm{SAF}(\pi, \lambda) \neq 0$.

It is an algorithm on quadratic fields

Theorem (Boshernitzan 88)

For λ in a quadratic number field $\mathbb{Q}[\sqrt{D}]^d$ the semi-algorithm always terminate

- *SAF=0 implies completely periodic,*
- *minimal implies uniquely ergodic.*